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COMBINATORIAL ASPECTS OF COVERS
OF GROUPS BY COSETS OR SUBGROUPS

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Abstract. If a group $G$ is the union of finitely many left cosets $a_1G_1, \ldots, a_kG_k$ of subgroups $G_1, \ldots, G_k$, then the system $\{a_iG_i\}_{i=1}^k$ is said to be a cover of $G$. In this talk we give a survey of results on extremal problems concerning covers of groups, and introduce progress on the famous Herzog-Schönheim conjecture which states that if $\{a_iG_i\}_{i=1}^k (1 < k < \infty)$ is a partition of a group $G$ by left cosets then the (finite) indices $[G : G_1], \ldots, [G : G_k]$ cannot be distinct. We will also mention some new challenging conjectures in the field.

1. Basic Concepts and a Fundamental Theorem

Let $G$ be a multiplicative group with identity $e$. For a subgroup $H$ of $G$, the index of $H$ in $G$ is $[G : H] = \{|gH : g \in G\}|$. Note that a right coset $Hg$ of $H$ is also a left coset $g(g^{-1}Hg)$ of the conjugate subgroup $g^{-1}Hg$, and $[G : g^{-1}Hg] = [G : H]$.

Let $H$ be a subgroup of a group $G$ with $k = [G : H] < \infty$. It is well known that we can partition $G$ into $k$ left cosets $a_1H, \ldots, a_kH$ of $H$ (such a partition is called a left coset decomposition of $G$ by $H$). If $G$ is the additive group $\mathbb{Z}$ of integers, then $H = n\mathbb{Z}$ for some $n \in \mathbb{Z}^+ = \{1, 2, \ldots\}$.
and \( \{r + n\mathbb{Z}\}_{r=0}^{n-1} \) is a partition of \( \mathbb{Z} \) into residue classes modulo \( n \). Note that \( \mathbb{Z} \) is an infinite cyclic group generated by 1, and \( [\mathbb{Z} : n\mathbb{Z}] = |\mathbb{Z}/n\mathbb{Z}| = n \) for all \( n \in \mathbb{Z}^+ \).

In the 1930s P. Erdős introduced covers of \( \mathbb{Z} \) by residue classes when he studied odd numbers not of the form \( 2^n + p \) where \( p \) is a prime.

**Definition of Covers of \( \mathbb{Z} \).** Let \( A = \{a_s + n_s\mathbb{Z}\}_{s=1}^{k} \) be a finite system of residue classes with \( a_s \in \mathbb{Z} \) and \( n_s \in \mathbb{Z}^+ \). If \( \bigcup_{s=1}^{k} (a_s + n_s\mathbb{Z}) = \mathbb{Z} \), then we call \( A \) a cover of \( \mathbb{Z} \) or a covering system. If \( A \) is a cover of \( \mathbb{Z} \) but \( \{a_s + n_s\mathbb{Z}\}_{s \neq t} \) is not, then we say that \( A \) forms a cover of \( \mathbb{Z} \) with \( a_t + n_t\mathbb{Z} \) irredundant. A cover of \( \mathbb{Z} \) with all the members irredundant is called a minimal cover of \( \mathbb{Z} \). If \( A \) is a cover of \( \mathbb{Z} \) and also the \( k \) residue classes in \( A \) are pairwise disjoint, then \( A \) is said to be an exact cover or a disjoint cover of \( \mathbb{Z} \). For system \( A \) we set

\[
N_A = \text{lcm}[n_1, \ldots, n_k] = \left[ \mathbb{Z} : \bigcap_{s=1}^{k} n_s\mathbb{Z} \right].
\]

Clearly, when \( x \equiv y \) (mod \( N_A \)), \( A = \{a_s + n_s\mathbb{Z}\}_{s=1}^{k} \) covers \( x \) if and only if \( A \) covers \( y \).

**Example 1.1.** The system

\[
A_0 = \{1 + 2\mathbb{Z}, 2 + 2^2\mathbb{Z}, \ldots, 2^{k-2} + 2^{k-1}\mathbb{Z}, 2^{k-1}\mathbb{Z}\}
\]

is a disjoint cover of \( \mathbb{Z} \) (by \( k \) residue classes) with \( N_{A_0} = 2^{k} \), while

\[
A_1 = \{2\mathbb{Z}, 3\mathbb{Z}, 1 + 4\mathbb{Z}, 5 + 6\mathbb{Z}, 7 + 12\mathbb{Z}\}
\]

is a minimal cover of \( \mathbb{Z} \) with the moduli distinct and \( N_{A_1} = 12 \).
Let $G$ be a multiplicative group, and let

$$\mathcal{A} = \{a_s G_s\}_{s=1}^k$$

be a finite system of left cosets in $G$ (where $a_1, \ldots, a_k \in G$, and $G_1, \ldots, G_k$ are subgroups of $G$). For any $I \subseteq [1, k] = \{1, \ldots, k\}$ we define the index map $I^*$ from $G$ to the power set of $[1, k]$ as follows:

$$I^*(x) = \{i \in I : x \in a_i G_i\} \quad (x \in G).$$

$m(\mathcal{A}) = \inf_{x \in G} |[1, k]^*(x)|$ is said to be the covering multiplicity of $\mathcal{A}$.

For a positive integer $m$, if $m(\mathcal{A}) \geq m$ (respectively, $|[1, k]^*(x)| = m$ for all $x \in G$) then we call $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ an $m$-cover (resp. exact $m$-cover) of $G$. If $\mathcal{A}$ forms an $m$-cover of $G$ but none of its proper subsystems does, then it is said to be a minimal (or an irredundant) $m$-cover of $G$. A cover of $G$ refers to a 1-cover of $G$, and a disjoint cover (or partition) of $G$ means an exact 1-cover of $G$. Obviously an exact $m$-cover is a minimal $m$-cover and any minimal $m$-cover has covering multiplicity $m$.

**Definition of Regular Covers** (Sun, 2005). Let $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ be a finite system of left cosets in a group $G$. If

$$I^*(G) = \{I^*(g) : g \in G\} \not\subseteq [1, k]^*(G) = \{[1, k]^*(x) : x \in G\}$$

for all $I \subset [1, k]$ (i.e., whenever $I \subset [1, k]$ there is a $g \in G$ such that $I^*(g) \neq [1, k]^*(x)$ for all $x \in G$), then we call $\mathcal{A}$ a regular cover of $G$.

Note that a regular cover $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ of a group $G$ must be a cover of $G$ since $[1, k]^*(G) \not\supseteq \emptyset^*(G) = \{\emptyset\}$. 
When $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ is a minimal $m$-cover of a group $G$, it forms a regular cover of $G$ because for any $I \subset [1,k]$ there is a $g \in G$ such that $|I^*(g)| < m$ while $[1,k]^*(x) \geq m$ for all $x \in G$.

Let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be a minimal cover of a group $G$ by left cosets. By a geometric consideration, B. H. Neumann [Publ. Math. Debrecen, 1954] showed that all the $[G:G_s]$ are finite and that $[G:\bigcap_{s=1}^k G_s] \leq c_k$ where $c_k$ only depends on $k$. Later M. J. Tomkinson [Comm. Algebra, 1987] proved that we can take $c_k = k!$ and the bound $k!$ is best possible. In 2005 Z. W. Sun [Internat. J. Math., arXiv:math.GR/0501451] introduced the concept of regular cover and extended Tomkinson’s result to regular covers of groups.

**A Fundamental Theorem** due to Efforts of B. H. Neumann (1954), M. J. Tomkinson (1987) and Z. W. Sun (2005). Let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be a regular cover of a group $G$ by left cosets of subgroups $G_1, \ldots, G_k$. Then we have

$$[G:\bigcap_{s=1}^k G_s] \leq k! < \infty.$$  

**Proof.** Let $F = \bigcap_{s=1}^k G_s$. We use induction on $|I|$ to show that

$$\left[\bigcap_{j \in \bar{I}} G_j : F \right] \leq |I|! \quad \text{for all } I \subseteq [1,k],$$

where $\bar{I} = [1,k] \setminus I$, and $\bigcap_{j \in \emptyset} G_j$ refers to $G$.

Clearly $[\bigcap_{j \in \emptyset} G_j : F] = 1 = |\emptyset|!$.

Now let $\emptyset \neq I \subseteq [1,k]$, and assume that $[\bigcap_{j \in I_0} G_j : F] \leq |I_0|!$ for all $I_0 \subset [1,k]$ with $|I_0| < |I|$. Since $\mathcal{A}$ is regular and $\bar{I} \neq [1,k]$, there exists a
For each $x \in \bigcap_{j \in I} g_j$, as $\bar{I}^*(gx) = \bar{I}^*(g) \neq [1, k]^*(gx)$, we must have $gx \in \bigcup_{i \in I} a_i G_i$. So
\[
g \left( \bigcap_{j \in I} G_j \right) \subseteq \bigcup_{i \in I} a_i G_i.
\]
For each $i \in I$, $\{i\} \cup \bar{I}$ is the complement of $I \setminus \{i\}$ in $[1, k]$; if $a_i G_i \cap g(\bigcap_{j \in I} G_j)$ is nonempty then it contains exactly $[G_i \cap \bigcap_{j \in I} G_j : F]$ left cosets of $F$. As
\[
g \left( \bigcap_{j \in I} G_j \right) = \bigcup_{i \in I} \left( a_i G_i \cap g \left( \bigcap_{j \in I} G_j \right) \right),
\]
we have
\[
\left[ \bigcap_{j \in I} G_j : F \right] \leq \sum_{i \in I} \left[ \bigcap_{j \in I \setminus \{i\}} G_j : F \right].
\]
By the induction hypothesis,
\[
\left[ \bigcap_{j \in I \setminus \{i\}} G_j : F \right] \leq |I \setminus \{i\}|! = (|I| - 1)! \quad \text{for all } i \in I.
\]
Therefore
\[
\left[ \bigcap_{j \in I} G_j : F \right] \leq \sum_{i \in I} (|I| - 1)! = |I|!.
\]
This concludes the induction step. □

Example 1.2. Let $H$ be any subgroup of a group $G$ with $k = |G : H| < \infty$.
Let $\{Ha_s\}_{s=1}^k$ be a right coset decomposition of $G$ by $H$. Set $G_s = a_s^{-1} Ha_s$ for $s = 1, \ldots, k$. Then $\{a_s G_s\}_{s=1}^k$ is a disjoint cover of $G$ with
\[
\bigcap_{s=1}^k G_s = \bigcap_{s=1}^k \bigcap_{h \in H} a_s^{-1} h^{-1} Ha_s = \bigcap_{g \in G} g^{-1} H g = H_G
\]
where \( H_G = \bigcap_{g \in G} g^{-1} H g \) (the core of \( H \) in \( G \)) is the largest normal subgroup of \( G \) contained in \( H \). By the Fundamental Theorem, \(|G/H_G| = [G : H_G] \leq k! = [G : H]!\). In fact, it is known in group theory that the quotient group \( G/H_G \) can be embedded into the symmetric group \( S_k \) on \( \{1, \ldots, k\} \).

If \( G = S_k \) and \( H = \{ \sigma \in G : \sigma(1) = 1 \} \) (the stabilizer of 1), then \( \{H, H(12), \ldots, H(1k)\} \) is a partition of \( G \), also \( H_G = \{e\} \) because \( G_i = (1i)^{-1} H(1i) \) is the stabilizer of \( i \); therefore \([G : \bigcap_{s=1}^k G_i] = |G/H_G| = k!\) as noted by Tomkinson in 1987. So the upper bound in the Fundamental Theorem is best possible.

2. Mycielski’s function and a group-theoretic function introduced by Korec and Sun

**Definition of Mycielski’s Function.** The Mycielski function \( f : \mathbb{Z}^+ \rightarrow \mathbb{N} = \{0, 1, \ldots\} \) is given by

\[
f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = \sum_{t=1}^{r} \alpha_t(p_t - 1),
\]

where \( p_1, \ldots, p_r \) are distinct primes, and \( \alpha_1, \ldots, \alpha_r \in \mathbb{N} \).

Note that \( f(1) = 0, f(p) = p - 1 \) for any prime \( p \), and \( f(mn) = f(m) + f(n) \) for all \( m, n \in \mathbb{Z}^+ \).

If \( n \in \mathbb{Z}^+ \) has the primary factorization \( \prod_{t=1}^{r} p_t^{\alpha_t} \), then

\[
1 + f(n) = 1 + \sum_{t=1}^{r} \alpha_t(p_t - 1) \leq \prod_{t=1}^{r} (1 + \alpha_t(p_t - 1)) \leq \prod_{t=1}^{r} (1 + (p_t - 1))^{\alpha_t} = n
\]

and

\[
n = \prod_{t=1}^{r} p_t^{\alpha_t} \leq \prod_{t=1}^{r} (2^{p_t-1})^{\alpha_t} = 2^{f(n)}.
\]
So
\[ n - 1 \geq f(n) \geq \log_2 n \quad \text{for all } n = 1, 2, 3, \ldots . \]

Let \( G \) be a group. A subgroup \( H \) of \( G \) is said to be subnormal if there is a chain \( H_0 = H \subseteq H_1 \subseteq \cdots \subseteq H_n = G \) of subgroups of \( G \) such that \( H_i \) is normal in \( H_{i+1} \) for all \( 0 \leq i < n \).

**Definition of a Group-theoretic Function** (I. Korec and Z. W. Sun).

Let \( H \) be a subnormal subgroup of a group \( G \) with finite index, and
\[
H_0 = H \subset H_1 \subset \cdots \subset H_n = G
\]
be a composition series from \( H \) to \( G \) (i.e. \( H_i \) is maximal normal in \( H_{i+1} \) for each \( 0 \leq i < n \)). If the length \( n \) is zero (i.e. \( H = G \)), then we set \( d(G, H) = 0 \), otherwise we put
\[
d(G, H) = \sum_{i=0}^{n-1} ([H_{i+1} : H_i] - 1).
\]

When \( H \) is a normal subgroup of a group with finite index, \( d(G, H) \) was first introduced by I. Korec [Fund. Math. 1974]. The current general notion is due to Z. W. Sun [Fund. Math. 1990], and he viewed \( d(G, H) \) as a ‘distance’.

Let \( H \) be a subnormal subgroup of a group \( G \) with \([G : H] < \infty \). By the Jordan–Hölder theorem in group theory, the definition \( d(G, H) \) does not depend on the choice of the composition series from \( H \) to \( G \). Clearly \( d(G, H) = 0 \) if and only if \( H = G \). If \( K \) is a subnormal subgroup of \( H \) with \([H : K] < \infty \), then
\[
d(G, H) + d(H, K) = d(G, K).
\]

$$[G : H] - 1 \geq d(G, H) \geq f([G : H]) \geq \log_2[G : H].$$

Moreover, $d(G, H) = f([G : H])$ if and only if $G/H_G$ is solvable.

3. Inequalities of Mycielski’s Type

In a regular cover $\{a_s G_s\}_{s=1}^k$ of a group $G$ by left cosets, if $\bigcap_{s=1}^k G_s$ equals a given subgroup $H$ of $G$ with finite index, what is the lower bound of $k$? (Of course, $k! \geq [G : H]$ by the Fundamental Theorem.)

**Mycielski’s Conjecture** [Fund. Math. 1966]. Let $G$ be an abelian group and $G_1, \cdots, G_k$ be subgroups of $G$ (with finite indices). If $A = \{a_s G_s\}_{s=1}^k$ forms an exact cover of $G$ for some $a_1, \ldots, a_k \in G$, then

$$k \geq 1 + f([G : G_s]) \quad \text{for every } s = 1, \cdots, k.$$

Š. Znám [Colloq. Math. 1966]: Mycielski’s conjecture holds for the additive group $\mathbb{Z}$ of integers. Equivalently, if $A = \{a_s + n_s \mathbb{Z}\}_{s=1}^k$ forms a disjoint cover of $\mathbb{Z}$ by residue classes then $k \geq 1 + f(n_t)$ for each $t = 1, \ldots, k$.

Znám [Colloq. Math. 1969]: If $A = \{a_s + n_s \mathbb{Z}\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ and the residue class $a_t + n_t \mathbb{Z}$ is disjoint with all the remaining residue classes, then we have $k \geq 1 + f(n_t)$. 
Znám [Acta Arith. 1975]: If $A = \{a_s + n_s \mathbb{Z}\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ with $a_t(n_t)$ irredundant, then there are $1 + f(n_t)$ integers no two of which belong to the same member of $A$.

**Znám’s Conjecture** [Coll. Math. Soc. János Bolyai, 1968; Acta Arith. 1975]. If $A = \{a_s + n_s \mathbb{Z}\}_{s=1}^k$ is a disjoint cover or just a minimal cover of $\mathbb{Z}$ by residue classes, then

$$k \geq 1 + f(N_A) \quad \text{and hence} \quad N_A \leq 2^{k-1} \leq k!.$$

(Recall that $N_A = [n_1, \ldots, n_k] = [\mathbb{Z} : \bigcap_{s=1}^k n_s \mathbb{Z}]$.)

I. Korec [Fund. Math. 1974]: Let $\{a_s G_s\}_{s=1}^k$ be a partition of a group $G$ into finitely many left cosets of normal subgroups $G_1, \ldots, G_k$. Then $[G : \bigcap_{s=1}^k G_s] < \infty$ and $k \geq 1 + f([G : \bigcap_{s=1}^k G_s])$.

**Sun’s Further Extension** [Fund. Math., 134(1990); European J. Combin. 22(2001)]. Let $G$ be a group and $A = \{a_s G_s\}_{s=1}^k$ be an exact $m$-cover of $G$ by left cosets with all the $G_s$ subnormal in $G$. Then

$$k \geq m + d\left(G_1 \bigcap_{s=1}^k G_s\right) \quad \text{(*)}$$

where the lower bound can be attained. Moreover, for any subgroup $K$ of $G$ not contained in all the $G_s$ we have

$$|\{1 \leq s \leq k : K \nsubseteq G_s\}| \geq 1 + d\left(K, K \cap \bigcap_{s=1}^k G_s\right).$$

A key step in Sun’s proof is to show that under the condition, for any maximal normal subgroup $H$ of $G$, either any left coset of $H$ contains one
of the $k$ members in $\mathcal{A}$ or none of the left cosets of $H$ contains a member in $\mathcal{A}$.

The lower bound in (*) can be attained as shown by the following example.

**Example 3.1.** Let $H$ be a subnormal subgroup of a group $G$ with finite index, and let $H = H_0 \subset H_1 \subset \cdots \subset H_n = G$ be a composition series from $H$ to $G$. Write

$$H_{i+1} \setminus H_i = \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i$$ for $i = 0, 1, \ldots, n - 1$.

Z. W. Sun [Fund. Math. 134(1990)] observed that the following $1+d(G,H)$ cosets

$$H_0, b_j^{(i)} H_i \quad (0 \leq i < n, \ 1 \leq j < [H_{i+1}:H_i])$$

form a partition of $G$ (in fact, $G = H_0 \cup (H_1 \setminus H_0) \cup \cdots \cup (H_n \setminus H_{n-1})$).

These cosets, together with $m - 1$ copies of $G$, form an exact $m$-cover of $G$ with the number $k$ of cosets being $m + d(G,H)$ and the intersection of the $k$ subnormal subgroups being $H$.

In view of Example 1.2 and Sun’s above result on exact $m$-covers of groups by cosets of subnormal subgroups, we have the following interesting application.

**Inequalities on Cores of Subnormal Subgroups.** Let $H$ be a subnormal subgroup of a group $G$ with $[G:H] < \infty$.

(i) [Z. W. Sun, Fund. Math. 134(1990)] We have

$$[G:H] - 1 \geq d(G,H_G) \geq f(|G/H_G|)$$
and hence \( |G/H_G| \leq 2^{[G:H]-1} \).

(ii) [Z. W. Sun, European J. Combin. 22(2001)] \( H \) is normal in \( G \) if and only if

\[
|N_G(H)/H| + d(H, H_G) \geq [G : H],
\]

where \( N_G(H) = \{ g \in G : gH = Hg \} \) is the normalizer of \( H \) in \( G \).

M. A. Berger, A. Felzenbaum and A. S. Fraenkel [Coll. Math. 1988]:
If \( \{a_sG_s\}_{s=1}^k \) is a disjoint cover of a finite solvable group \( G \) by left cosets, then \( k \geq 1 + f([G : G_t]) \) for \( t = 1, \ldots, k \).

Z. W. Sun [European J. Combin. 2001]: Let \( \{a_sG_s\}_{s=1}^k \) be an exact \( m \)-cover of a group \( G \). For any \( 1 \leq t \leq k \), whenever \( G/(G_t)_G \) is solvable we have \( k \geq m + f([G : G_t]) \) and hence \([G : G_t] \leq 2^{k-m}\).

**A Conjecture of Sun.** Let \( \{a_sG_s\}_{s=1}^k \) be an exact \( m \)-cover of a group \( G \) with all the \( G_s \) \( G \)-solvable. Then \( k \geq m + f(N) \) where \( N \) is the least common multiple of the indices \([G : G_1], \ldots, [G : G_k]\).

R. J. Simpson [Acta Arith. 1985]: Let \( A = \{a_s + n_sZ\}_{s=1}^k \) be a minimal cover of \( Z \) with \( a_s \in Z \) and \( n_s \in Z^+ \). Then for any divisor \( d \) of \( N_A \) with \( 0 < d < N_A \) we have

\[
|\{1 \leq s \leq k : n_s \nmid d\}| \geq 1 + f(N_A/d).
\]

Letting \( d = 1 \) we then obtain \( k \geq 1 + f(N_A) \).

M. A. Berger, A. Felzenbaum and A. S. Fraenkel [Coll. Math. 1988]:
Let \( \{a_sG_s\}_{s=1}^k \) be a minimal cover of a group \( G \) of squarefree order. If all the \( G_s \) are normal in \( G \) then \( k \geq 1 + f([G : \bigcap_{s=1}^k G_s]) \).
A Result of Sun on Regular Covers [Internat. J. Math., in press, arXiv:math.GR/0501451]. If \( \mathcal{A} = \{a_s G_s\}_{s=1}^k \) is a regular cover of a group \( G \) by left cosets, and for any \( i, j = 1, \ldots, k \) either \( G_i \) and \( G_j \) are subnormal in \( G \) with \( [G : G_i] \) relatively prime to \( [G_i : G_i \cap G_j] \), or \( G_i \) and \( G_j \) are normal in \( G \) with \( G/(G_i \cap G_j) \) cyclic, then we have the inequality

\[
k \geq m(\mathcal{A}) + d\left(G, \bigcap_{s=1}^k G_s\right).
\]

The conditions of this result are essentially indispensable as shown by the following example:

Example 3.2. Let \( G \) be the group \( C_p \times C_p \) where \( p \) is a prime and \( C_p \) is the cyclic group of order \( p \). Then any element \( a \neq e \) of \( G \) has order \( p \). Let \( G_1, \ldots, G_k \) be all the distinct subgroups of \( G \) with order \( p \). If \( 1 \leq s < t \leq k \) then \( G_s \cap G_t = \{e\} \). Clearly \( \{G_s\}_{s=1}^k \) forms a minimal cover of \( G \) by normal subgroups whose intersection is \( H = \{e\} \). Since \( 1 + k(p-1) = |\bigcup_{s=1}^k G_s| = |G| = p^2 \), we have

\[
k = p + 1 \leq 2p - 1 = 1 + f([G : H]) = 1 + d\left(G, \bigcap_{s=1}^k G_s\right).
\]

When \( p > 2 \) the last inequality becomes strict. We remark that both \( G/G_s \) and \( G_s/(G_s \cap G_t) \) \( (t \neq s) \) have order \( p \).

Let \( G \) be a finite multiplicative abelian group with identity \( e \). If \( aH \) is a left coset in \( G \) with \( e \not\in aH \) (i.e. \( a \not\in H \)), then \( aH \) is called a proper coset in \( G \). During their study of zero-sum problems on abelian groups, W. D. Gao and A. Geroldinger [European J. Combin. 24(2003)] defined
$s(G)$ to be the smallest number of proper cosets that can cover $G \setminus \{e\}$. For left cosets $a_1G_1, \ldots, a_kG_k$ in group $G$, the system $\{a_sG_s\}_{s=1}^k$ is a cover of $G \setminus \{e\}$ if and only if $\{a_sG_s\}_{s=0}^k$ forms a cover of $G$ with the coset $a_0G_0$ irredundant where $a_0 = e$ and $G_0 = \{e\}$ (and hence $\bigcap_{s=0}^k G_s = \{e\}$). Thus, by the previous results on covers of groups, we have $s(G) \leq f(|G|)$, and $s(G) = f(|G|)$ if $G$ is cyclic or of squarefree order.

By using algebraic number theory and characters of abelian groups, G. Lettl and Z. W. Sun [arXiv:math.GR/0411144] obtained in 2004 the following result for minimal $m$-covers of a general abelian group.

**A Result of Lettl and Sun on Covers of Abelian Groups.** Let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be a minimal $m$-cover of an abelian group $G$ by left cosets.

Then $k \geq m + f([G : G_t])$ for any $t = 1, \ldots, k$.

By Example 3.2, even for $G = C_p \times C_p$ we cannot replace $f([G : G_t])$ in the above result by $f([G : \bigcap_{s=1}^k G_s])$. But the speaker believes that $f([G : G_t])$ in the lower bound of $k$ can be replaced by $f(N)$ where $N$ is the least common multiple of $[G : G_1], \ldots, [G : G_k]$.

**Another Conjecture of Sun.** Let $\mathcal{A} = \{a_sG_s\}_{s=1}^k$ be an $m$-cover of a group $G$ by left cosets. For any $s = 1, \ldots, k$ with $a_sG_s$ irredundant, if $G_s$ is subnormal in $G$ then $k \geq m + d(G, G_s)$; if $G/(G_s)G$ is solvable then $k \geq m + f([G : G_s])$.

4. Covering a group by subgroups

Any finite non-cyclic group $G$ can be covered by finitely many proper
subgroups because $G = \bigcup_{x \in G} \langle x \rangle$, but no field can be covered by finitely many proper subfields as shown by A. Bialynicki-Birula, J. Browkin and A. Schinzel [Colloq. Math. 7(1959)].

**Example 4.1.** Let $G$ be a group with $G/Z(G)$ finite, where $Z(G)$ is the center of $G$. For $x, y \in G$, if $xy \neq yx$ then $(x^{-1}y)x \neq x(x^{-1}y)$ and hence $xZ(G) \neq yZ(G)$. Let $X = \{x_1, \ldots, x_k\}$ be a maximal set of pairwise non-commuting elements of $G$. Then $k = |X| \leq |G/Z(G)|$, and $\{C_G(x_i)\}_{i=1}^k$ forms a minimal cover of $G$ by centralizers with $\bigcap_{i=1}^k C_G(x_i) = Z(G)$ (recall that $C_G(x) = \{g \in G : gx = xg\}$), and $|G/Z(G)| \leq c^k$ for some absolute constant $c > 0$ [L. Pyber, J. London Math. Soc. 35(1987)]. D. R. Mason [Math. Proc. Cambridge Philos. Soc. 83(1978)] proved that $|G| \geq 2k - 2$, which was conjectured by Erdős and E. G. Straus in 1975.

Recall that a group $G$ is said to be **perfect** if it coincides with its derived group $G' = \langle x^{-1}y^{-1}xy : x, y \in G \rangle$.

M. A. Brodie, R. F. Chamberlain and L.-C. Kappe [Proc. Amer. Math. Soc. 104(1998)]: If $\{G_{s}\}_{s=1}^k$ is a minimal cover of a group $G$ by finitely many normal subgroups, then $G/\bigcap_{s=1}^k G_s$ is solvable and all perfect normal subgroups of $G$ are contained in each of $G_1, \ldots, G_k$.

Z. W. Sun [Internat. J. Math., arxiv:math.GR/0501451] **Suppose that** $\{G_{s}\}_{s=1}^k$ **is a minimal $m$-cover of a group $G$ by subnormal subgroups. Then there is a composition series from $\bigcap_{s=1}^k G_s$ to $G$ whose factors are of prime orders (equivalently, $G/\bigcap_{s=1}^k G_s)_G$ is solvable), and all the $G_s$ contain every perfect subgroup of $G$.**
Concerning covers of a group by subnormal subgroups, the speaker and his student Song Guo made the following conjecture in 2004.

**A Conjecture of S. Guo and Z. W. Sun.** Let \( \{G_s\}_{s=1}^k \) be a minimal \( m \)-cover of a group \( G \) by finitely many subnormal subgroups. Assume that \( [G : \bigcap_{i=1}^k G_s] = \prod_{t=1}^r p_t^{\alpha_t} \) where \( p_1, \ldots, p_r \) are distinct primes and \( \alpha_1, \ldots, \alpha_r \) are positive integers. Then

\[
k > m + \sum_{t=1}^r (\alpha_t - 1)(p_t - 1).
\]

5. Unions of Cosets and Disjoint Cosets

In number theory, a theorem of C. A. Rogers asserts that if \( a_s \in \mathbb{Z} \) and \( n_s \in \mathbb{Z}^+ \) for \( s = 1, \ldots, k \) then

\[
\left| \left\{ 0 \leq x < N : x \in \bigcup_{s=1}^k a_s + n_s \mathbb{Z} \right\} \right| \geq \left| \left\{ 0 \leq x < N : x \in \bigcup_{s=1}^k n_s \mathbb{Z} \right\} \right|,
\]

where \( N = [n_1, \ldots, n_k] \) is the least common multiple of \( n_1, \ldots, n_k \).

Inspired by this, the speaker conjectured in 1990 that for any finitely many left cosets \( a_1 G_1, \ldots, a_k G_k \) in a finite group \( G \) we always have the inequality \( |\bigcup_{s=1}^k a_s G_s| \geq |\bigcup_{s=1}^k G_s| \). When \( G \) is a finite cyclic group this reduces to Rogers’ result. But soon Tomkinson pointed out that the conjecture is not true for the Klein group \( C_2 \times C_2 \). In fact, if \( G = \{1, -1\} \times \{1, -1\} \),

\[
G_1 = \{ e = (1,1) \}, \ G_2 = \{ e, (1, -1) \}, \ G_3 = \{ e, (-1, 1) \}, \ G_4 = \{ e, (-1, -1) \},
\]
then

\[ G_1 \cup G_2 \cup G_3 \cup (1, -1)G_4 = \{ e, (1, -1), (-1, 1) \} \subset \bigcup_{s=1}^{4} G_s = G. \]

For a subgroup \( H \) of a finite group \( G \), if \(|H|\) is relatively prime to \([G : H]\) (i.e., \( \gcd(|H|, [G : H]) = 1 \)) then \( H \) is called a Hall subgroup of \( G \). A Sylow \( p \)-subgroup of a finite group \( G \) is just a Hall \( p \)-subgroup of \( G \).

**A Result of Sun on Unions of Cosets** [Internat. J. Math., in press]. Let \( G \) be a finite group and let \( G_1, \ldots, G_k \) be normal Hall subgroups of \( G \). Then, for any \( a_1, \ldots, a_k \in G \), we have

\[ \left| \bigcup_{s=1}^{k} a_s G_s \right| \geq \left| \bigcup_{s=1}^{k} G_s \right|. \]

The following conjecture seems very challenging.

**A Conjecture of Sun on Disjoint Cosets** [Internat. J. Math., in press]. Let \( a_1 G_1, \ldots, a_k G_k \) \((k > 1)\) be finitely many pairwise disjoint left cosets in a group \( G \) with \([G : G_s] < \infty\) for all \( s = 1, \ldots, k \). Then we have \( \gcd([G : G_i], [G : G_j]) \geq k \) for some \( 1 \leq i < j \leq k \).

This conjecture is true when \( G \) is a \( p \)-group with \( p \) a prime. In fact, under the condition of the above conjecture, clearly

\[ [G : \bigcap_{s=1}^{k} G_s] \geq \left[ \bigcup_{i=1}^{k} a_i G_i : \bigcap_{s=1}^{k} G_s \right] = \sum_{i=1}^{k} [G_i : \bigcap_{s=1}^{k} G_s] \]

and hence \( \sum_{i=1}^{k} [G : G_i]^{-1} \leq 1 \). Suppose that \([G : G_1] \leq \cdots \leq [G : G_k]\).

Since \( \sum_{i=1}^{k-1} [G : G_i]^{-1} < 1 \), there is an \( i \in [1, k-1] \) such that \([G : G_i] \leq \cdots \leq [G : G_k] \)
$k - 1$ and hence $[G : G_k] \geq [G : G_i] \geq k$. If $[G : G_k]$ is divisible by all those $[G : G_1], \ldots, [G : G_{k-1}]$ (this happens if $G$ is a $p$-group with $p$ a prime), then $\gcd([G : G_i], [G : G_k]) = [G : G_i] \geq k$.

The conjecture is also true for $k = 2$. In fact, when $H$ and $K$ are two subgroups of a group $G$ with finite index, it is easy to see that

$$\gcd([G : H], [G : K]) = 1 \implies HK = G \iff xH \cap yK \neq \emptyset \text{ for all } x, y \in G.$$  

The conjecture for the infinite cyclic group $\mathbb{Z}$ has been proved to be true for $k < 5$ by the speaker [Chinese Ann. Math. Ser. A 13(1992)], and for $k \leq 20$ by K. O'Bryant [arXiv:math.NT/0604347] quite recently.

6. On extended Herzog-Schönheim Conjecture

Soon after his invention of the concept of cover of $\mathbb{Z}$, Erdős made the following conjecture: If $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ and $1 < n_1 < \cdots < n_k$, then $A$ cannot be a partition of $\mathbb{Z}$.

This conjecture of Erdős was soon confirmed independently by H. Davenport, L. Mirsky, D. Newman and R. Rado in the 1950s.

The Davenport-Mirsky-Newman-Rado Result. Let $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ be an exact cover of $\mathbb{Z}$ with $1 \leq n_1 \leq \cdots \leq n_{k-1} \leq n_k$. Then we must have $n_{k-1} = n_k$.

Proof. Without loss of generality we assume that $0 \leq a_s < n_s$ for all $s \in [1, k]$. For $|z| < 1$ we have

$$\sum_{s=1}^{k} \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{s=1}^{k} \sum_{q=0}^{\infty} z^{a_s + q n_s} = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.$$
If \( n_{k-1} < n_k \) then

\[
\infty = \lim_{|z| < 1} \frac{z^{a_k}}{1 - z^{n_k}} = \lim_{|z| < 1} \left( \frac{1}{1 - z} - \sum_{s=1}^{k-1} \frac{z^{a_s}}{1 - z^{n_s}} \right) < \infty,
\]
a contradiction! □

Recall that \( \{1(2), 2(2^2), \ldots, 2^{k-2}(2^{k-1}), 0(2^{k-1})\} \) is a disjoint cover of \( \mathbb{Z} \) by \( k \) residue classes whose first \( k - 1 \) moduli are distinct.


**Herzog-Schönheim Conjecture.** Let \( A = \{a_s G_s\}_{s=1}^k \) be a partition of a group \( G \) into \( k > 1 \) left cosets of subgroups \( G_1, \ldots, G_k \). Then the finite indices \( [G : G_1], \ldots, [G : G_k] \) cannot be distinct.

M. A. Berger, A. Felzenbaum and A. S. Fraenkel [Canad. Math. Bull. 29(1986); Fund. Math. 128(1987)] proved the Herzog-Schönheim conjecture for finite nilpotent groups and pyramidal groups. (A finite group \( G \) is said to be *pyramidal* if it contains a chain \( \{e\} = H_0 \subset H_1 \subset \cdots \subset H_n = G \) of subgroups such that \( [H_1 : H_0] \geq \cdots \geq [H_n : H_{n-1}] \) are primes in non-ascending order, such a group must be solvable.)

A finite system \( A = \{a_s G_s\}_{s=1}^k \) of left cosets in a group \( G \) is called a *uniform cover* of \( G \) if it covers all elements of \( G \) the same number of times. Note that \( A \) is a uniform cover of \( G \) if and only if it is an exact \( m \)-cover of \( G \) for some \( m = 1, 2, \ldots \).

Here is recent progress [Z. W. Sun, J. Algebra 273(2004)] on the extended Herzog-Schönheim conjecture for uniform covers of groups.
Sun’s Result on Herzog-Schönheim Conjecture for Uniform Covers. Let $\mathcal{A} = \{a_s G_s\}_{s=1}^k$ be a uniform cover of a group $G$ with $G_1, \ldots, G_k$ not all equal to $G$ and

$$n_1 = [G : G_1] \leq \cdots \leq n_k = [G : G_k].$$

Suppose that all the $G_i$ are subnormal in $G$, or $G/H$ is a solvable group having a normal Sylow $p$-subgroup where $H = (\bigcap_{s=1}^k G_s)_G$, and $p$ is the largest prime divisor of $|G/H|$. Then the indices $n_1, \ldots, n_k$ cannot be pairwise distinct. Moreover, if $|\{1 \leq i \leq k : n_i = n\}| \leq M$ for all $n \in \mathbb{Z}^+$, then we have

$$\log n_1 \leq \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log \log M),$$

where the logarithm has the natural base $e = 2.718...$, $\gamma = 0.577...$ is the Euler constant and the $O$-constant is absolute.

The proof of this result is long and sophisticated; it involves combinatorics and group theory, as well as analytic number theory. One of the basic lemma used in the proof is the following arithmetical property of indices.

**A Lemma on Divisibility of Indices** [Z. W. Sun, European J. Combin. 22(2001)]. Let $G_1, \ldots, G_k$ be subnormal subgroups of a group $G$ with finite index. Then

$$\left[ G : \bigcap_{s=1}^k G_s \right] \bigg| \prod_{s=1}^k [G : G_s]$$

and hence

$$P\left( \left[ G : \bigcap_{s=1}^k G_s \right] \right) = \bigcup_{s=1}^k P([G : G_s]),$$
where $P(n)$ denotes the set of prime divisors of $n$.

If $G_1, \ldots, G_k$ are subgroups of a group $G$ with finite index, then $[G : \bigcap_{s=1}^k G_s] \leq \prod_{s=1}^k [G : G_s] < \infty$ by Poincaré’s theorem. The above lemma can be viewed as an important number-theoretic property of subnormality, it is the main reason why covers involving subnormal subgroups are better behaved than general covers.

In view of Example 1.2 and the above lemma, if $H$ is a subnormal subgroup of a group $G$ with $[G : H] < \infty$ then

$$P(|G/H_G|) = P([G : H]).$$

It should be mentioned that this result and the divisibility lemma are quite useful but could not been found in group theory references before the speaker worked them out.