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COVERING SYSTEMS AND THEIR CONNECTIONS TO ZERO-SUMS

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Abstract. A finite system of residue classes is called a covering system if every integer belongs to one of the residue classes. Paul Erdős invented this concept and initiated the study of this fascinating topic. On the basis of known connections between covering systems and unit fractions, the speaker recently found that covering systems are closely related to zero-sum problems on abelian groups (another interesting topic initiated by P. Erdős). In this talk we will introduce some recent results connecting covering systems and zero-sums, as well as several related open problems. We will also give a survey of results on covers of a group by cosets, and talk about Gao and Geroldinger's reduction of some zero-sum problems to the study of coverings of a subset of a group, as well as Dimitrov's new discovery of the connection between the Davenport constant of $\mathbb{Z}_n^r$ and covers of $\mathbb{Z}_n^r$.

1. Properties of covers of $\mathbb{Z}$ related to unit fractions

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ we call

$$a(n) = a + n\mathbb{Z} = \{a + nx : \ x \in \mathbb{Z}\}$$

a residue class with modulus n.

For a finite system

$$A = \{a_s(n_s)\}_{s=1}^k = \{a_1(n_1), \ldots , a_k(n_k)\}$$

of residue classes, if $\bigcup_{s=1}^k a_s(n_s) = \mathbb{Z}$ then we call A a *covering system* of $\mathbb{Z}$, or a *cover* of $\mathbb{Z}$ in short. This concept was first introduced by Paul Erdős in the early 1930's. Clearly A forms a cover of $\mathbb{Z}$ if and only if it covers $0, 1, \cdots, N_A - 1$, where N_A is the least common multiple of the moduli $n_1, \cdots, n_k$. If A covers every integer exactly once, then we call A an *exact cover* of $\mathbb{Z}$ or a *disjoint cover* of $\mathbb{Z}$.

As any integer can be written uniquely in the form $nq + r$ with $q \in \mathbb{Z}$ and $r \in R(n) = \{0, 1, \cdots, n-1\}$, the finite system $\{r(n)\}_{r=0}^{n-1}$ is a disjoint cover of $\mathbb{Z}$. The first nontrivial cover of $\mathbb{Z}$ with distinct moduli was the following one discovered by P. Erdős.

$$B = \{0(2),\ 0(3),\ 1(4),\ 5(6),\ 7(12)\}.$$

Note that $N_B = 12$ and B covers $0, 1, \ldots, 11$.

Clearly each residue class $a_s(n_s)$ in system A covers exactly N_A/n_s integers in $R(N_A) = \{0, 1, \ldots, N_A - 1\}$. Thus, if A is a cover of $\mathbb{Z}$ then

$$|R(N_A)| \leqslant \sum_{s=1}^k \frac{N_A}{n_s} \quad \text{and hence} \quad \sum_{s=1}^k \frac{1}{n_s} \geqslant 1;$$

if A is a disjoint cover of $\mathbb{Z}$ then $\sum_{s=1}^k 1/n_s = 1$. Observe that the sum of reciprocals of the moduli in the cover B equals

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{12} = 1\frac{1}{3}.$$

In 1989, by using the Riemann zeta function, M. Z. Zhang [J. Sichuan Univ. (Nat. Sci. Ed.)] showed the following surprising result: *If system*

A is a cover of $\mathbb{Z}$ then $\sum_{s\in I} 1/n_s \in \mathbb{Z}^+$ for some $I \subseteq [1,k] = \{1,\cdots,k\}$. The starting point of Zhang is that A forms a cover of $\mathbb{Z}$ if any only if

$$\prod_{s=1}^{k}\left(1-e^{2\pi i(n+a_s)/n_s}\right)=0 \quad \text{for all } n=1,2,3,\cdots.$$

The crucial trick in Zhang's proof is that for a real number c the series

$$\sum_{n=1}^{+\infty}\frac{e^{2\pi icn}}{n}$$

diverges if and only if c is an integer.

The *covering function* of $A=\{a_s(n_s)\}_{s=1}^{k}$ is defined by

$$w_A(x)=|\{1\leqslant s\leqslant k\colon x\in a_s(n_s)\}|,$$

it is periodic modulo $N_A=[n_1,\cdots,n_k]$. *The arithmetic average of the covering function in a period equals the sum $\sum_{s=1}^{k}1/n_s$.* In fact, letting $[x\equiv a_s(\text{mod } n_s)]$ be the characteristic function of the residue class $a_s(n_s)$ we then have

$$\begin{aligned}\sum_{x=0}^{N_A-1}w_A(x)&=\sum_{x=0}^{N_A-1}\sum_{s=1}^{k}[x\equiv a_s \pmod{n_s}]\\&=\sum_{s=1}^{k}\sum_{x=0}^{N_A-1}[x\equiv a_s \pmod{n_s}]=\sum_{s=1}^{k}\frac{N_A}{n_s}.\end{aligned}$$

Let m be a positive integer. If $w_A(x)\geqslant m$ for all $x\in\mathbb{Z}$, then we call system A an m-*cover* of $\mathbb{Z}$; if $w_A(x)=m$ for all $x\in\mathbb{Z}$, then we call system A an *exact m-cover* of $\mathbb{Z}$. By the above, $\sum_{s=1}^{k}1/n_s\geqslant m$ for any m-cover A, and $\sum_{s=1}^{k}1/n_s=m$ if A is an exact m-cover of $\mathbb{Z}$.

In 1976 Š. Porubský asked whether every exact m-cover is a union of m disjoint covers. Choi supplied the following exact 2-cover

$$\{1(2); 0(3); 2(6); 0,4,6,8(10); 1,2,4,7,10,13(15); 5,11,12,22,23,29(30)\},$$

which is not a union of two exact covers. In 1991, using a graph-theoretic argument M. Z. Zhang [J. Sichuan Univ. (Nat. Sci. Ed.)] proved that for each $m = 2, 3, \cdots$ there are infinitely many exact m-covers of $\mathbb{Z}$ which cannot be a union of an n-cover and an $(m-n)$-cover with $0 < n < m$.

Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact m-cover of $\mathbb{Z}$. Then $\sum_{s=1}^k 1/n_s = m \in \mathbb{Z}^+$. In 1992 Z. W. Sun [Israel J. Math.] proved that for every $n = 0, 1, \cdots, m$ there are at least $\binom{m}{n}$ subsets I of $[1,k]$ with $\sum_{s\in I} 1/n_s = n$. The initial idea of the proof is the identity

$$\prod_{s=1}^k \left(1 - r^{1/n_s} e^{2\pi i a_s/n_s}\right) = (1-r)^m \quad (r \geqslant 0).$$

In 1997 Sun [Acta Arith] showed further that for any $t = 1, \ldots, k$ and $a = 0, 1, 2, \ldots$ we have

$$\left|\left\{ I \subseteq [1,k] \setminus \{t\} : \sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_t} \right\}\right| \geqslant \binom{m-1}{\lfloor a/n_t \rfloor}.$$

In 1995, by a mixed use of tools from analysis, linear algebra, number theory and combinatorics, Z. W. Sun obtained the first substantial characterization of m-covers.

Theorem 1.1 (Z. W. Sun, Acta Arith, 1995). $A = \{a_s(n_s)\}_{s=1}^k$ *forms an m-cover of $\mathbb{Z}$ if and only if we have*

$$\sum_{\substack{I\subseteq[1,k] \\ \{\sum_{s\in I} 1/n_s\} = \theta}} (-1)^{|I|} \binom{\lfloor \sum_{s\in I} 1/n_s \rfloor}{n} e^{2\pi i \sum_{s\in I} a_s/n_s} = 0$$

for all $\theta \in [0,1)$ *and* $n = 0, 1, \cdots, m-1$.

By the way, the new approach made Sun obtained the following local-global result: $A = \{a_s(n_s)\}_{s=1}^k$ *forms an* m*-cover of* $\mathbb{Z}$ *if and only if it covers* $|S(A)|$ *consecutive integers at least* m *times, where*

$$S(A) = \left\{ \left\{ \sum_{s\in I} \frac{1}{n_s} \right\} : I \subseteq [1,k] \right\}$$

and $\{\alpha\}$ stands for the fractional part of a real number α. This is stronger than a conjecture of P. Erdős which says that system A is a cover of $\mathbb{Z}$ if it covers integers from 1 to 2^k.

Here are some further properties of m-covers related to unit fractions.

Theorem 1.2. *Let* $\{a_s(n_s)\}_{s=1}^k$ *be an* m*-cover of* $\mathbb{Z}$ *and let* $m_1, \cdots, m_k$ *be positive integers.*

(i) (Z. W. Sun, Trans. Amer. Math. Soc. 1996) *There are at least* m *positive integers in the form* $\sum_{s\in I} m_s/n_s$ *with* $I \subseteq [1,k]$.

(ii) (Z. W. Sun, Proc. Amer. Math. Soc. 1999) *If the residue class* $a_t(n_t)$ *is irredundant (i.e., with* $a_t(n_t)$ *omitted (1.1) would fail to be an* m*-cover of* $\mathbb{Z}$*), and* $(m_s, n_s) = 1$ *for all* $s \in [1,k] \setminus \{t\}$*, then there is an* $\alpha_t \in [0,1)$ *such that*

$$\left\{ \left\{ \sum_{s\in I} \frac{m_s}{n_s} \right\} : I \subseteq [1,k] \setminus \{t\},\ \left\lfloor \sum_{s\in I} \frac{m_s}{n_s} \right\rfloor \geqslant m-1 \right\} \supseteq \left\{ \frac{\alpha_t + r}{n_t} : r \in R(n_t) \right\}.$$

The following result concerning covering functions is also remarkable.

Theorem 1.3 (Z. W. Sun, Combinatorica 2003). *Let* $A = \{a_s(n_s)\}_{s=1}^k$.

(i) $m(A)=\max_{x\in\mathbb{Z}} w_A(x)$ *can be written in the form* $\sum_{s=1}^k m_s/n_s$ *with* $m_1,\ldots,m_k\in\mathbb{Z}^+$.

(ii) *If* $w_A(x)$ *is periodic modulo* $n_0\in\mathbb{Z}^+$*, then for any* $t=1,\ldots,k$ *we have*

$$\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq[1,k]\setminus\{t\}\right\}\supseteq\left\{\frac{r}{n_t}:\ r\in\mathbb{N}\ \&\ r<\frac{n_t}{(n_0,n_t)}\right\}.$$

2. Connections between zero-sum problems and covers of $\mathbb{Z}$

In 1961 P. Erdős, A. Ginzburg and A. Ziv [Bull. Research Council. Israel] established the following celebrated theorem which initiated the study of zero-sums.

The EGZ Theorem. *For any* $c_1,\cdots,c_{2n-1}\in\mathbb{Z}$*, there is an* $I\subseteq[1,2n-1]$ *with* $|I|=n$ *such that* $\sum_{s\in I}c_s\equiv 0\ (\mathrm{mod}\ n)$*. In other words, given* $2n-1$ *(not necessarily distinct) elements of* $\mathbb{Z}_n=\mathbb{Z}/n\mathbb{Z}$*, we can select* n *of them with the sum vanishing.*

The EGZ theorem can be easily reduced to the case where n is a prime (and hence $\mathbb{Z}_n$ is a field), and then deduced from the well-known Cauchy-Davenport theorem or the Chevalley-Warning theorem.

For a finite abelian group G (written additively), the *Davenport constant* $D(G)$ is defined as the smallest positive integer k such that any sequence $\{c_s\}_{s=1}^k$ (repetition allowed) of elements of G has a subsequence $c_{i_1},\cdots,c_{i_l}$ $(i_1<\cdots<i_l)$ with zero-sum (i.e. $c_{i_1}+\cdots+c_{i_l}=0$). In 1966 Davenport showed that if K is an algebraic number field with ideal

class group G, then $D(G)$ is the maximal number of prime ideals (counting multiplicity) in the decomposition of an irreducible integer in K. In 1969 Olson [J. Number Theory] used the knowledge of group rings to show that the Davenport constant of an abelian p-group $G \cong \mathbb{Z}_{p^{h_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{h_l}}$ is

$$L(G) = 1 + \sum_{t=1}^{l} (p^{h_t} - 1).$$

Olson's Theorem. *Let p be a prime and let G be an additive abelian p-group. Then, for any $c_1, \cdots, c_{L(G)} \in G$, there exists a nonempty $I \subseteq [1, L(G)]$ with $\sum_{s\in I} c_s = 0$.*

Observe that the additive group of the finite field with p^l elements is isomorphic to $\mathbb{Z}_p^l$, the direct sum of l copies of the ring $\mathbb{Z}_p$.

In 1994 W. R. Alford, A. Granville and C. Pomerance [Ann. Math.] employed an upper bound for the Davenport constant of the unit group of the ring $\mathbb{Z}_n$ to prove that there are infinitely many Carmichael numbers which are those composites m such that $a^{m-1} \equiv 1 \pmod{m}$ for any $a \in \mathbb{Z}$ with $(a, m) = 1$.

What is the smallest integer $k = s(\mathbb{Z}_n^2)$ such that every sequence of k elements in $\mathbb{Z}_n^2$ contains a zero-sum subsequence of length n? In 1983 Kemnitz [Ars Combin.] conjectured that $s(\mathbb{Z}_n^2) = 4n - 3$. In 1993 Alon and Dubiner showed that $s(\mathbb{Z}_n^2) \leqslant 6n-5$. In 2000 Rónyai [Combinatorica] was able to prove that $s(\mathbb{Z}_p^2) \leqslant 4p - 2$ for every prime p, in 2001 W. D. Gao [J. Combin. Theory Ser. A] used Olson's result to deduce that $s(\mathbb{Z}_q^2) \leqslant 4q - 2$ for any prime power q. All these results were obtained by various ingenious **algebraic methods**.

The following lemma plays an indispensable role in the study of the Kemnitz conjecture.

The Alon-Dubiner Lemma. *Let q be a prime power, and let $c_1, \cdots, c_{3q}$ be elements of $\mathbb{Z}_q^2$ with $c_1 + \cdots + c_{3q} = 0$. Then there is an $I \subseteq [1, 3q]$ with $|I| = q$ such that $\sum_{s \in I} c_s = 0$.*

Here is a useful formula due to Zhi-Wei Sun.

Theorem 2.1 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]. *Let R be a ring with identity, and let $f(x_1, \cdots, x_k)$ be a polynomial over R. If $J \subseteq [1, k]$ and $|J| \geqslant \deg f$, then we have the formula*

$$\sum_{I \subseteq J} (-1)^{|J|-|I|} f([1 \in I], \cdots, [k \in I]) = \left[\prod_{j \in J} x_j\right] f(x_1, \cdots, x_k)$$

where $[x_1^{i_1} \cdots x_k^{i_k}] f(x_1, \cdots, x_k)$ denotes the coefficient of the monomial $x_1^{i_1} \cdots x_k^{i_k}$ in the polynomial $f(x_1, \cdots, x_k)$, and we let $[i \in I]$ be 1 or 0 according to whether $i \in I$ or not.

The EGZ theorem and Olson's theorem **are easy consequences of the above formula!**

By using the powerful formula, Sun [Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`] obtained the following result concerning the Kemnitz conjecture.

Theorem 2.2 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]. *Let p be a prime and let $h > 0$ be an integer. Let $a_s, b_s \in \mathbb{Z}$ for $s = 1, \cdots, 4p^h - 2$.*

(i) *Set* $\mathcal{I} = \{I \subseteq [1, 4p^h - 2]\colon \sum_{s\in I} a_s \equiv \sum_{s\in I} b_s \equiv 0 \pmod{p^h}\}$. *Then*

$$|\{I \in \mathcal{I}\colon |I| = p^h\}| \equiv |\{I \in \mathcal{I}\colon |I| = 3p^h\}| + 2 \pmod{p}.$$

(ii) *Suppose that*

$$\sum_{\substack{I,J\subseteq[1,4p^h-3]\\ |I|=|J|=p^h-1\\ I\cap J=\emptyset}} \Big(\prod_{i\in I} a_i\Big)\Big(\prod_{j\in J} b_j\Big) \not\equiv 2 \pmod{p}.$$

Then there exists an $I \subseteq [1, 4p^h - 3]$ *with* $|I| = p^h$ *such that* $\sum_{s\in I} a_s \equiv \sum_{s\in I} b_s \equiv 0 \pmod{p^h}$.

Applying Theorem 2.2(i) with $a_{4p-2} = b_{4p-2} = 0$, we obtain

Corollary 2.1. *Let* p *be a prime and let* $h > 0$ *be an integer. Let* $a_s, b_s \in \mathbb{Z}$ *for* $s = 1, \cdots, 4p^h - 3$. *Set* $\mathcal{I} = \{I \subseteq [1, 4p^h - 3]\colon \sum_{s\in I} a_s \equiv \sum_{s\in I} b_s \equiv 0 \pmod{p^h}\}$. *Then*

$$\begin{aligned} &|\{I \in \mathcal{I}\colon |I| = p^h\}| + |\{I \in \mathcal{I}\colon |I| = p^h - 1\}| \\ \equiv &|\{I \in \mathcal{I}\colon |I| = 3p^h\}| + |\{I \in \mathcal{I}\colon |I| = 3p^h - 1\}| + 2 \pmod{p}. \end{aligned}$$

Although we have essentially exhausted the power of algebraic methods to attack the Kemnitz conjecture, the conjecture remained open until October in 2003.

Quite recently, using a **combinatorial argument**, C. Reiher got the following sophisticated result.

Reiher's Lemma. *Let p be a prime and let $a_s, b_s \in \mathbb{Z}$ for $s = 1, \cdots, 4p-3$. Set*

$$\mathcal{I} = \left\{ I \subseteq [1, 4p-3] \colon \sum_{s \in I} a_s \equiv \sum_{s \in I} b_s \equiv 0 \pmod{p} \right\}.$$

Then, either $\{I \in \mathcal{I} \colon |I| = p\} \neq \emptyset$ or

$$|\{I \in \mathcal{I} \colon |I| = p-1\}| \equiv |\{I \in \mathcal{I} \colon |I| = 3p-1\}| \pmod{p}.$$

We remark that the prime power version of this lemma also holds.

Combining Reiher's Lemma, the Alon-Dubiner lemma and Corollary 2.1, we immediately obtain the following result of Reiher.

Theorem 2.3 (C. Reiher, 2003)**.** *The Kemnitz conjecture is true, that is, $s(\mathbb{Z}_n^2) = 4n - 3$.*

What does Reiher's solution teach us? **A mixed use of algebraic methods and combinatorial methods might be more powerful**!

In 2003 Z. W. Sun established connections between covers of $\mathbb{Z}$ and some classical theorems on zero-sums such as $D(\mathbb{Z}_m) = m$, the EGZ theorem, the Alon-Dubiner lemma and Olson's theorem. The following theorem in the case $n_1 = \cdots = n_k = 1$ reduces to known results on zero-sums.

Theorem 2.4 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]**.** *Let $A = \{a_s(n_s)\}_{s=1}^k$ and let q be a prime power.*

(i) *If A forms a q-cover of $\mathbb{Z}$, then for any $m_1, \ldots, m_k \in \mathbb{Z}$ there exists a nonempty $I \subseteq [1, k]$ such that $\sum_{s \in I} m_s / n_s \in q\mathbb{Z}$.*

(ii) *If* $\{w_A(x) : x \in \mathbb{Z}\} \subseteq \{2q-1, 2q\}$, *then for any* $c_1, \cdots, c_k \in \mathbb{Z}_q$ *there exists an* $I \subseteq [1,k]$ *such that* $\sum_{s\in I} 1/n_s = q$ *and* $\sum_{s\in I} c_s = 0$.

(iii) *If* A *is an exact* $3q$*-cover of* $\mathbb{Z}$, *then for any* $c_1, \cdots, c_k \in \mathbb{Z}_q^2$ *with* $c_1 + \cdots + c_k = 0$, *there exists an* $I \subseteq [1,k]$ *such that* $\sum_{s\in I} 1/n_s = q$ *and* $\sum_{s\in I} c_s = 0$.

(iv) *Let* G *be an additive abelian group of order* q. *Suppose that* A *is an* $L(G)$*-cover of* $\mathbb{Z}$. *Then, for any* $m_1, \cdots, m_k \in \mathbb{Z}$ *and* $c_1, \cdots, c_k \in G$, *there is a nonempty* $I \subseteq [1,k]$ *such that* $\sum_{s\in I} c_s = 0$ *and* $\sum_{s\in I} m_s/n_s \in \mathbb{Z}$.

The following conjecture seems very difficult.

Sun's Conjecture 2.1. *If we replace the prime power* q *in Theorem 2.4 by a general positive integer, and use* $D(G)$ *instead of* $L(G)$ *in part (iii), then the new version of Theorem 2.4 still holds.*

It seems that we cannot have a similar extension of the (confirmed) Kemnitz conjecture.

3. Covers of groups and Davenport's constant $D(\mathbb{Z}_n^r)$

As $a(n) = a + n\mathbb{Z}$ is just a coset of the subgroup $n\mathbb{Z}$ of the additive group $\mathbb{Z}$, instead of covers of $\mathbb{Z}$ by residue classes one may study covers of an abstract group G by left cosets of subgroups.

A fundamental result on covers of groups is as follows:

Theorem 3.1 (B. H. Neumann, Publ. Math. Debrecen, 1954; M. J. Tomkinson, Comm. Algebra, 1987)**.** *Let* $a_1G_1, \ldots, a_kG_k$ *be left cosets in a group* G. *If* $\mathcal{A} = \{a_iG_i\}_{i=1}^k$ *forms a cover of* G *but none of its proper* *subsystems does (in this case $\mathcal{A}$ is called a minimal cover of G), then*

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \le k! < \infty$$

where the bound $k!$ is best possible.

In a cover $\{a_iG_i\}_{i=1}^k$ of group G, if $\bigcap_{i=1}^k G_i$ equals a given subgroup H of G with finite index, what is the lower bound of k? In this direction we need two functions.

In 1966 J. Mycielski [Fund. Math.] introduced the following function $f : \mathbb{Z}^+ \to \mathbb{N}$ which is called Mycielski function now.

$f(p) = p-1$ for any prime p and $f(mn) = f(m)+f(n)$ for all $m, n \in \mathbb{Z}^+$.

Evidently

$$f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \sum_{i=1}^{r} \alpha_i(p_i - 1),$$

where $p_1, \cdots, p_r$ are distinct primes and $\alpha_1, \cdots, \alpha_r$ are nonnegative integers.

Let G be a group. A subgroup H of G is said to be *subnormal* if there is a chain $H_0 = H \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ of subgroups of G such that H_i is normal in H_{i+1} for all $0 \leqslant i < n$.

Let H be a subnormal subgroup of a group G with finite index, and

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

be a composition series from H to G (i.e. H_i is maximal normal in H_{i+1} for each $0 \leqslant i < n$). If the length n is zero (i.e. $H = G$), then we set

$d(G,H)=0$, otherwise we put

$$d(G,H)=\sum_{i=0}^{n-1}([H_{i+1}:H_i]-1).$$

By the Jordan–Hölder theorem, $d(G,H)$ does not depend on the choice of the composition series from H to G. Clearly $d(G,H)=0$ if and only if $H=G$. If K is a subnormal subgroup of H with $[H:K]<\infty$, then

$$d(G,H)+d(H,K)=d(G,K).$$

When H is normal in G, the 'distance' $d(G,H)$ was first introduced by I. Korec [Fund. Math. 1974]. The current general notion is due to Z. W. Sun [Fund. Math. 1990].

Theorem 3.2 (Z. W. Sun, Fund. Math. 1990; European J. Combin. 2001)**.** *Let G be a group and H a subnormal subgroup of G with finite index. Then*

$$[G:H]-1\geqslant d(G,H)\geqslant f([G:H])\geqslant \log_2[G:H].$$

Moreover, $d(G,H)=f([G:H])$ if and only if G/H_G is solvable where $H_G=\bigcap_{g\in G} gHg^{-1}$ is the largest normal subgroup of G contained in H.

Mycielski's Conjecture [1966, Fund. Math.]: *Let G be an abelian group and $G_1,\cdots,G_k$ be subgroups of G (with finite indices). If $A=\{a_iG_i\}_{i=1}^k$ forms an exact cover of G then*

$$k\geqslant 1+f([G:G_s])\quad \textit{for every}\ \ s=1,\cdots,k.$$

Š. Znám [1966, Colloq. Math.]: *Mycielski's conjecture holds for the additive group* $\mathbb{Z}$ *of integers. Equivalently, if* $A = \{a_s(n_s)\}_{s=1}^k$ *forms a disjoint cover of* $\mathbb{Z}$ *then* $k \geqslant 1 + f(n_s)$ *for* $s = 1, \ldots, k$.

Znam's Conjecture [1968, Coll. Math. Soc. János Bolyai]: *If* $A = \{a_s(n_s)\}_{s=1}^k$ *is a disjoint cover of* $\mathbb{Z}$ *then*

$$k \geqslant 1 + f(N_A) \quad \text{and hence } N_A \leqslant 2^{k-1},$$

where $N_A = [n_1, \ldots, n_k] = [\mathbb{Z} : \bigcap_{s=1}^k n_s\mathbb{Z}]$.

I. Korec [1974, Fund. Math.]: *Let* $\{a_iG_i\}_{i=1}^k$ *be a partition of a group into left cosets of normal subgroups. Then* $[G : \bigcap_{i=1}^k G_i] < \infty$ *and* $k \geqslant 1 + f([G : \bigcap_{i=1}^k G_i])$.

Theorem 3.3 [Sun, Fund. Math., 134(1990); European J. Combin. 22(2001)]**.** *Let* G *be a group and* $\{a_iG_i\}_{i=1}^k$ *be an exact* m*-cover of* G *with all the* G_i *subnormal in* G. *Then*

$$k \geqslant m + d\Big(G, \bigcap_{i=1}^k G_i\Big), \tag{$*$}$$

where the lower bound can be attained. Moreover, for any subgroup K *of* G *not contained in all the* G_i *we have*

$$|\{1 \leqslant i \leqslant k : K \not\subseteq G_i\}| \geqslant 1 + d\Big(K, K \cap \bigcap_{i=1}^k G_i\Big).$$

The lower bound in $(*)$ can be attained as shown by the following example.

Example 3.1. Let H be a subnormal subgroup of a group G with finite index, and let $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ be a composition series from H to G. Let $a \in G$ and

$$H_{i+1} \setminus H_i = \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i \quad \text{for } i = 0, 1, \ldots, n-1.$$

Z. W. Sun [Fund. Math. 134(1990)] observed that the following $1+d(G,H)$ cosets

$$aH_0,\ ab_j^{(i)} H_i\ (0 \leqslant i < n,\ 1 \leqslant j < [H_{i+1}] : H_i])$$

form a partition of G. These cosets, together with $m-1$ copies of G, form an exact m-cover of G with the number k of cosets being $m+d(G,H)$ and the intersection of the k subnormal subgroups being H.

Berger-Felzenbaum-Fraenkel [1988, Coll. Math.]: *If $\{a_iG_i\}_{i=1}^k$ is a disjoint cover of a finite solvable group G, then $k \geqslant 1 + f([G : G_i])$ for $i = 1, \cdots, k$.*

Theorem 3.4 [Z. W. Sun, European J. Combin. 2001]**.** *Let $\{a_iG_i\}_{i=1}^k$ be an exact m-cover of a group G. For any $i = 1, \leqslant, k$, whenever $G/(G_i)_G$ is solvable we have $k \geqslant m + f([G : G_i])$ and hence $[G : G_i] \leqslant 2^{k-m}$.*

We have a further conjecture.

Sun's Conjecture 3.1. *Let $\{a_iG_i\}_{i=1}^k$ be an exact m-cover of a group G with all the $G/(G_i)_G$ solvable. Then $k \geqslant m + f(N)$ where N is the least common multiple of the indices $[G : G_1], \cdots, [G : G_k]$.*

Znám [1969, Colloq. Math.]: *$k \geqslant 1 + f(n_t)$ if $A = \{a_s(n_s)\}_{s=1}^k$ forms a cover in which $a_t(n_t)$ is disjoint with all the remaining classes.*

Znám [1975, Acta Arith.]: *If* $A = \{a_s(n_s)\}_{s=1}^k$ *is a cover with* $a_t(n_t)$ *irredundant then* $k \geqslant 1 + f(n_t)$.

Znám's Conjecture [1975, Acta Arith.]: *If* $A = \{a_s(n_s)\}_{s=1}^k$ *forms a minimal cover of* $\mathbb{Z}$, *then* $k \geqslant 1 + f(N_A)$.

R.J. Simpson [1985, Acta Arith.]: *Let* $A = \{a_s(n_s)\}_{s=1}^k$ *be a minimal cover of* $\mathbb{Z}$. *Then for any divisor* d *of* N_A *with* $d < N_A$ *we have*

$$|\{1 \leqslant i \leqslant k : \ n_i \nmid d\}| \geqslant 1 + f(N_A/d).$$

Letting $d = 1$ *we then obtain* $k \geqslant 1 + f(N_A)$.

In 1990 Z. W. Sun extended Simpson's result to minimal m-cover of cyclic groups. Simpson's result cannot be extended to minimal covers of abelian groups.

Berger-Felzenbaum-Fraenkel [1988, Coll. Math.]: *Let* $\{a_iG_i\}_{i=1}^k$ *be a minimal cover of a group* G *of squarefree order. If all the* G_i *are normal in* G *then* $k \geqslant 1 + f([G : \bigcap_{i=1}^k G_i])$.

Z. W. Sun [preprint]: *Let* $\{a_iG_i\}_{i=1}^k$ *be a minimal* m*-cover of a group* G *with all the* G_i *subnormal in* G. *Suppose that for any* $i, j = 1, \ldots, k$ *either* $[G : G_i]$ *is relatively prime to* $[G_i : G_i \cap G_j]$, *or* G_i *and* G_j *are normal in* G *with* $G/(G_i \cap G_j)$ *cyclic. Then we have* $k \geqslant m + d(G, \bigcap_{i=1}^k G_i)$.

Example 3.2. Let G be the group $C_p \times C_p$ where p is a prime and C_p is the cyclic group of order p. Then any element $a \neq e$ of G has order p. Let $G_1, \ldots, G_k$ be all the distinct subgroups of G with order p. If $1 \leqslant i < j \leqslant k$ then $G_i \cap G_j = \{e\}$. Clearly $\{G_i\}_{i=1}^k$ forms a minimal cover of G by normal subgroups whose intersection is $H = \{e\}$. Since

$1 + k(p-1) = |\bigcup_{i=1}^{k} G_i| = |G| = p^2$, we have

$$k = p + 1 \leqslant 2p - 1 = 1 + f([G : H]) = 1 + d\Big(G, \bigcap_{i=1}^{k} G_i\Big).$$

When $p > 2$ the last inequality becomes strict. We remark that both G/G_i and $G_i/(G_i \cap G_j)$ $(j \neq i)$ have order p.

Sun's Conjecture 3.2. *Let G be a group and $G_1, \cdots, G_k$ its subnormal subgroups such that system $\{G_i\}_{i=1}^{k}$ forms a minimal m-cover of G where $m \in \mathbb{Z}^+$. Assume that $[G : \bigcap_{i=1}^{k} G_i] = \prod_{t=1}^{r} p_t^{\alpha_t}$ where $p_1, \cdots, p_r$ are distinct primes and $\alpha_1, \cdots, \alpha_r$ are positive integers. Then*

$$k \geqslant m + \sum_{t=1}^{r} p_t(\alpha_t - 1).$$

Let G be a finite multiplicative abelian group with identity e. If aH is a left coset in G with $e \notin aH$ (i.e. $a \notin H$), then aH is called a *proper coset* in G. W. D. Gao and A. Geroldinger [European J. Combin. 24(2003)] defined $\mathbf{s}(G)$ to be the smallest number of proper cosets that can cover $G \setminus \{e\}$. For left cosets $a_1G_1, \ldots, a_kG_k$ in group G, the system $\{a_iG_i\}_{i=1}^{k}$ is a cover of $G \setminus \{e\}$ if and only if $\{a_iG_i\}_{i=0}^{k}$ forms a cover of G with the coset a_0G_0 irredundant where $a_0 = e$ and $G_0 = \{e\}$ (and hence $\bigcap_{i=0}^{k} G_i = \{e\}$). Thus, by the previous results on covers of groups, we have $\mathbf{s}(G) \leqslant f(|G|)$, and $\mathbf{s}(G) = f(|G|)$ if G is cyclic or of squarefree order. This was also deduced by Gao and Geroldinger who were not aware of previous results on covers of groups. Furthermore, they proved the following remarkable result.

Theorem 3.5 [Gao and Geroldinger, European J. Combin. 2003]**.**

(i) *If G is an elementary abelian p-group, then $\mathbf{s}(G) = f(|G|)$. Equivalently, $\mathbf{s}(\mathbb{Z}_p^r) = r(p-1)$ for any prime p.*

(ii) *Let $G = \mathbb{Z}_n^r$. Suppose that $1 + f(|G|) = 1 + rf(n)$ is the smallest positive integer k such that there are k proper cosets in $\mathbb{Z}_n^{r(n-1)+1}$ with their union containing $V \setminus \{0\}$, where*

$$V = \{\vec{x} = \langle x_1, \dots, x_{r(n-1)+1}\rangle \in \mathbb{Z}_n^{r(n-1)+1} :\ x_i \in \{0,1\} \text{ for each } i\}.$$

Then the well-known conjecture $D(\mathbb{Z}_n^r) = 1 + r(n-1)$ is true.

It seems that $\mathbf{s}(\mathbb{Z}_n^r) = f(|\mathbb{Z}_n^r|) = rf(n)$. We have a further conjecture.

Sun's Conjecture 3.3[1]**.** *Let $\mathcal{A} = \{a_iG_i\}_{i=1}^k$ be an m-cover of a group G by left cosets. For any $s = 1, \dots, k$ with a_sG_s irredundant, if G_s is subnormal in G then $k \geqslant m + d(G, G_s)$; if $G/(G_s)_G$ is solvable then $k \geqslant m + f([G : G_s])$.*

Inspired by the connections between zero-sum problems and covers of $\mathbb{Z}$ revealed by Z. W. Sun, V. Dimitrov found the following new connection between the conjecture $D(\mathbb{Z}_n^r) = 1 + r(n-1)$ and covers of $\mathbb{Z}_n^r$.

Theorem 3.6 [V. Dimitrov, *Zero-sum problems in finite groups*, 2003]**.** *Let n and r be positive integers. Denote by $c(n, r)$ the smallest integer k such that for any $k \times r$ matrix $M = (m_{ij})$ over $\mathbb{Z}_n$ there are $a_1, \dots, a_k \in \mathbb{Z}_n^r$ such that $\{a_iG_i\}_{i=1}^k$ forms a cover of the group $G = \mathbb{Z}_n^r$, where G_i is the subgroup $\{\langle x_1, \dots, x_r\rangle \in \mathbb{Z}_n^r : \sum_{j=1}^r m_{ij}x_j = 0\}$. Then $D(\mathbb{Z}_n^r) \leqslant c(n, r)$, and $D(\mathbb{Z}_n^r) = 1 + r(n-1)$ if $c(n, r) = 1 + r(n-1)$.*

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[1]Soon after this talk the conjecture was confirmed by G. Lettl and Z. W. Sun in the most important case.