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PROBLEMS AND RESULTS ON COVERING SYSTEMS

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Abstract. This is an introduction to the important aspects of covers of $\mathbb{Z}$ by residue classes and covers of groups by cosets or subgroups. The field is connected with number theory, combinatorics, algebra and analysis. It is quite fascinating, and also very difficult (but the results can be easily understood). Many problems and conjectures remain open, some nice theorems and applications will be introduced.

Perhaps my favorite problem of all concerns covering systems.

—Paul Erdős (1995)

1. Main Problems and Related Results

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$, we let

$$a(n) = a + n\mathbb{Z} = \{x \in \mathbb{Z} : \ x \equiv a \pmod n\}.$$

This is called a residue class (with *modulus* n) or an arithmetic sequence (with *common difference* n).

A finite system

$$A = \{a_s(n_s)\}_{s=1}^k = \{a_1(n_1), \dots , a_k(n_k)\}$$

of residue classes is said to be a *covering system* (or a *cover* of $\mathbb{Z}$ in short) if each integer lies in at least one of the members in A.

The concept of cover of $\mathbb{Z}$ was first introduced by P. Erdős in his solution to a question of Romanoff given in 1934. Since the Chinese Remaider Theorem tells us when $\bigcap_{s=1}^k a_s(n_s) \neq \emptyset$, it is fundamental to study when we have $\bigcup_{s=1}^k a_s(n_s) = \mathbb{Z}$.

Clearly A forms a cover of $\mathbb{Z}$ if and only if it covers $0, 1, \cdots , N_A - 1$, where N_A is the least common multiple of the moduli $n_1, \cdots , n_k$.

The first nontrivial cover of $\mathbb{Z}$ with distinct moduli is the following one discovered by P. Erdős:

$$B = \{0(2),\ 0(3),\ 1(4),\ 5(6),\ 7(12)\}.$$

Note that $N_B = 12$ and B covers $0, 1, \dots , 11$.

If A covers every integer exactly once, then we call A an *exact cover* of $\mathbb{Z}$ or a *disjoint cover* of $\mathbb{Z}$.

As any integer can be written uniquely in the form $nq + r$ with $q \in \mathbb{Z}$ and $r \in [0, n - 1] = \{0, 1, \cdots , n - 1\}$, the finite system $\{r(n)\}_{r=0}^{n-1}$ is a disjoint cover of $\mathbb{Z}$. Since $0(2^n)$ is a disjoint union of the residue classes $2^n(2^{n+1})$ and $0(2^{n+1})$, the systems

$$A_1 = \{1(2), 0(2)\},\ A_2 = \{1(2), 2(4), 0(4)\},\ A_3 = \{1(2), 2(4), 4(8), 0(8)\},$$
$$\cdots\cdots,\ A_k = \{1(2), 2(2^2), \dots , 2^{k-1}(2^k), 0(2^k)\},\ \cdots\cdots$$

are disjoint covers of $\mathbb{Z}$.

Clearly each residue class $a_s(n_s)$ in system A covers exactly N_A/n_s integers in $[0, N_A-1]$. Thus, if A is a cover of $\mathbb{Z}$ then

$$|[0, N_A-1]| \leqslant \sum_{s=1}^{k} \frac{N_A}{n_s} \quad \text{and hence} \quad \sum_{s=1}^{k} \frac{1}{n_s} \geqslant 1;$$

if A is a disjoint cover of $\mathbb{Z}$ then $\sum_{s=1}^{k} 1/n_s = 1$.

By Example 3 of Z. W. Sun [Trans. Amer. Math. Soc. 348(1996)], if $n > 1$ is odd then

$$\{1(2), 2(2^2), \ldots, 2^{n-2}(2^{n-1}), 2^{n-1}(n), 2^{n-1}2(2n), \ldots, 2^{n-1}n(2^{n-1}n)\}$$

forms a cover of $\mathbb{Z}$ with distinct moduli, and the sum of reciprocals of the moduli is less than $1 + 2/n$ which tends to 1 as $n \to +\infty$.

Now we introduce three main conjectures concerning covers of $\mathbb{Z}$.

Erdős' Conjecture. *For any arbitrarily large $c > 0$, there exists a cover of $\mathbb{Z}$ whose moduli are distinct and greater than c.*

P. Erdős offers \$1000 for a solution of this conjecture. Up to now, no substantial progress has been made. Erdős' conjecture implies the well-known fact that $\sum_{n=1}^{\infty} 1/n$ diverges, because we can construct infinitely many covers

$$A^{(i)} = \{a_s^{(i)}(n_s^{(i)})\}_{s=1}^{k_i} \quad (i = 1, 2, 3, \ldots)$$

such that

$$n_1^{(1)} < \cdots < n_{k_1}^{(1)} < n_1^{(2)} < \cdots < n_{k_2}^{(2)} < \cdots\cdots$$

and hence
$$\sum_{i=1}^{m}\sum_{s=1}^{k_i}\frac{1}{n_s^{(i)}}\geqslant m \qquad \text{for } m=1,2,3,\ldots.$$

In 1981 R. Morikawa [Bull. Fac. Lib. Arts; MR 84j:10064] constructed a cover $\{a_s(n_s)\}_{s=1}^k$ of $\mathbb{Z}$ with $n_1 = 24 < n_2 < \cdots < n_k$. On the other hand, Z. W. Sun [J. Algebra 273(2004)] proved that *for any $M > 1$ if $\{a_s(n_s)\}_{s=1}^k$ covers every integer the same number of times and each of the moduli occurs at most M times, then the smallest modulus n_1 has an upper bound in terms of M, namely,*

$$\log n_1 \leqslant \frac{e^\gamma}{\log 2} M\log^2 M + O(M\log M\log\log M),$$

where γ is Euler's constant and the O-constant is absolute.

Erdős-Selfridge Conjecture. *Let $A = \{a_s(n_s)\}_{s=1}^k$ be a cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$. Then $n_1,\ldots,n_k$ cannot be all odd.*

P. Erdös offers \$25 for a positive answer, and Selfridge offers \$900 for an counterexample.

Let $A = \{a_s(n_s)\}_{s=1}^k$ be a cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$. Recently S. Guo and Z. W. Sun [Adv. Appl. Math., to appear] showed that if $n_1,\ldots,n_k$ are odd and squarefree, then $N_A = [n_1,\ldots,n_k]$ has at least 22 prime divisors. In contrast with the Erdős-Selfridge conjecture, Z. W. Sun [J. Number Theory, 111(2005), 190-196] proved that A *cannot cover every integer an odd number of times.*

Schinzel's Conjecture. *If $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$, then there is a modulus n_t dividing another modulus n_s.*

A. Schinzel [Acta Arith. 1967, MR 36#2596]: The Erdős-Selfridge conjecture is stronger than and Schinzel's conjecture is weaker than the following proposition: *For any polynomial* $P(x) \in \mathbb{Z}[x]$ *with* $P(0) \neq 0$, $P(1) \neq -1$ *and* $P(x) \neq 1$, *there exists an infinite arithmetic progression of positive integers such that* $x^n + P(x)$ *is irreducible over the rational field* $\mathbb{Q}$ *for every* n *in the progression.*

L. J. Stockmeyer and A. R. Meyer [Proc. 5th. Ann. ACM Symp. on Theory of Computing, Assoc. for Computing Machinery, 1973]: *The question whether a given* $A = \{a_s(n_s)\}_{s=1}^k$ *is a cover of* $\mathbb{Z}$ *is co-NP-complete. Thus,* $NP = P$ *if and only if we can decide whether* $A = \{a_s(n_s)\}_{s=1}^k$ *is a cover of* $\mathbb{Z}$ *in polynomial time.*

Now we mention some curious applications of covers.

In 1934, by using the cover

$$\{0(2),\ 0(3),\ 1(4),\ 3(8),\ 7(12),\ 23(24)\},$$

Erdős [Summa Brasil. Math. 1950; MR 13,437] proved that there is an infinite arithmetic progression of positive odd integers no term of which is of the form $2^n + p$, where n is a positive integer and p is an odd prime. Later R. Crocker [Pacific J. Math. 1971] proved further that there are infinitely many positive odd integers not of the form $2^a + 2^b + p$ where $a, b \in \mathbb{Z}^+$ and p is an odd prime.

In 2001 Z. W. Sun and M. H. Le [Acta Arith. 99(2001)] improved a result of Schinzel and Crocker by showing that for each $n = 4, 5, \ldots$ the number $2^{2^n} - 1$ cannot be written as the sum of two distinct powers of 2 and a prime power.

On the basis of the work of Corcker, and Sun and Le, P. Z. Yuan [Acta Arith. 2004] confirmed a conjecture of Sun.

Theorem 1.1 (conjectured by Z. W. Sun and proved by P. Z. Yuan). *For any positive integer c, there are infinitely many positive odd integers not of the form* $c(2^a + 2^b) + p^\alpha$ *where* $a, b, \alpha \in \mathbb{N} = \{0, 1, 2, \dots\}$ *and p is an odd prime.*

Erdös [1952, Mat. Lapok; MR 17,14] observed that the positive solution of his conjecture implies that for every $m \geqslant 1$ there exists an infinite arithmetic progression of positive odd integers no term of which is of the form $2^n + \theta_m$ where θ_m has at most m distinct prime factors. In 1950 Yu. V. Linnik proved that *there is a positive integer l such that every large even integer can be written a sum of two primes and at most l powers of 2*, recently J. Pintz and I. Z. Ruzsa [Acta Arith. 109(2003)] (and independently, D. R. Heath-Brown and J. C. Puchta [Asian J. Math. 6(2002)]) proved that one can take $l = 7$ under the Generalized Riemann Hypothesis, they announced further that $l = 8$ is okay unconditionally.

With help of covers, F. Cohen and J. L. Selfridge [1975, Math. Comput.; MR 51#12758] proved that not every positive integer is the sum or difference of two prime powers. By introducing a method to avoid a bunch of extra congruences, Z. W. Sun was able to establish the following explicit result.

Theorem 1.2 [Z. W. Sun, Proc. Amer. Math. Soc. 2000]. *Let P be the*

26-digit prime 47867742232066880047611079*, and let* M *be the 29-digit number given by*

$$\prod_{p\leqslant 19} p\times 31\times 37\times 41\times 61\times 73\times 97\times 109\times 151\times 241\times 257\times 331$$
$$= 66483084961588510124010691590.$$

If $x \equiv P \pmod{M}$*, then* x *is not of the form* $\pm p^a \pm q^b$*, where* p, q *are primes and* a, b *are nonnegative integers.*

Here is another remarkable application of covers.

Theorem 1.3. (i) (W. Sierpinki, 1960) *There are infinitely many positive odd integers* k *such that* $k\times 2^n+1$ *are composite for all* $n = 1, 2, 3, \ldots$. *(Such integers* k *is called Sierpinski numbers.)*

(ii) (J. L. Selfridge) $78557\cdot 2^n+1$ *always has a prime divisor in the set* $\{3, 5, 7, 13, 17, 19, 73\}$*, and perhaps* 78557 *is the smallest Sierpinski number.*

After lots of computations, now it is known that Sierpinski numbers less than 78557 can only be among the following candidates:

$$4847, 10223, 19249, 21181, 22699, 24737, 27653, 28433, 33661, 55459, 67607.$$

2. The Davenport-Mirsky-Newman-Rado Result and its Generalizations

Let $A = \{a_s(n_s)\}_{s=1}^k$ be a cover of $\mathbb{Z}$. Remember that $\sum_{s=1}^k 1/n_s \geqslant 1$, and equality holds if and only if A is an exact cover of $\mathbb{Z}$.

Soon after his invention of the concept of covering system, Erdős made the following conjecture: *If* $A=\{a_s(n_s)\}_{s=1}^k$ *is a cover of* $\mathbb{Z}$ *with* $1<n_1<\cdots<n_k$*, then* $\sum_{s=1}^k 1/n_s>1$*, i.e. A covers some integer more than once.*

Theorem 2.1 (H. Davenport, L. Mirsky, D. Newman, R. Rado, 1950s)**.** *Let* $A=\{a_s(n_s)\}_{s=1}^k$ *be an exact cover of* $\mathbb{Z}$ *with* $1<n_1\leqslant\cdots\leqslant n_{k-1}\leqslant n_k$*. Then we must have* $n_{k-1}=n_k$*.*

Proof. Without loss of generality we assume that $0\leqslant a_s<n_s$ for all $s\in[1,k]$. For $|z|<1$ we have

$$\sum_{s=1}^k\frac{z^{a_s}}{1-z^{n_s}}=\sum_{s=1}^k\sum_{q=0}^\infty z^{a_s+qn_s}=\sum_{n=0}^\infty z^n=\frac{1}{1-z}.$$

If $n_{k-1}<n_k$ then

$$\infty=\lim_{\substack{z\to e^{2\pi i/n_k}\\ |z|<1}}\frac{z^{a_k}}{1-z^{n_k}}=\lim_{\substack{z\to e^{2\pi i/n_k}\\ |z|<1}}\left(\frac{1}{1-z}-\sum_{s=1}^{k-1}\frac{z^{a_s}}{1-z^{n_s}}\right)<\infty,$$

a contradiction!

Observe that $\{a_s(n_s)\}_{s=1}^k$ is an exact cover if and only if $\sum_{s=1}^k\chi_s(x)=1$ for all $x\in\mathbb{Z}$, where $\chi_s(x)$ is the characteristic function $[x\in a_s(n_s)]$ of the residue class $a_s(n_s)$.

Z. W. Sun [J. Nanjing Univ. (Nat. Sci. Edi.), 1991]: *For* $s=1,\cdots,k$ *let* ψ_s *be an arithmetical function periodic mod* n_s *such that* $\sum_{r=0}^{n_s-1}\psi_s(r)\zeta^r\neq 0$ *for some primitive* n_s*-th root* ζ *of unity. If* $[n_1,\ldots,n_k]$ *is not the smallest positive period of the function* $\psi=\psi_1+\cdots+\psi_k$*, then there must exist some* s,t *such that* $n_s=n_t$ *and* $\psi_s\neq\psi_t$*.*

Clearly a residue class $a(n) = a + n\mathbb{Z}$ is a coset of the subgroup $n\mathbb{Z}$ of the additive cyclic group $\mathbb{Z}$, and the modulus n is just the index $[\mathbb{Z} : n\mathbb{Z}]$ of $n\mathbb{Z}$ in $\mathbb{Z}$. Instead of finite covers of $\mathbb{Z}$ by residue classes, one may also investigate covers of a general group by finitely many cosets. If $\{a_iG_i\}_{i=1}^k$ is a cover of a group G by left cosets but none of its proper subsystems, then $n_i = [G : G_i] < \infty$ for all $i \in [1, k]$ (B. H. Neumann [J. London Math. Soc. 1954]), and $N = [G : \bigcap_{i=1}^k G_i] \leqslant k!$ (M. J. Tomkinson [Comm. Algebra 1987]) where the bound $k!$ is best possible.

Here is a nice generalization of the above conjecture of Erdős suggested by M. Herzog and J. Schönheim [Canad. Math. Bull. 1974].

Herzog-Schönheim Conjecture. *Let $\{a_iG_i\}_{i=1}^k$ $(k > 1)$ be a partition of a group G into left cosets of subgroups $G_1, \ldots, G_k$. Then the indices $n_1 = [G : G_1], \ldots, n_k = [G : G_k]$ cannot be distinct.*

M. A. Berger, A. Felzenbaum and A. S. Fraenkel [Canad. Math. Bull. 1986; Fund. Math. 1987] proved that the Herzog-Schönheim conjecture holds for finite nilpotent groups and supersolvable groups.

Here is a recent progress on the Herzog-Schönheim conjecture.

Theorem 2.2 [Z. W. Sun, J. Algebra 273(2004)]**.** *Let G be a group, and let $a_1G_1, \ldots, a_kG_k$ be left cosets of subgroups of G such that $\{a_iG_i\}_{i=1}^k$ covers all the elements of G the same number of times but not all the G_i equal G. Let p be the largest prime divisor of $N = [[G : G_1], \ldots, [G : G_k]]$, If all those G_i with $[G : G_i] \geqslant p$ are subnormal in G, or $G/(\bigcap_{i=1}^k G_i)_G$ is a solvable group having a normal Sylow p-subgroup (where H_G denotes* *the largest normal subgroup of G contained in a subgroup H of G). Then there is a pair $\{i,j\}$ with $1 \leqslant i < j \leqslant k$ such that $[G:G_i]=[G:G_j] \equiv 0 \pmod p$.*

The following refinement of the Davenport-Mirsky-Newman-Rado result was first conjectured by Š. Znám in 1969 and then confirmed by M. Newman [Math. Ann. 191(1971)]: *If $A=\{a_s(n_s)\}_{s=1}^k$ is an exact cover of $\mathbb{Z}$ with $n_1 \leqslant \cdots \leqslant n_{k-1} < n_{k-l+1} = \cdots = n_k$, then $l \geqslant p(n_k)$ where $p(n_k)$ is the least prime divisor of n_k.*

Here is a further result in this direction.

Theorem 2.3. *Let $A=\{a_s(n_s)\}_{s=1}^k$ and $w(x)=\sum_{x\in a_s(n_s)}\lambda_s$ where $\lambda_s \in \mathbb{C}$.*

(i) [Z. W. Sun, Chin. Quart. J. Math. 6(1991)] *Let $n_0 \in \mathbb{Z}^+$ be the smallest period of the function $w(x)$. If $d \in \mathbb{Z}^+$ does not divide n_0 and $\sum_{\substack{1\leqslant s\leqslant k\\ d\mid n_s}}\lambda_s/n_s \neq 0$, then*

$$|\{a_s \bmod d : 1 \leqslant s \leqslant k \ \&\ d \mid n_s\}| \geqslant \min_{\substack{0\leqslant s\leqslant k\\ d\nmid n_s}} \frac{d}{(d,n_s)} \geqslant p(d), \qquad (*)$$

where (d,n_s) is the greatest common divisor of d and n_s. In particular, if $n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k$ and $n_k \nmid n_0$, then

$$l \geqslant \min_{0\leqslant s\leqslant k-l} \frac{n_k}{(n_s,n_k)} \geqslant p(n_k).$$

(ii) [Z. W. Sun, J. Number Theory 111(2005), 190-196] *Let $n_0 \in \mathbb{Z}^+$ be the smallest positive period of $w(x)$ mod $m \in \mathbb{Z}$. Suppose that $d \in \mathbb{Z}^+$ does not divide n_0 but $I(d)=\{1 \leqslant s \leqslant k : d \mid n_s\} \neq \emptyset$. If $\lambda_1,\ldots,\lambda_k \in$*

$\mathbb{Z}$, and m does not divide $[n_1,\dots,n_k]\sum_{s\in I(d)}\lambda_s/n_s$, then $(*)$ also holds. Consequently, if $k>1$ and $n_1,\dots,n_k$ are distinct, then the range of the function $w_A(x)=|\{1\leqslant s\leqslant k: x\in a_s(n_s)\}|$ is not contained in any residue class with modulus greater than one.

3. m-covers of $\mathbb{Z}$ and Unit fractions

The *covering function* of $A=\{a_s(n_s)\}_{s=1}^k$ defined by

$$w_A(x)=|\{1\leqslant s\leqslant k\colon x\in a_s(n_s)\}|,$$

is obviously periodic modulo $N_A=[n_1,\cdots,n_k]$. *The arithmetic average of the covering function in a period equals the sum $\sum_{s=1}^k 1/n_s$.* In fact,

$$\begin{aligned}\sum_{x=0}^{N_A-1} w_A(x) &= \sum_{x=0}^{N_A-1}\sum_{s=1}^{k}[x\equiv a_s \pmod{n_s}]\\ &=\sum_{s=1}^{k}\sum_{x=0}^{N_A-1}[x\equiv a_s \pmod{n_s}]=\sum_{s=1}^{k}\frac{N_A}{n_s}.\end{aligned}$$

Here is a uniqueness theorem.

Theorem 3.1 (S. K. Stein, Š. Znám, Z. W. Sun)**.** *If $A=\{a_s(n_s)\}_{s=1}^k$ and $B=\{b_t(m_t)\}_{t=1}^l$ are both systems with distinct moduli, and that $w_A(x)\equiv w_B(x) \pmod{m}$ for all $x\in\mathbb{Z}$ where m is an integer not dividing $N=[n_1,\dots,n_k,m_1,\dots,m_l]$, then systems A and B are identical.*

In the case $m=0$, this was proved by Stein [Math. Ann. 1958] under the condition that both A and B are disjoint. The case $m=0$ was showed by Znám [Acta Arith. 1975], and the current general version is due to Z. W. Sun [J. Number Theory 111(2005), 190-196].

Let m be a positive integer. If $w_A(x) \geqslant m$ for all $x \in \mathbb{Z}$, then we call system A an *m-cover* of $\mathbb{Z}$; if $w_A(x) = m$ for all $x \in \mathbb{Z}$, then we call system A an *exact m-cover* of $\mathbb{Z}$. It is easy to see that $\sum_{s=1}^k 1/n_s \geqslant m$ for any m-cover $A = \{a_s(n_s)\}_{s=1}^k$, and $\sum_{s=1}^k 1/n_s = m$ if $A = \{a_s(n_s)\}_{s=1}^k$ is an exact m-cover of $\mathbb{Z}$.

In 1976 Š. Porubský asked whether every exact m-cover is a union of m disjoint covers. S. G. Choi supplied the following exact 2-cover

$$\{1(2); 0(3); 2(6); 0,4,6,8(10); 1,2,4,7,10,13(15); 5,11,12,22,23,29(30)\},$$

which is not a union of two exact covers. In 1991, using a graph-theoretic argument M. Z. Zhang [J. Sichuan Univ. (Nat. Sci. Ed.)] proved that for each $m = 2, 3, \cdots$ there are infinitely many exact m-covers of $\mathbb{Z}$ which cannot be a union of an n-cover and an $(m-n)$-cover with $0 < n < m$.

In 1989, by using the Riemann zeta function, M. Z. Zhang [J. Sichuan Univ. (Nat. Sci. Ed.)] showed the following surprising result: *If $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ then $\sum_{s\in I} 1/n_s \in \mathbb{Z}^+$ for some $I \subseteq [1,k]$, which is stronger than the inequality $\sum_{s=1}^k 1/n_s \geqslant 1$.* The starting point of Zhang is that $A = \{a_s(n_s)\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ if any only if

$$\prod_{s=1}^k \left(1 - e^{2\pi i(n+a_s)/n_s}\right) = 0 \quad \text{for all } n = 1, 2, 3, \cdots.$$

The crucial trick in Zhang's proof is that for a real number c the series $\sum_{n=1}^{+\infty} e^{2\pi icn}/n$ diverges if and only if c is an integer.

Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact m-cover of $\mathbb{Z}$. Then $\sum_{s=1}^k 1/n_s = m \in \mathbb{Z}^+$. In 1992 Z. W. Sun [Israel J. Math.] proved that for every $n =$

$0,1,\cdots,m$ there are at least $\binom{m}{n}$ subsets I of $[1,k]$ with $\sum_{s\in I}1/n_s=n$. The initial idea of the proof is the identity

$$\prod_{s=1}^{k}\left(1-r^{1/n_s}e^{2\pi i a_s/n_s}\right)=(1-r)^m\quad(r\geqslant 0).$$

Here is a result of Z. W. Sun concerning exact m-covers of $\mathbb{Z}$.

Theorem 3.1 [Z. W. Sun, Acta Arith. 1995, 1997]**.** *Let $A=\{a_s(n_s)\}_{s=1}^k$ be an exact m-cover of $\mathbb{Z}$. If $\emptyset\neq J\subset[1,k]$, then there is an $I\subseteq[1,k]$ with $I\neq J$ such that $\sum_{s\in I}1/n_s=\sum_{s\in J}1/n_s$. Also, for any $a=0,1,2,\dots$ we have*

$$\left|\left\{I\subseteq[1,k-1]:\sum_{s\in I}\frac{1}{n_s}=\frac{a}{n_k}\right\}\right|\geqslant\binom{m-1}{\lfloor a/n_k\rfloor}$$

where the lower bound is best possible.

Further results appeared in Z. W. Sun [`arXiv:math.NT/0403271`].

In 1995, by a mixed use of tools from analysis, linear algebra, number theory and combinatorics, Z. W. Sun obtained the first substantial characterization of m-covers.

Theorem 3.2 [Z. W. Sun, Acta Arith. 1995]**.** *$A=\{a_s(n_s)\}_{s=1}^k$ forms an m-cover of $\mathbb{Z}$ if and only if we have*

$$\sum_{\substack{I\subseteq[1,k]\\ \{\sum_{s\in I}1/n_s\}=\theta}}(-1)^{|I|}\binom{\lfloor\sum_{s\in I}1/n_s\rfloor}{n}e^{2\pi i\sum_{s\in I}a_s/n_s}=0$$

for all $\theta\in[0,1)$ and $n=0,1,\cdots,m-1$.

By the way, Sun's approach made him obtain the following local-global result: $A=\{a_s(n_s)\}_{s=1}^k$ *forms an m-cover of $\mathbb{Z}$ if and only if it covers*

$|S(A)|$ consecutive integers at least m times, where

$$S(A)=\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}: I\subseteq[1,k]\right\}$$

and $\{\alpha\}$ stands for the fractional part of a real number α. This is stronger than a conjecture of P. Erdős which says that $A=\{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ if it covers integers from 1 to 2^k.

Theorem 3.3. *Let $A=\{a_s(n_s)\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$ and $m_1,\ldots,m_k$ be any positive integers.*

(i) [Z. W. Sun, Trans. Amer. Math. Soc. 348(1996)] *There are at least m positive integers in the form $\sum_{s\in I}m_s/n_s$ with $I\subseteq[1,k]$.*

(ii) [Z. W. Sun, Proc. Amer. Math. Soc. 127(1999] *For any $J\subseteq[1,k]$ we have*

$$\left|\left\{I\subseteq[1,k]:\ I\neq J\ \&\ \sum_{s\in I}\frac{m_s}{n_s}-\sum_{s\in J}\frac{m_s}{n_s}\in\mathbb{Z}\right\}\right|\geqslant m.$$

(iii) [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003)] *If m is a prime power, then for any $J\subseteq[1,k]$ there is an $I\subseteq[1,k]$ with $I\neq J$ such that $\sum_{s\in I}m_s/n_s-\sum_{s\in J}m_s/n_s\in m\mathbb{Z}$.*

(iv) [Z. W. Sun, Trans. Amer. Math. Soc. 348(1996)] If $n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k$, then either $\sum_{s=1}^{k-l}1/n_s\geqslant m$ or $l\geqslant n_k/n_{k-l}$.

Here parts (i)–(iii) are different extensions of Zhang's result in 1989. We conjecture that the condition in part (iii) is unnecessary. Part (iv) in the case $l=1$ is stronger than the Davenport-Mirsky-Newman-Radó result.

An improvement of Theorem 3.3(ii) can be extended to integral rings of certain algebraic number fields including cyclotomic fields and quadratic fields.

Theorem 3.4 [H. Pan and Z. W. Sun, `arXiv:math.NT/0504413`]. *Let K be an algebraic number field and O_K be the ring of algebraic integers in K. Suppose that K has an integral basis $1,\gamma,\ldots,\gamma^{n-1}$ with $\gamma\in O_K$ and $n=[K:\mathbb{Q}]$, and $\mathcal{A}=\{\alpha_s+\beta_sO_K\}_{s=1}^k$ forms an m-cover of O_K with $\alpha_s,\beta_s\in O_K$. Then, for any $\mu\in K$, either the set*

$$\left\{I\subseteq\{1,\ldots,k\}:\sum_{s\in I}\frac{1}{\beta_s}\in\mu+O_K\right\}$$

is empty or it has at least 2^m elements.

Theorem 3.5. *Let $A=\{a_s(n_s)\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$ with $a_k(n_k)$ irredundant.*

(i) [Z. W. Sun, Proc. AMS 127(1999); `arXiv:math.NT/0305369`] *Let $m_1,\cdots,m_{k-1}$ be positive integers relatively prime to $n_1,\cdots,n_{k-1}$ respectively. Then there is an $\alpha\in[0,1)$ such that for any $r=0,1,\ldots,n_k-1$ we have*

$$\left|\left\{\left\lfloor\sum_{s\in I}\frac{m_s}{n_s}\right\rfloor : I\subseteq[1,k-1]\ \text{and}\ \left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}=\frac{\alpha+r}{n_k}\right\}\right|\geqslant m.$$

(ii) [Z. W. Sun, `arXiv:math.NT/0411305`] *If n_k is a period of the covering function $w_A(x)$, then for any $r=0,1,\ldots,n_k-1$ we have*

$$\left|\left\{\left\lfloor\sum_{s\in I}\frac{1}{n_s}\right\rfloor : I\subseteq[1,k-1]\ \text{and}\ \left\{\sum_{s\in I}\frac{1}{n_s}\right\}=\frac{r}{n_k}\right\}\right|\geqslant m.$$

The following result concerning covering function is also remarkable.

Theorem 3.6 [Z. W. Sun, Combinatorica 2003]. *Let* $A = \{a_s(n_s)\}_{s=1}^k$.

(i) $M(A) = \max_{x\in\mathbb{Z}} w_A(x)$ *can be written in the form* $\sum_{s=1}^k m_s/n_s$ *with* $m_1, \ldots, m_k \in \mathbb{Z}^+$.

(ii) *If* $w_A(x)$ *is periodic modulo* $n_0 \in \mathbb{Z}^+$, *then*

$$\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq[1,k-1]\right\}\supseteq\left\{\frac{r}{n_k}:\ r\in\mathbb{N}\ \&\ r<\frac{n_k}{(n_0,n_k)}\right\}.$$

The following theorem is related to zero-sum problems.

Theorem 3.7 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]. *Let* $A = \{a_s(n_s)\}_{s=1}^k$ *and let* q *be a prime power.*

(i) *If* $\{w_A(x) : x \in \mathbb{Z}\} \subseteq \{2q-1, 2q\}$, *then for any* $c_1, \cdots, c_k \in \mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ *there exists an* $I \subseteq [1,k]$ *such that* $\sum_{s\in I} 1/n_s = q$ *and* $\sum_{s\in I} c_s = 0$.

(ii) *If* A *is an exact* $3q$*-cover of* $\mathbb{Z}$, *then for any* $c_1, \cdots, c_k \in \mathbb{Z}_q \oplus \mathbb{Z}_q$ *with* $c_1 + \cdots + c_k = 0$, *there exists an* $I \subseteq [1,k]$ *such that* $\sum_{s\in I} 1/n_s = q$ *and* $\sum_{s\in I} c_s = 0$.

Z. W. Sun conjectured that we can replace the prime power q in Theorem 3.7 by a general positive integer.

Part (i) in the case $n_1 = \cdots = n_k = 1$, reduces to the famous Erdős-Ginzburg-Ziv theorem [Bull. Research Council. Israel, 1961]: *For any* $c_1, \cdots, c_{2n-1} \in \mathbb{Z}$, *there is an* $I \subseteq [1, 2n-1]$ *with* $|I| = n$ *such that* $\sum_{s\in I} c_s \equiv 0 \pmod n$. Part (ii) in the case $n_1 = \cdots = n_k = 1$ reduces to the following result of N. Alon and Dubiner: *Let* q *be a prime power, and let* $c_1, \cdots, c_{3q}$ *be elements of* $\mathbb{Z}_q \oplus \mathbb{Z}_q$ *with* $c_1 + \cdots + c_{3q} = 0$. *Then there is an* $I \subseteq [1, 3q]$ *with* $|I| = q$ *such that* $\sum_{s\in I} c_s = 0$.

4. Covers of Groups by Cosets or Subgroups

In a cover $\{a_iG_i\}_{i=1}^k$ of a group G by left cosets, if $\bigcap_{i=1}^k G_i$ equals a given subgroup H of G with finite index, what is the lower bound of k? In this direction we need two functions.

In 1966 J. Mycielski [Fund. Math.] introduced the following function $f : \mathbb{Z}^+ \to \mathbb{N}$ which is now called *Mycielski's function*.

$f(p) = p-1$ for any prime p and $f(mn) = f(m)+f(n)$ for all $m, n \in \mathbb{Z}^+$.

Evidently

$$f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \sum_{t=1}^{r} \alpha_t(p_t - 1),$$

where $p_1, \cdots, p_r$ are distinct primes, and $\alpha_1, \cdots, \alpha_r \in \mathbb{N}$.

Let G be a group. A subgroup H of G is said to be *subnormal* if there is a chain $H_0 = H \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ of subgroups of G such that H_i is normal in H_{i+1} for all $0 \leqslant i < n$.

Let H be a subnormal subgroup of a group G with finite index, and

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

be a composition series from H to G (i.e. H_i is maximal normal in H_{i+1} for each $0 \leqslant i < n$). If the length n is zero (i.e. $H = G$), then we set $d(G, H) = 0$, otherwise we put

$$d(G, H) = \sum_{i=0}^{n-1} ([H_{i+1} : H_i] - 1).$$

When H is normal in G, the 'distance' $d(G, H)$ was first introduced by I. Korec [Fund. Math. 1974]. The current general notion is due to Z. W.

Sun [Fund. Math. 1990]. Z. W. Sun [Fund. Math. 1990; European J. Combin. 2001] showed that

$$[G:H]-1 \geqslant d(G,H) \geqslant f([G:H]) \geqslant \log_2[G:H],$$

and $d(G,H)=f([G:H])$ if and only if G/H_G is solvable.

Here is a conjecture of Mycielski posed in [Fund. Math. 1966].

Mycielski's Conjecture. *Let $A=\{a_iG_i\}_{i=1}^k$ be a disjoint cover of an abelian group G by left cosets. Then*

$$k \geqslant 1+f([G:G_i]) \quad \textit{for every} \ \ i=1,\cdots,k.$$

In the case $G=\mathbb{Z}$, this was confirmed by Š. Znám [Colloq. Math.] in 1966.

Theorem 4.1 (I. Korec, Z. W. Sun)**.** *Let G be a group and $\{a_iG_i\}_{i=1}^k$ be an exact m-cover of G with all the G_i subnormal in G. Then*

$$k \geqslant m + d\left(G, \bigcap_{i=1}^k G_i\right),$$

where the lower bound can be attained.

When $m=1$ and all the G_i are normal in G, this was proved by I. Korec [Fund. Math. 1974] and conjectured by Š. Znám in 1969. The current version is due to Z. W. Sun [European J. Combin. 22(2001)].

Theorem 4.2 [G. Lettl & Z. W. Sun, 2004, `arXiv:math.GR/0411144`]**.** *Let G be an abelian group and $\{a_iG_i\}_{i=1}^k$ be an m-cover of G with a_kG_k irredundant. Then we have $k \geqslant m+f([G:G_k])$.*

In the case $G_k = \{e\}$, this was conjectured by W. D. Gao and A. Geroldinger in 2003.

For a subgroup H of a group G, if $(|H|, [G : H]) = 1$ then H is called a *Hall subgroup* of G.

Theorem 4.3 [Z. W. Sun, 2005, `arXiv:math.GR/0501451`]. *Let G be a group and $a_1G_1, \ldots, a_kG_k$ be left cosets in G such that $\{a_iG_i\}_{i=1}^k$ forms a minimal cover of G (i.e., it is an m-cover of G but none of its proper subsystems is). If G is cyclic, or G is finite and $G_1, \ldots, G_k$ are normal Hall subgroups of G, then we have*

$$k \geqslant m + d\left(G, \bigcap_{i=1}^k G_i\right).$$

Recall that a group G is said to be *perfect* if it coincides with its derived group G'.

Theorem 4.4 [Z. W. Sun, 2005, `arXiv:math.GR/0501451`]. *If $\{G_i\}_{i=1}^k$ forms a minimal m-cover of a group G by subnormal subgroups, then there is a composition series from $\bigcap_{i=1}^k G_i$ to G whose factors are of prime orders, and all the G_i contain every perfect subgroup of G.*

When $m = 1$ and all the G_i are normal in G, this was essentially obtained by M. A. Brodie, R. F. Chamberlain and L.-C. Kappe [Proc. Amer. Math. Soc. 1988].

Theorem 4.5 [Z. W. Sun, `arXiv:math.GR/0501451`]. *Let $G_1, \ldots, G_k$ be normal Hall subgroups of a finite group G. Then, for any $a_1, \ldots, a_k \in G$,* *we have*

$$\left|\bigcup_{i=1}^{k} a_iG_i\right| \geqslant \left|\bigcup_{i=1}^{k} G_i\right|.$$

Finally we mention two open conjectures.

Conjecture 4.1 (S. Guo and Z. W. Sun, 2004). *Let $\{G_i\}_{i=1}^k$ be a minimal m-cover of a group G by finitely many subnormal subgroups. Assume that $[G : \bigcap_{i=1}^k G_i] = \prod_{t=1}^r p_t^{\alpha_t}$ where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r$ are positive integers. Then*

$$k > m + \sum_{t=1}^{r} (\alpha_t - 1)(p_t - 1).$$

Conjecture 4.2 (Z. W. Sun, 2004). *Let $a_1G_1, \ldots, a_kG_k$ be pairwise disjoint left cosets of a group G with $[G : G_i] < \infty$ for all $i = 1, \ldots, k$. Then $([G : G_i], [G : G_j]) \geqslant k$ for some $1 \leqslant i < j \leqslant k$.*

Conjecture 4.2 is known true for $k = 2$ and for p-groups, but it remains open even for the additive cyclic group $\mathbb{Z}$. I'd like to offer a prize of $200 for a proof of Conjecture 4.2.