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COMBINATORIAL ASPECTS OF COVERS OF GROUPS BY COSETS OR SUBGROUPS

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Abstract. If a group G is the union of finitely many left cosets $a_1G_1, \ldots ,$ a_kG_k of subgroups $G_1, \ldots , G_k$, then the system $\{a_sG_s\}_{i=1}^k$ is said to be a cover of G. In this talk we give a survey of results on extremal problems concerning covers of groups, and introduce progress on the famous Herzog-Schönheim conjecture which states that if $\{a_sG_s\}_{s=1}^k$ $(1 < k < \infty)$ is a partition of a group G by left cosets then the (finite) indices $[G : G_1], \ldots , [G : G_k]$ cannot be distinct. We will also mention some new challenging conjectures in the field.

1. Basic Concepts and a Fundamental Theorem

Let G be a multiplicative group with identity e. For a subgroup H of G, the *index* of H in G is $[G : H] = |\{gH : g \in G\}|$. Note that a right coset Hg of H is also a left coset $g(g^{-1}Hg)$ of the conjugate subgroup $g^{-1}Hg$, and $[G : g^{-1}Hg] = [G : H]$.

Let H be a subgroup of a group G with $k = [G : H] < \infty$. It is well known that we can partition G into k left cosets $a_1H, \ldots , a_kH$ of H (such a partition is called a left coset decomposition of G by H). If G is the additive group $\mathbb{Z}$ of integers, then $H = n\mathbb{Z}$ for some $n \in \mathbb{Z}^+ = \{1, 2, \ldots \}$

and $\{r+n\mathbb{Z}\}_{r=0}^{n-1}$ is a partition of $\mathbb{Z}$ into residue classes modulo n. Note that $\mathbb{Z}$ is an infinite cyclic group generated by 1, and $[\mathbb{Z}:n\mathbb{Z}]=|\mathbb{Z}/n\mathbb{Z}|=n$ for all $n\in\mathbb{Z}^+$.

In the 1930s P. Erdős introduced covers of $\mathbb{Z}$ by residue classes when he studied odd numbers not of the form 2^n+p where p is a prime.

Definition of Covers of $\mathbb{Z}$. Let $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ be a finite system of residue classes with $a_s\in\mathbb{Z}$ and $n_s\in\mathbb{Z}^+$. If $\bigcup_{s=1}^k(a_s+n_s\mathbb{Z})=\mathbb{Z}$, then we call A a *cover of* $\mathbb{Z}$ or a *covering system.* If A is a cover of $\mathbb{Z}$ but $\{a_s+n_s\mathbb{Z}\}_{s\neq t}$ is not, then we say that A forms a cover of $\mathbb{Z}$ with $a_t+n_t\mathbb{Z}$ *irredundant.* A cover of $\mathbb{Z}$ with all the members irredundant is called a *minimal cover* of $\mathbb{Z}$. If A is a cover of $\mathbb{Z}$ and also the k residue classes in A are pairwise disjoint, then A is said to be an *exact cover* or a *disjoint cover* of $\mathbb{Z}$. For system A we set

$$N_A=\mathrm{lcm}[n_1,\ldots,n_k]=\left[\mathbb{Z}:\bigcap_{s=1}^k n_s\mathbb{Z}\right].$$

Clearly, when $x\equiv y\pmod{N_A}$, $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ covers x if and only if A covers y.

Example 1.1. The system

$$A_0=\{1+2\mathbb{Z},\ 2+2^2\mathbb{Z},\ \ldots,\ 2^{k-2}+2^{k-1}\mathbb{Z},\ 2^{k-1}\mathbb{Z}\}$$

is a disjoint cover of $\mathbb{Z}$ (by k residue classes) with $N_{A_0}=2^k$, while

$$A_1=\{2\mathbb{Z},\ 3\mathbb{Z},\ 1+4\mathbb{Z},\ 5+6\mathbb{Z},\ 7+12\mathbb{Z}\}$$

is a minimal cover of $\mathbb{Z}$ with the moduli distinct and $N_{A_1}=12$.

Let G be a multiplicative group, and let

$$\mathcal{A} = \{a_s G_s\}_{s=1}^k$$

be a finite system of left cosets in G (where $a_1, \ldots, a_k \in G$, and $G_1, \ldots, G_k$ are subgroups of G). For any $I \subseteq [1,k] = \{1, \ldots, k\}$ we define the *index map* I^* from G to the power set of $[1,k]$ as follows:

$$I^*(x) = \{i \in I : \ x \in a_i G_i\} \quad (x \in G).$$

$m(\mathcal{A}) = \inf_{x \in G} |[1,k]^*(x)|$ is said to be the *covering multiplicity* of $\mathcal{A}$.

For a positive integer m, if $m(\mathcal{A}) \geqslant m$ (respectively, $|[1,k]^*(x)| = m$ for all $x \in G$) then we call $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ an *m-cover* (resp. *exact m-cover*) of G. If $\mathcal{A}$ forms an m-cover of G but none of its proper subsystems does, then it is said to be a *minimal* (or an *irredundant*) *m-cover* of G. A *cover* of G refers to a 1-cover of G, and a *disjoint cover* (or *partition*) of G means an exact 1-cover of G. Obviously an exact m-cover is a minimal m-cover and any minimal m-cover has covering multiplicity m.

Definition of Regular Covers (Sun, 2005). Let $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ be a finite system of left cosets in a group G. If

$$I^*(G) = \{I^*(g) : g \in G\} \not\subseteq [1,k]^*(G) = \{[1,k]^*(x) \colon x \in G\}$$

for all $I \subset [1,k]$ (i.e., whenever $I \subset [1,k]$ there is a $g \in G$ such that $I^*(g) \neq [1,k]^*(x)$ for all $x \in G$), then we call $\mathcal{A}$ a *regular cover* of G.

Note that a regular cover $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ of a group G must be a cover of G since $[1,k]^*(G) \not\ni \emptyset^*(G) = \{\emptyset\}$.

When $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ is a minimal m-cover of a group G, it forms a regular cover of G because for any $I \subset [1, k]$ there is a $g \in G$ such that $|I^*(g)| < m$ while $[1, k]^*(x) \geqslant m$ for all $x \in G$.

Let $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ be a minimal cover of a group G by left cosets. By a geometric consideration, B. H. Neumann [Publ. Math. Debrecen, 1954] showed that all the $[G : G_s]$ are finite and that $[G : \bigcap_{s=1}^k G_s] \leqslant c_k$ where c_k only depends on k. Later M. J. Tomkinson [Comm. Algebra, 1987] proved that we can take $c_k = k!$ and the bound $k!$ is best possible. In 2005 Z. W. Sun [Internat. J. Math., `arXiv:math.GR/0501451`] introduced the concept of regular cover and extended Tomkinson's result to regular covers of groups.

A Fundamental Theorem due to Efforts of B. H. Neumann (1954), M. J. Tomkinson (1987) and Z. W. Sun (2005). *Let $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ be a regular cover of a group G by left cosets of subgroups $G_1, \ldots, G_k$. Then we have*

$$\left[G : \bigcap_{s=1}^k G_s\right] \leqslant k! < \infty.$$

Proof. Let $F = \bigcap_{s=1}^k G_s$. We use induction on $|I|$ to show that

$$\left[\bigcap_{j \in \bar{I}} G_j : F\right] \leqslant |I|! \qquad \text{for all } I \subseteq [1, k],$$

where $\bar{I} = [1, k] \setminus I$, and $\bigcap_{j \in \emptyset} G_j$ refers to G.

Clearly $[\bigcap_{j \in \bar{\emptyset}} G_j : F] = 1 = |\emptyset|!$.

Now let $\emptyset \neq I \subseteq [1, k]$, and assume that $[\bigcap_{j \in \bar{I}_0} G_j : F] \leqslant |I_0|!$ for all $I_0 \subset [1, k]$ with $|I_0| < |I|$. Since $\mathcal{A}$ is regular and $\bar{I} \neq [1, k]$, there exists a

$g \in G$ such that

$$\bar{I}^*(g) = \{j \in \bar{I} : g \in a_j G_j\} \notin [1,k]^*(G).$$

For each $x \in \bigcap_{j\in\bar{I}} G_j$, as $\bar{I}^*(gx) = \bar{I}^*(g) \neq [1,k]^*(gx)$, we must have $gx \in \bigcup_{i\in I} a_i G_i$. So

$$g\Big(\bigcap_{j\in\bar{I}} G_j\Big) \subseteq \bigcup_{i\in I} a_i G_i.$$

For each $i \in I$, $\{i\} \cup \bar{I}$ is the complement of $I \setminus \{i\}$ in $[1,k]$; if $a_i G_i \cap g(\bigcap_{j\in\bar{I}} G_j)$ is nonempty then it contains exactly $[G_i \cap \bigcap_{j\in\bar{I}} G_j : F]$ left cosets of F. As

$$g\Big(\bigcap_{j\in\bar{I}} G_j\Big) = \bigcup_{i\in I}\Big(a_i G_i \cap g\Big(\bigcap_{j\in\bar{I}} G_j\Big)\Big),$$

we have

$$\Big[\bigcap_{j\in\bar{I}} G_j : F\Big] \leqslant \sum_{i\in I}\Big[\bigcap_{j\in\overline{I\setminus\{i\}}} G_j : F\Big].$$

By the induction hypothesis,

$$\Big[\bigcap_{j\in\overline{I\setminus\{i\}}} G_j : F\Big] \leqslant |I\setminus\{i\}|! = (|I|-1)! \qquad \text{for all } i \in I.$$

Therefore

$$\Big[\bigcap_{j\in\bar{I}} G_j : F\Big] \leqslant \sum_{i\in I}(|I|-1)! = |I|!.$$

This concludes the induction step. □

Example 1.2. Let H be any subgroup of a group G with $k = [G : H] < \infty$. Let $\{Ha_s\}_{s=1}^k$ be a right coset decomposition of G by H. Set $G_s = a_s^{-1} H a_s$ for $s = 1, \ldots, k$. Then $\{a_s G_s\}_{s=1}^k$ is a disjoint cover of G with

$$\bigcap_{s=1}^k G_s = \bigcap_{s=1}^k \bigcap_{h\in H} a_s^{-1} h^{-1} H h a_s = \bigcap_{g\in G} g^{-1} H g = H_G$$

where $H_G = \bigcap_{g\in G} g^{-1}Hg$ (the *core* of H in G) is the largest normal subgroup of G contained in H. By the Fundamental Theorem, $|G/H_G| = [G : H_G] \leqslant k! = [G : H]!$. In fact, it is known in group theory that the quotient group G/H_G can be embedded into the symmetric group S_k on $\{1, \dots, k\}$.

If $G = S_k$ and $H = \{\sigma \in G : \sigma(1) = 1\}$ (the stabilizer of 1), then $\{H, H(12), \dots, H(1k)\}$ is a partition of G, also $H_G = \{e\}$ because $G_i = (1i)^{-1}H(1i)$ is the stabilizer of i; therefore $[G : \bigcap_{s=1}^k G_i] = |G/H_G| = k!$ as noted by Tomkinson in 1987. So the upper bound in the Fundamental Theorem is best possible.

2. Mycielski's function and a group-theoretic function introduced by Korec and Sun

Definition of Mycielski's Function. The Mycielski function $f : \mathbb{Z}^+ \to \mathbb{N} = \{0, 1, \dots\}$ is given by

$$f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \sum_{t=1}^{r} \alpha_t(p_t - 1),$$

where $p_1, \dots, p_r$ are distinct primes, and $\alpha_1, \dots, \alpha_r \in \mathbb{N}$.

Note that $f(1) = 0$, $f(p) = p - 1$ for any prime p, and $f(mn) = f(m) + f(n)$ for all $m, n \in \mathbb{Z}^+$.

If $n \in \mathbb{Z}^+$ has the primary factorization $\prod_{t=1}^r p_t^{\alpha_t}$, then

$$1+f(n) = 1+\sum_{t=1}^{r} \alpha_t(p_t-1) \leqslant \prod_{t=1}^{r}(1+\alpha_t(p_t-1)) \leqslant \prod_{t=1}^{r}(1+(p_t-1))^{\alpha_t} = n$$

and

$$n = \prod_{t=1}^{n} p_t^{\alpha_t} \leqslant \prod_{t=1}^{r} \left(2^{p_t-1}\right)^{\alpha_t} = 2^{f(n)}.$$

So

$$n-1 \geqslant f(n) \geqslant \log_2 n \qquad \text{for all } n = 1, 2, 3, \ldots .$$

Let G be a group. A subgroup H of G is said to be *subnormal* if there is a chain $H_0 = H \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ of subgroups of G such that H_i is normal in H_{i+1} for all $0 \leqslant i < n$.

Definition of a Group-theoretic Function (I. Korec and Z. W. Sun). Let H be a subnormal subgroup of a group G with finite index, and

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

be a composition series from H to G (i.e. H_i is maximal normal in H_{i+1} for each $0 \leqslant i < n$). If the length n is zero (i.e. $H = G$), then we set $d(G, H) = 0$, otherwise we put

$$d(G, H) = \sum_{i=0}^{n-1}([H_{i+1} : H_i] - 1).$$

When H is a normal subgroup of a group with finite index, $d(G, H)$ was first introduced by I. Korec [Fund. Math. 1974]. The current general notion is due to Z. W. Sun [Fund. Math. 1990], and he viewed $d(G, H)$ as a 'distance'.

Let H be a subnormal subgroup of a group G with $[G : H] < \infty$. By the Jordan–Hölder theorem in group theory, the definition $d(G, H)$ does not depend on the choice of the composition series from H to G. Clearly $d(G, H) = 0$ if and only if $H = G$. If K is a subnormal subgroup of H with $[H : K] < \infty$, then

$$d(G, H) + d(H, K) = d(G, K).$$

Connection between the Functions d **and** f [Z. W. Sun, Fund. Math. 1990; European J. Combin. 2001]. *Let* H *be a subnormal subgroup of a group* G *with* $[G:H]<\infty$. *Then*

$$[G:H]-1\geqslant d(G,H)\geqslant f([G:H])\geqslant \log_2[G:H].$$

Moreover, $d(G,H)=f([G:H])$ *if and only if* G/H_G *is solvable.*

3. Inequalities of Mycielski's Type

In a regular cover $\{a_sG_s\}_{s=1}^k$ of a group G by left cosets, if $\bigcap_{s=1}^k G_s$ equals a given subgroup H of G with finite index, what is the lower bound of k? (Of course, $k!\geqslant[G:H]$ by the Fundamental Theorem.)

Mycielski's Conjecture [Fund. Math. 1966]. *Let* G *be an abelian group and* $G_1,\cdots,G_k$ *be subgroups of* G *(with finite indices). If* $A=\{a_sG_s\}_{s=1}^k$ *forms an exact cover of* G *for some* $a_1,\ldots,a_k\in G$, *then*

$$k\geqslant 1+f([G:G_s])\qquad \text{for every}\quad s=1,\cdots,k.$$

Š. Znám [Colloq. Math. 1966]: *Mycielski's conjecture holds for the additive group* $\mathbb{Z}$ *of integers. Equivalently, if* $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ *forms a disjoint cover of* $\mathbb{Z}$ *by residue classes then* $k\geqslant 1+f(n_t)$ *for each* $t=1,\ldots,k$.

Znám [Colloq. Math. 1969]: *If* $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ *forms a cover of* $\mathbb{Z}$ *and the residue class* $a_t+n_t\mathbb{Z}$ *is disjoint with all the remaining residue classes, then we have* $k\geqslant 1+f(n_t)$.

Znám [Acta Arith. 1975]: *If* $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ *forms a cover of* $\mathbb{Z}$ *with* $a_t(n_t)$ *irredundant, then there are* $1 + f(n_t)$ *integers no two of which belong to the same member of* A*.*

Znám's Conjecture [Coll. Math. Soc. János Bolyai, 1968; Acta Arith. 1975]. *If* $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ *is a disjoint cover or just a minimal cover of* $\mathbb{Z}$ *by residue classes, then*

$$k \geqslant 1 + f(N_A) \quad \text{and hence} \quad N_A \leqslant 2^{k-1} \leqslant k!.$$

(Recall that $N_A = [n_1, \ldots, n_k] = [\mathbb{Z} : \bigcap_{s=1}^k n_s\mathbb{Z}]$.)

I. Korec [Fund. Math. 1974]: *Let* $\{a_sG_s\}_{s=1}^k$ *be a partition of a group* G *into finitely many left cosets of normal subgroups* $G_1, \ldots, G_k$*. Then* $[G : \bigcap_{s=1}^k G_s] < \infty$ *and* $k \geqslant 1 + f([G : \bigcap_{s=1}^k G_s])$*.*

Sun's Further Extension [Fund. Math., 134(1990); European J. Combin. 22(2001)]**.** *Let* G *be a group and* $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ *be an exact* m*-cover of* G *by left cosets with all the* G_s *subnormal in* G*. Then*

$$k \geqslant m + d\left(G, \bigcap_{s=1}^k G_s\right), \qquad (*)$$

where the lower bound can be attained. Moreover, for any subgroup K *of* G *not contained in all the* G_s *we have*

$$|\{1 \leqslant s \leqslant k : K \not\subseteq G_s\}| \geqslant 1 + d\left(K, K \cap \bigcap_{s=1}^k G_s\right).$$

A key step in Sun's proof is to show that under the condition, for any maximal normal subgroup H of G, either any left coset of H contains one of the k members in $\mathcal{A}$ or none of the left cosets of H contains a member in $\mathcal{A}$.

The lower bound in $(*)$ can be attained as shown by the following example.

Example 3.1. Let H be a subnormal subgroup of a group G with finite index, and let $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ be a composition series from H to G. Write

$$H_{i+1} \setminus H_i = \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i \quad \text{for } i = 0, 1, \ldots, n-1.$$

Z. W. Sun [Fund. Math. 134(1990)] observed that the following $1+d(G,H)$ cosets

$$H_0,\ b_j^{(i)} H_i\ (0 \leqslant i < n,\ 1 \leqslant j < [H_{i+1} : H_i])$$

form a partition of G (in fact, $G = H_0 \cup (H_1 \setminus H_0) \cup \cdots \cup (H_n \setminus H_{n-1})$). These cosets, together with $m-1$ copies of G, form an exact m-cover of G with the number k of cosets being $m + d(G,H)$ and the intersection of the k subnormal subgroups being H.

In view of Example 1.2 and Sun's above result on exact m-covers of groups by cosets of subnormal subgroups, we have the following interesting application.

Inequalities on Cores of Subnormal Subgroups. *Let H be a subnormal subgroup of a group G with $[G:H] < \infty$.*

(i) [Z. W. Sun, Fund. Math. 134(1990)] *We have*

$$[G:H] - 1 \geqslant d(G, H_G) \geqslant f(|G/H_G|)$$

and hence $|G/H_G| \leqslant 2^{[G:H]-1}$.

(ii) [Z. W. Sun, European J. Combin. 22(2001)] *H is normal in G if and only if*

$$|N_G(H)/H| + d(H, H_G) \geqslant [G : H],$$

where $N_G(H) = \{g \in G : gH = Hg\}$ *is the normalizer of H in G.*

M. A. Berger, A. Felzenbaum and A. S. Fraenkel [Coll. Math. 1988]: *If* $\{a_sG_s\}_{s=1}^k$ *is a disjoint cover of a finite solvable group G by left cosets, then* $k \geqslant 1 + f([G : G_t])$ *for* $t = 1, \ldots, k$.

Z. W. Sun [European J. Combin. 2001]: *Let* $\{a_sG_s\}_{s=1}^k$ *be an exact m-cover of a group G. For any* $1 \leqslant t \leqslant k$, *whenever* $G/(G_t)_G$ *is solvable we have* $k \geqslant m + f([G : G_t])$ *and hence* $[G : G_t] \leqslant 2^{k-m}$.

A Conjecture of Sun. *Let* $\{a_sG_s\}_{s=1}^k$ *be an exact m-cover of a group G with all the* $G/(G_s)_G$ *solvable. Then* $k \geqslant m + f(N)$ *where N is the least common multiple of the indices* $[G : G_1], \ldots, [G : G_k]$.

R. J. Simpson [Acta Arith. 1985]: *Let* $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ *be a minimal cover of $\mathbb{Z}$ with* $a_s \in \mathbb{Z}$ *and* $n_s \in \mathbb{Z}^+$. *Then for any divisor d of N_A with* $0 < d < N_A$ *we have*

$$|\{1 \leqslant s \leqslant k : \ n_s \nmid d\}| \geqslant 1 + f(N_A/d).$$

Letting $d = 1$ *we then obtain* $k \geqslant 1 + f(N_A)$.

M. A. Berger, A. Felzenbaum and A. S. Fraenkel [Coll. Math. 1988]: *Let* $\{a_sG_s\}_{s=1}^k$ *be a minimal cover of a group G of squarefree order. If all the G_s are normal in G then* $k \geqslant 1 + f([G : \bigcap_{s=1}^k G_s])$.

A Result of Sun on Regular Covers [Internat. J. Math., in press, `arXiv:math.GR/0501451`]. If $\mathcal{A}=\{a_sG_s\}_{s=1}^k$ is a regular cover of a group G by left cosets, and for any $i,j=1,\ldots,k$ either G_i and G_j are subnormal in G with $[G:G_i]$ relatively prime to $[G_i:G_i\cap G_j]$, or G_i and G_j are normal in G with $G/(G_i\cap G_j)$ cyclic, then we have the inequality

$$k\geqslant m(\mathcal{A})+d\bigg(G,\bigcap_{s=1}^k G_s\bigg).$$

The conditions of this result are essentially indispensable as shown by the following example:

Example 3.2. Let G be the group $C_p\times C_p$ where p is a prime and C_p is the cyclic group of order p. Then any element $a\neq e$ of G has order p. Let $G_1,\ldots,G_k$ be all the distinct subgroups of G with order p. If $1\leqslant s<t\leqslant k$ then $G_s\cap G_t=\{e\}$. Clearly $\{G_s\}_{s=1}^k$ forms a minimal cover of G by normal subgroups whose intersection is $H=\{e\}$. Since $1+k(p-1)=|\bigcup_{s=1}^k G_s|=|G|=p^2$, we have

$$k=p+1\leqslant 2p-1=1+f([G:H])=1+d\bigg(G,\bigcap_{s=1}^k G_s\bigg).$$

When $p>2$ the last inequality becomes strict. We remark that both G/G_s and $G_s/(G_s\cap G_t)$ $(t\neq s)$ have order p.

Let G be a finite multiplicative abelian group with identity e. If aH is a left coset in G with $e\notin aH$ (i.e. $a\notin H$), then aH is called a *proper coset* in G. During their study of zero-sum problems on abelian groups, W. D. Gao and A. Geroldinger [European J. Combin. 24(2003)] defined

$\mathbf{s}(G)$ to be the smallest number of proper cosets that can cover $G\setminus\{e\}$. For left cosets $a_1G_1,\ldots,a_kG_k$ in group G, the system $\{a_sG_s\}_{s=1}^k$ is a cover of $G\setminus\{e\}$ if and only if $\{a_sG_s\}_{s=0}^k$ forms a cover of G with the coset a_0G_0 irredundant where $a_0=e$ and $G_0=\{e\}$ (and hence $\bigcap_{s=0}^k G_s=\{e\}$). Thus, by the previous results on covers of groups, we have $\mathbf{s}(G)\leqslant f(|G|)$, and $\mathbf{s}(G)=f(|G|)$ if G is cyclic or of squarefree order.

By using algebraic number theory and characters of abelian groups, G. Lettl and Z. W. Sun [`arXiv:math.GR/0411144`] obtained in 2004 the following result for minimal m-covers of a general abelian group.

A Result of Lettl and Sun on Covers of Abelian Groups. *Let $\mathcal{A}=\{a_sG_s\}_{s=1}^k$ be a minimal m-cover of an abelian group G by left cosets. Then $k\geqslant m+f([G:G_t])$ for any $t=1,\ldots,k$.*

By Example 3.2, even for $G=C_p\times C_p$ we cannot replace $f([G:G_t])$ in the above result by $f([G:\bigcap_{s=1}^k G_s])$. But the speaker believes that $f([G:G_t])$ in the lower bound of k can be replaced by $f(N)$ where N is the least common multiple of $[G:G_1],\ldots,[G:G_k]$.

Another Conjecture of Sun. *Let $\mathcal{A}=\{a_sG_s\}_{s=1}^k$ be an m-cover of a group G by left cosets. For any $s=1,\ldots,k$ with a_sG_s irredundant, if G_s is subnormal in G then $k\geqslant m+d(G,G_s)$; if $G/(G_s)_G$ is solvable then $k\geqslant m+f([G:G_s])$.*

4. Covering a group by subgroups

Any finite non-cyclic group G can be covered by finitely many proper subgroups because $G=\bigcup_{x\in G}\langle x\rangle$, but no field can be covered by finitely many proper subfields as shown by A. Bialynicki-Birula, J. Browkin and A. Schinzel [Colloq. Math. 7(1959)].

Example 4.1. Let G be a group with $G/Z(G)$ finite, where $Z(G)$ is the center of G. For $x,y\in G$, if $xy\neq yx$ then $(x^{-1}y)x\neq x(x^{-1}y)$ and hence $xZ(G)\neq yZ(G)$. Let $X=\{x_1,\ldots,x_k\}$ be a maximal set of pairwise non-commuting elements of G. Then $k=|X|\leqslant|G/Z(G)|$, and $\{C_G(x_i)\}_{i=1}^k$ forms a minimal cover of G by centralizers with $\bigcap_{i=1}^k C_G(x_i)=Z(G)$ (recall that $C_G(x)=\{g\in G:\ gx=xg\}$), and $|G/Z(G)|\leqslant c^k$ for some absolute constant $c>0$ [L. Pyber, J. London Math. Soc. 35(1987)]. D. R. Mason [Math. Proc. Cambridge Philos. Soc. 83(1978)] proved that $|G|\geqslant 2k-2$, which was conjectured by Erdös and E. G. Straus in 1975.

Recall that a group G is said to be *perfect* if it coincides with its derived group $G'=\langle x^{-1}y^{-1}xy:\ x,y\in G\rangle$.

M. A. Brodie, R. F. Chamberlain and L.-C. Kappe [Proc. Amer. Math. Soc. 104(1998)]: *If $\{G_s\}_{s=1}^k$ is a minimal cover of a group G by finitely many normal subgroups, then $G/\bigcap_{s=1}^k G_s$ is solvable and all perfect normal subgroups of G are contained in each of $G_1,\ldots,G_k$.*

Z. W. Sun [Internat. J. Math., `arxiv:math.GR/0501451`] *Suppose that $\{G_s\}_{s=1}^k$ is a minimal m-cover of a group G by subnormal subgroups. Then there is a composition series from $\bigcap_{s=1}^k G_s$ to G whose factors are of prime orders (equivalently, $G/(\bigcap_{s=1}^k G_s)_G$ is solvable), and all the G_s contain every perfect subgroup of G.*

Concerning covers of a group by subnormal subgroups, the speaker and his student Song Guo made the following conjecture in 2004.

A Conjecture of S. Guo and Z. W. Sun. *Let* $\{G_s\}_{s=1}^k$ *be a minimal* m*-cover of a group* G *by finitely many subnormal subgroups. Assume that* $[G : \bigcap_{i=1}^k G_s] = \prod_{t=1}^r p_t^{\alpha_t}$ *where* $p_1, \ldots, p_r$ *are distinct primes and* $\alpha_1, \ldots, \alpha_r$ *are positive integers. Then*

$$k > m + \sum_{t=1}^{r} (\alpha_t - 1)(p_t - 1).$$

5. Unions of cosets and Disjoint Cosets

In number theory, a theorem of C. A. Rogers asserts that if $a_s \in \mathbb{Z}$ and $n_s \in \mathbb{Z}^+$ for $s = 1, \ldots, k$ then

$$\left|\left\{0 \leqslant x < N : x \in \bigcup_{s=1}^{k} a_s + n_s\mathbb{Z}\right\}\right| \geqslant \left|\left\{0 \leqslant x < N : x \in \bigcup_{s=1}^{k} n_s\mathbb{Z}\right\}\right|,$$

where $N = [n_1, \ldots, n_k]$ is the least common multiple of $n_1, \ldots, n_k$.

Inspired by this, the speaker conjectured in 1990 that for any finitely many left cosets $a_1G_1, \ldots, a_kG_k$ in a finite group G we always have the inequality $|\bigcup_{s=1}^k a_sG_s| \geqslant |\bigcup_{s=1}^k G_s|$. When G is a finite cyclic group this reduces to Rogers' result. But soon Tomkinson pointed out that the conjecture is not true for the Klein group $C_2 \times C_2$. In fact, if $G = \{1, -1\} \times \{1, -1\}$,

$$G_1 = \{e = (1, 1)\},\ G_2 = \{e, (1, -1)\},\ G_3 = \{e, (-1, 1)\},\ G_4 = \{e, (-1, -1)\},$$

then

$$G_1 \cup G_2 \cup G_3 \cup (1,-1)G_4 = \{e, (1,-1), (-1,1)\} \subset \bigcup_{s=1}^{4} G_s = G.$$

For a subgroup H of a finite group G, if $|H|$ is relatively prime to $[G:H]$ (i.e., $\gcd(|H|,[G:H])=1$) then H is called a *Hall subgroup* of G. A Sylow p-subgroup of a finite group G is just a Hall p-subgroup of G.

A Result of Sun on Unions of Cosets [Internat. J. Math., in press]**.** *Let G be a finite group and let $G_1,\ldots,G_k$ be normal Hall subgroups of G. Then, for any $a_1,\ldots,a_k \in G$, we have*

$$\left|\bigcup_{s=1}^{k} a_s G_s\right| \geqslant \left|\bigcup_{s=1}^{k} G_s\right|.$$

The following conjecture seems very challenging.

A Conjecture of Sun on Disjoint Cosets [Internat. J. Math., in press]**.** *Let $a_1G_1,\ldots,a_kG_k$ $(k>1)$ be finitely many pairwise disjoint left cosets in a group G with $[G:G_s]<\infty$ for all $s=1,\ldots,k$. Then we have $\gcd([G:G_i],[G:G_j]) \geqslant k$ for some $1 \leqslant i < j \leqslant k$.*

This conjecture is true when G is a p-group with p a prime. In fact, under the condition of the above conjecture, clearly

$$\left[G : \bigcap_{s=1}^{k} G_s\right] \geqslant \left[\bigcup_{i=1}^{k} a_i G_i : \bigcap_{s=1}^{k} G_s\right] = \sum_{i=1}^{k} \left[G_i : \bigcap_{s=1}^{k} G_s\right]$$

and hence $\sum_{i=1}^{k}[G:G_i]^{-1} \leqslant 1$. Suppose that $[G:G_1] \leqslant \cdots \leqslant [G:G_k]$. Since $\sum_{i=1}^{k-1}[G:G_i]^{-1} < 1$, there is an $i \in [1,k-1]$ such that $[G:G_i] \not\leqslant$

$k-1$ and hence $[G:G_k] \geqslant [G:G_i] \geqslant k$. If $[G:G_k]$ is divisible by all those $[G:G_1],\ldots,[G:G_{k-1}]$ (this happens if G is a p-group with p a prime), then $\gcd([G:G_i],[G:G_k]) = [G:G_i] \geqslant k$.

The conjecture is also true for $k=2$. In fact, when H and K are two subgroups of a group G with finite index, it is easy to see that

$$\gcd([G:H],[G:K]) = 1 \Longrightarrow HK = G \iff xH \cap yK \neq \emptyset \text{ for all } x,y \in G.$$

The conjecture for the infinite cyclic group $\mathbb{Z}$ has been proved to be true for $k<5$ by the speaker [Chinese Ann. Math. Ser. A 13(1992)], and for $k \leqslant 20$ by K. O'Bryant [`arXiv:math.NT/0604347`] quite recently.

6. On extended Herzog-Schönheim Conjecture

Soon after his invention of the concept of cover of $\mathbb{Z}$, Erdős made the following conjecture: *If $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ and $1<n_1<\cdots<n_k$, then A cannot be a partition of $\mathbb{Z}$.*

This conjecture of Erdős was soon confirmed independently by H. Davenport, L. Mirsky, D. Newman and R. Rado in the 1950s.

The Davenport-Mirsky-Newman-Rado Result. *Let $A=\{a_s+n_s\mathbb{Z}\}_{s=1}^k$ be an exact cover of $\mathbb{Z}$ with $1 \leqslant n_1 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$. Then we must have $n_{k-1}=n_k$.*

Proof. Without loss of generality we assume that $0 \leqslant a_s < n_s$ for all $s \in [1,k]$. For $|z|<1$ we have

$$\sum_{s=1}^{k} \frac{z^{a_s}}{1-z^{n_s}} = \sum_{s=1}^{k}\sum_{q=0}^{\infty} z^{a_s+qn_s} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

If $n_{k-1} < n_k$ then

$$\infty = \lim_{\substack{z\to e^{2\pi i/n_k}\\ |z|<1}} \frac{z^{a_k}}{1-z^{n_k}} = \lim_{\substack{z\to e^{2\pi i/n_k}\\ |z|<1}} \left(\frac{1}{1-z} - \sum_{s=1}^{k-1} \frac{z^{a_s}}{1-z^{n_s}}\right) < \infty,$$

a contradiction! □

Recall that $\{1(2), 2(2^2), \ldots, 2^{k-2}(2^{k-1}), 0(2^{k-1})\}$ is a disjoint cover of $\mathbb{Z}$ by k residue classes whose first $k-1$ moduli are distinct.

In 1974 M. Herzog and J. Schönheim [Canad. Math. Bull. 17(1974)] extended Erdős' conjecture to partitions of groups.

Herzog-Schönheim Conjecture. *Let $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ be a partition of a group G into $k > 1$ left cosets of subgroups $G_1, \ldots, G_k$. Then the finite indices $[G : G_1], \ldots, [G : G_k]$ cannot be distinct.*

M. A. Berger, A. Felzenbaum and A. S. Fraenkel [Canad. Math. Bull. 29(1986); Fund. Math. 128(1987)] proved the Herzog-Schönheim conjecture for finite nilpotent groups and pyramidal groups. (A finite group G is said to be *pyramidal* if it contains a chain $\{e\} = H_0 \subset H_1 \subset \cdots \subset H_n = G$ of subgroups such that $[H_1 : H_0] \geqslant \cdots \geqslant [H_n : H_{n-1}]$ are primes in non-ascending order, such a group must be solvable.)

A finite system $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ of left cosets in a group G is called a *uniform cover* of G if it covers all elements of G the same number of times. Note that $\mathcal{A}$ is a uniform cover of G if and only if it is an exact m-cover of G for some $m = 1, 2, \ldots$.

Here is recent progress [Z. W. Sun, J. Algebra 273(2004)] on the extended Herzog-Schönheim conjecture for uniform covers of groups.

Sun's Result on Herzog-Schönheim Conjecture for Uniform Covers. *Let* $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ *be a uniform cover of a group* G *with* $G_1, \ldots, G_k$ *not all equal to* G *and*

$$n_1 = [G : G_1] \leqslant \cdots \leqslant n_k = [G : G_k].$$

Suppose that all the G_i *are subnormal in* G*, or* G/H *is a solvable group having a normal Sylow* p*-subgroup where* $H = (\bigcap_{s=1}^k G_s)_G$*, and* p *is the largest prime divisor of* $|G/H|$*. Then the indices* $n_1, \ldots, n_k$ *cannot be pairwise distinct. Moreover, if* $|\{1 \leqslant i \leqslant k : n_i = n\}| \leqslant M$ *for all* $n \in \mathbb{Z}^+$*, then we have*

$$\log n_1 \leqslant \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log\log M),$$

where the logarithm has the natural base $e = 2.718...$*,* $\gamma = 0.577...$ *is the Euler constant and the* O*-constant is absolute.*

The proof of this result is long and sophisticated; it involves combinatorics and group theory, as well as analytic number theory. One of the basic lemma used in the proof is the following arithmetical property of indices.

A Lemma on Divisibility of Indices [Z. W. Sun, European J. Combin. 22(2001)]. *Let* $G_1, \ldots, G_k$ *be subnormal subgroups of a group* G *with finite index. Then*

$$\left[G : \bigcap_{s=1}^k G_s\right] \,\Big|\, \prod_{s=1}^k [G : G_s]$$

and hence

$$P\left(\left[G : \bigcap_{s=1}^k G_s\right]\right) = \bigcup_{s=1}^k P([G : G_s]),$$

where $P(n)$ denotes the set of prime divisors of n.

If $G_1, \ldots, G_k$ are subgroups of a group G with finite index, then $[G : \bigcap_{s=1}^k G_s] \leqslant \prod_{s=1}^k [G : G_s] < \infty$ by Poincaré's theorem. The above lemma can be viewed as an important number-theoretic property of subnormality, it is the main reason why covers involving subnormal subgroups are better behaved than general covers.

In view of Example 1.2 and the above lemma, *if H is a subnormal subgroup of a group G with $[G : H] < \infty$ then*

$$P(|G/H_G|) = P([G : H]).$$

It should be mentioned that this result and the divisibility lemma are quite useful but could not been found in group theory references before the speaker worked them out.