On Covers of Groups by Cosets

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Jan. 12, 2018

Abstract

If a group G is the union of finitely many left cosets $a_1 G_1, \ldots,$ $a_k G_k$ of subgroups $G_1, \ldots, G_k$, then the system $\{a_s G_s\}_{i=1}^k$ is said to be a cover of G. We will give a survey of problems and results on extremal problems concerning covers of groups, and introduce progress on the famous Herzog-Schönheim conjecture which states that if $\{a_s G_s\}_{s=1}^k$ $(k > 1)$ is a partition of a group G into finitely many left cosets then the (finite) indices $[G : G_1], \ldots, [G : G_k]$ cannot be distinct. We will also mention some new challenging conjectures in the field.

Part I. Basic Results on Covers of Groups

Disjoint covers of a group by left or right cosets

Let H be a subgroup of a group G with $[G : H] = k < \infty$. Then we can partition G into k left cosets $g_1 H, \ldots, g_k H$, and $\{g_i H\}_{i=1}^k$ forms a disjoint cover of G by left cosets.

Let $\{Ha_i\}_{i=1}^k$ be a right coset decomposition of G. Then $\{a_i G_i\}_{i=1}^k$ is a disjoint cover of G where $G_i = a_i^{-1} H a_i$. Observe that

$$\bigcap_{i=1}^k G_i = \bigcap_{i=1}^k \bigcap_{h \in H} a_i^{-1} h^{-1} H h a_i = \bigcap_{g \in G} g^{-1} H g$$

is the normal core H_G of H in G which is the largest normal subgroup of G contained in H.

In group theory, it is known that G/H_G can be embedded into the symmetric group $S_{[G:H]} = S_k$ and thus

$$\left[G : \bigcap_{i=1}^k G_i \right] = |G/H_G| \leqslant k!.$$

A basic theorem on covers of groups

An Example of M. J. Tomkinson. Let $k > 1$ be a positive integer, and let G be the symmetric group S_k and H be the stabilizer of 1. Then $G_i = (1i)^{-1}H(1i)$ is the stabilizer of i for each $i = 1, \ldots, k$. Clearly,

$$\{G_1, (12)G_2, \ldots, (1k)G_k\} = \{H, H(12), \ldots, H(1k)\}$$

forms a disjoint cover of G with $\bigcap_{i=1}^k G_i = H_G = \{e\}$. Note that $[G : \bigcap_{i=1}^k G_i] = |G| = k!$.

A Basic Theorem on Covers of Groups. Let $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ be a finite system of left cosets in a group G where $G_1, \ldots, G_k$ are subgroups of G. Suppose that $\mathcal{A}$ forms a minimal cover of G (i.e. $\mathcal{A}$ covers all the elements of G but none of its proper systems does).

(i) (B. H. Neumann, 1954) There is a constant c_k depending only on k such that $[G : G_i] \leqslant c_k$ for all $i = 1, \ldots, k$.
(ii) (M. J. Tomkinson, 1987) We have $[G : \bigcap_{i=1}^k G_i] \leqslant k!$, where the upper bound $k!$ is best possible.

Tomkinson's proof of the second part

We show by induction that

$$\left[\bigcap_{i\in I} G_i : \bigcap_{i=1}^{k} G_i\right] \leqslant (k-|I|)! \qquad (*_I)$$

for all $I \subseteq \{1,\dots,k\}$, where $\bigcap_{i\in\emptyset} G_i$ is regarded as G.

Clearly $(*_I)$ holds for $I = \{1,\dots,k\}$. Now let $I \subset \{1,\dots,k\}$ and assume $(*_J)$ for all $J \subseteq \{1,\dots,k\}$ with $|J| > |I|$. Since $\{a_i G_i\}_{i\in I}$ is not a cover of G, there is an $a \in G$ not covered by $\{a_i G_i\}_{i\in I}$. Clearly $a(\bigcap_{i\in I} G_i)$ is disjoint from the union $\bigcup_{i\in I} a_i G_i$ and hence contained in $\bigcup_{j\notin I} a_j G_j$. Thus

$$a\Big(\bigcap_{i\in I} G_i\Big) = \bigcup_{\substack{j\notin I \\ a_jG_j\cap a(\bigcap_{i\in I} G_i)\neq\emptyset}} \Big(a_j G_j \cap a\Big(\bigcap_{i\in I} G_i\Big)\Big),$$

$$\left[\bigcap_{i\in I} G_i : H\right] \leqslant \sum_{j\notin I}\left[G_j \cap \bigcap_{i\in I} G_i : H\right] \leqslant \sum_{j\notin I}(k-(|I|+1))! = (k-|I|)!$$

where $H = \bigcap_{i=1}^{k} G_i$. This concludes the induction proof.

m-covers and exactly m-covers

A right coset Ha in a group G is also a left coset $a(a^{-1}Ha)$. So we only consider left cosets.

Let $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, and let $A = \{a_i G_i\}_{i=1}^k$ be a finite system of left cosets in a group G. If each element of G is covered by A at least (resp., exactly) m times, then we call A an *m-cover* (resp., *exact m-cover*) of G. If A is an m-cover of G but none of its proper subsystems does, then A is said to be a *minimal m-cover* of G.

The Neumann-Tomkinson theorem can be extended to minimal m-covers of groups (cf. Corollary 1 of Z. W. Sun [Fund. Math. 134(1990)]); it also has applications in Galois theory, groups rings, Banach spaces, projective geometry and Riemann surfaces as pointed out by T. Soundararajan and K. Venkatachaliengar [Acta Math. Vietnam 19(1994)].

Part II. Extremal Problems for m-Covers

Mycielski's function and an example of Znám

The Mycielski function $f : \mathbb{Z}^+ \to \mathbb{N} = \{0, 1, 2, \ldots\}$ is given by $f(1) = 0$ and $f(mn) = f(m) + f(n)$ for $m, n \in \mathbb{Z}^+$.

An Example of Š. Znám. Let $n > 1$ be an integer with the factorization $\prod_{t=1}^{r} p_t^{\alpha_t}$, where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$. Then $0 (\bmod n)$ and the following $f(n) = \sum_{s=1}^{r} \alpha_s(p_s - 1)$ residue classes

$$jp_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha-1} (\bmod p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha})$$
$$(\alpha = 1, \ldots, \alpha_s;\ j = 1, \ldots, p_s - 1;\ s = 1, \ldots, r)$$

form a disjoint cover of $\mathbb{Z}$ whose moduli have the least common multiple n.

An extremal problem for exact m-covers of groups

Let $A = \{a_i G_i\}_{i=1}^k$ be an exact m-cover of a group G with $\bigcap_{i=1}^k G_i = H$. By the Neumann-Tomkinson theorem, $[G : H] \leqslant k!$. How to provide a sharp lower bound of k in terms of G and H?

An Example of Z.-W. Sun [Fund. Math. 134(1990)]. Let H be a subnormal subgroup of a group G with finite index. Let

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

be a composition series from H to G. For $i = 0, \ldots, n-1$, write

$$H_{i+1} \setminus H_i = \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i.$$

Then the following $d(G, H) = \sum_{i=0}^{n-1}([H_{i+1} : H_i] - 1)$ left cosets

$$b_j^{(i)} H_i \ (0 \leqslant i < n;\ 1 \leqslant j < [H_{i+1} : H_i]),$$

together with H and $m-1$ copies of G, form an exact m-cover of G by $m + d(G, H)$ left cosets of subgroups whose intersection is H. (In the case $H = G$ we define $d(G, H) = 0$.)

Relation between the Mycielski Function f and $d(G, H)$

Relation between the Mycielski Function f **and** $d(G, H)$ (Z.-W. Sun, Fund. Math. 1990; European J. Combin. 2001). Let H be any subnormal subgroup of G with finite index. Then

$$d(G, H) \geqslant f([G : H]) \geqslant \log_2[G : H].$$

Also, $d(G, H) = f([G : H])$ if and only if G/H_G is solvable.

Mycielski's conjecture and further extensions

Mycielski's Conjecture (J. Mycielski, 1966). If $\{a_iG_i\}_{i=1}^k$ is a disjoint cover of an abelian group G, then $k \geqslant 1 + f([G : G_i])$ for all $i = 1, \ldots, k$.

Related Results on Exact *m*-covers. Let $A = \{a_iG_i\}_{i=1}^k$ be an exact m-cover of a group G with $\bigcap_{i=1}^k G_i = H$.
(i) (I. Korec [Fund. Math., 1974]) If $m = 1$ and $G_1, \ldots, G_k$ are normal in G, then $k \geqslant 1 + f([G : H])$.
(ii) (Z.-W. Sun [European J. Combin., 2001]) If $G_1, \ldots, G_k$ are subnormal in G, then $k \geqslant m + d(G, H)$, with the lower bound best possible.

The proof is by induction, on the basis of the following key lemma.

A Lemma (Z.-W. Sun [European J. Combin., 2001]). Let $\mathcal{A} = \{a_iG_i\}_{i=1}^k$ be an exact m-cover of a group G by left cosets of subnormal subgroups $G_1, \ldots, G_k$. For any maximal normal subgroup H of G, we have

$$\{C \in G/H : C \supseteq a_iG_i \text{ for some } i = 1, \ldots, k\} = \emptyset \text{ or } G/H.$$

On the core of a subnormal subgroup

Note that Korec's result is stronger than Mycielski's conjecture, and also Sun's result has the following consequence.

Corollary (Sun [Fund. Math., 1990]). Let H be a subnormal subgroup of a group G with $[G:H] < \infty$. Then

$$[G:H] \geqslant 1 + d(G, H_G) \geqslant 1 + f([G:H_G])$$

and hence

$$|G/H_G| \leqslant 2^{[G:H]-1}.$$

Proof. Let $\{Ha_i\}_{i=1}^k$ be a right coset decomposition of G where $k = [G:H]$. Then $\{a_iG_i\}_{i=1}^k$ is a disjoint cover of G where all the $G_i = a_i^{-1}Ha_i$ are subnormal in G and $\bigcap_{i=1}^k G_i = H_G$. So the desired result follows.

On minimal m-covers

Korec's and Sun's results on exact m-covers can be extended to minimal m-covers of $\mathbb{Z}$, see R. J. Simpson [Acta Arith., 1985] for the case $m = 1$ and Z. W. Sun [Internat. J. Math. 17(2006)] for general $m \geqslant 1$. However, they cannot be extended to minimal m-covers of abelian groups as illustrated by the following example.

Example (G. Lettl and Z.-W. Sun [Acta Arith., 2008]). Let G be the abelian group $C_p \times C_p$ where p is a prime and C_p is the cyclic group of order p. Then any element $a \neq e$ of G has order p. Let $G_1, \ldots, G_k$ be all the distinct subgroups of G with order p. If $1 \leqslant i < j \leqslant k$, then $G_i \cap G_j = \{e\}$. Thus $\{G_s\}_{s=1}^k$ forms a minimal cover of G with $\bigcap_{s=1}^k G_s = \{e\}$. As $1 + k(p-1) = |\bigcup_{s=1}^k G_s|$ $= |G| = p^2$, we have

$$k = p + 1 \geqslant 1 + f([G : G_s]) = 1 + f(p) = p.$$

However, if $p > 2$ then

$$k = p + 1 < 2p - 1 = 1 + f([G : \{e\}]) = 1 + d\left(G, \bigcap_{s=1}^{k} G_s\right).$$

On *m*-covers of abelian groups

Let m and n be positive integers. Is $m + f(n)$ the smallest positive integer k such that for any abelian group having a subgroup of index n there is a minimal m-cover of G by k cosets of subgroups one of which has index n?

Theorem (G. Lettl and Z.-W. Sun [Acta Arith. 131(2008)]). Let $A = \{a_s G_s\}_{s=1}^k$ be a minimal m-cover of an abelian group G by left cosets. Then

$$k \geqslant m + f([G : G_t]) \qquad \text{for any} \quad t = 1, \ldots, k.$$

This theorem implies the following conjecture of W. D. Gao and A. Geroldinger [European J. Combin. 2003].

Gao-Geroldinger Conjecture. Let G be a finite abelian group with identity e. If $G \setminus \{e\}$ is a union of k cosets $a_1 G_1, \ldots, a_k G_k$, then we have $k \geqslant f(|G|)$.

In fact, if we set $a_0 = e$ and $G_0 = \{e\}$ then $\{a_s G_s\}_{s=0}^k$ forms a cover of G with $a_0 G_0$ irredundant, hence $k + 1 \geqslant 1 + f([G : G_0])$ and thus $k \geqslant f(|G|)$.

A lemma

The Lettl-Sun result cannot be shown in the way that we prove Korec's or Sun's result because we don't have the corresponding lemma for minimal m-covers of abelian groups. Thus, new ideas are needed!

The proof of the Lettl-Sun result was obtained via characters of abelian groups and algebraic number theory; below is a key lemma used in the proof.

Lemma (G. Lettl and Z.-W. Sun [Acta Arith. 131(2008)]). Let $n > 1$ be an integer. Then $f(n)$ is the smallest positive integer k such that there are roots of unity $\zeta_1, \ldots, \zeta_k$ different from 1 for which $\prod_{s=1}^{k}(1-\zeta_s) \equiv 0 \pmod{n}$ in the ring $\bar{\mathbb{Z}}$ of algebraic integers.

On cyclotomic polynomials

The nth cyclotomic polynomial is given by

$$\Phi_n(x) = \prod_{\substack{m=1 \\ (m,n)=1}}^{n} (x - e^{2\pi i m/n}).$$

$$x^n - 1 = \prod_{m=1}^{n} \left(x - e^{2\pi i m/n}\right) = \prod_{d|n} \prod_{\substack{c=1 \\ (c,d)=1}}^{d} \left(x - e^{2\pi i c/d}\right) = \prod_{d|n} \Phi_d(x).$$

Applying the Möbius inversion we obtain

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)} \in \mathbb{Z}[x].$$

If $n > 1$, then $\sum_{d|n} \mu(n/d) = \sum_{d|n} \mu(d) = 0$, thus

$$\Phi_n(x) = \prod_{d|n} \left(\frac{x^d - 1}{x - 1}\right)^{\mu(n/d)} = \prod_{d|n} (1 + x + \cdots + x^{d-1})^{\mu(n/d)}$$

and hence $\Phi_n(1) = \prod_{d|n} d^{\mu(n/d)}$.

Using the Mangoldt function

Recall that the Mangoldt function is given by

$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \text{ for some prime } p \text{ and } a \in \mathbb{Z}^+, \\ 0 & \text{otherwise.} \end{cases}$$

If n has the primary factorization $\prod_{i=1}^r p_i^{\alpha_i}$ where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{N}$, then

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{r} \sum_{\beta_i=0}^{\alpha_i} \Lambda(p_i^{\beta_i}) = \sum_{i=1}^{r} \alpha_i \log p_i = \log \prod_{i=1}^{r} p_i^{\alpha_i} = \log n.$$

Applying the Möbius inversion formula we get

$$\sum_{d|n} \mu\left(\frac{n}{d}\right) \log d = \Lambda(n).$$

Therefore

$$\log \Phi_n(1) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log d = \begin{cases} \log p & \text{if } n \text{ is a power of some prime } p, \\ 0 & \text{otherwise.} \end{cases}$$

Find roots $\zeta_1, \ldots, \zeta_{f(n)} \neq 1$ of unity with $n \mid \prod_{s=1}^{f(n)}(1-\zeta_s)$

So we have

$$\prod_{\substack{m=1 \\ (m,n)=1}}^{n} (1-e^{2\pi i m/n}) = \Phi_n(1) = \begin{cases} p & \text{if } n \text{ is a power of some prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

In particular,

$$\prod_{m=1}^{p-1}(1-e^{2\pi i m/p}) = p \quad \text{for any prime } p.$$

Note that

$$n = \prod_{p \mid n} p^{\mathrm{ord}_p(n)} = \prod_{p \mid n} \prod_{m=1}^{p-1} (1-e^{2\pi i m/p})^{\mathrm{ord}_p(n)}.$$

So there exist $f(n) = \sum_{p \mid n} \mathrm{ord}_p(n)(p-1)$ roots of unity $\zeta_1, \ldots, \zeta_{f(n)} \neq 1$ such that n divides $(1-\zeta_1)\cdots(1-\zeta_{f(n)})$ in the ring $\bar{\mathbb{Z}}$ of all algebraic integers.

Proof of the lemma of Lettl and Sun

Now suppose that $\prod_{s=1}^{k}(1-\zeta_s)\equiv 0 \pmod{n}$, where ζ_s is a primitive n_sth root of unity with $n_s>1$. Recall that

$$\prod_{\substack{r=1\\(r,n_s)=1}}^{n_s}(1-\zeta_s^r)=\prod_{\substack{m=1\\(m,n_s)=1}}^{n_s}(1-e^{2\pi im/n_s})$$

$$=\begin{cases} p & \text{if } n_s \text{ is a power of some prime } p,\\ 1 & \text{otherwise.}\end{cases}$$

Let N be the least common multiple of $n_1,\ldots,n_k$. Then

$$n^{\varphi(N)}\ \Big|\ \prod_{s=1}^{k}\left((1-\zeta_s)^{\varphi(n_s)}\right)^{\varphi(N)/\varphi(n_s)}\ \Big|\ \prod_{\substack{s=1\\ n_s \text{ is a prime power}}}^{k} p(n_s)^{\varphi(N)/\varphi(n_s)},$$

where $p(n_s)$ is the least prime divisor of n_s. So, for any prime p,

$$\mathrm{ord}_p(n)\varphi(N)\leqslant\sum_{\substack{s=1\\ n_s \text{ is a power of } p}}^{k}\frac{\varphi(N)}{\varphi(n_s)}$$

Proof of the lemma of Lettl and Sun

and hence

$$|\{1 \leqslant s \leqslant k : \ n_s \text{ is a power of } p\}| \geqslant \mathrm{ord}_p(n)(p-1).$$

It follows that

$$\begin{aligned} k &\geqslant \sum_{p|n} |\{1 \leqslant s \leqslant k : \ n_s \text{ is a power of } p\}| \\ &\geqslant \sum_{p|n} \mathrm{ord}_p(n)(p-1) = f(n). \end{aligned}$$

This concludes the proof of the lemma.

Proof of the Lettl-Sun result

For a finite abelian group G, let $\widehat{G}$ denote the group of all complex-valued characters of G. One has $\widehat{G} \cong G$. For any subgroup H of G let $H^{\perp}$ denote the group of those characters $\chi \in \widehat{G}$ with $\ker(\chi) = \{x \in G : \chi(x) = 1\}$ containing H. Then there is a canonical isomorphism $H^{\perp} \cong \widehat{G/H}$ by putting $\chi(aH) = \chi(a)$ for any $a \in G$ and any $\chi \in H^{\perp}$. Furthermore, for each $a \in G \setminus H$ there exists some $\chi \in H^{\perp}$ with $\chi(a) \neq 1$.

Proof of the Lettl-Sun Result. As $H = \bigcap_{s=1}^{k} G_s$ is of finite index in G. Instead of the minimal m-cover $A = \{a_s G_s\}_{s=1}^{k}$ of G, we may consider the minimal m-cover $\bar{A} = \{\bar{a}_s \bar{G}_s\}_{s=1}^{k}$ of the finite abelian group $\bar{G} = G/H$, where $\bar{a}_s = a_s H$ and $\bar{G}_s = G_s/H$ (hence $[\bar{G} : \bar{G}_s] = [G : G_s]$). Therefore, without any loss of generality, we can assume that G is finite.

Proof of the Lettl-Sun result (continued)

Fix $1 \leqslant t \leqslant k$. As $\{a_s G_s\}_{s \neq t}$ is not an m-cover of G, there is an $a \in a_t G_t$ such that it is covered by exactly m cosets in A and hence $J = \{1 \leqslant j \leqslant k : a \notin a_j G_j\}$ has cardinality $k - m$. For each $j \in J$ we may choose a $\chi_j \in G_j^\perp$ with $\zeta_j := \chi_j(a^{-1}a_j) \neq 1$. For any $x \in G \setminus G_t$ we have $ax \notin aG_t = a_t G_t$. Since A is an m-cover of G, there exists some $j \in J$ such that $ax \in a_j G_j$, and therefore $\chi_j(x) = \chi_j(a^{-1}a_j) = \zeta_j$ by the choice of χ_j and the definition of ζ_j.

For $x \in G$ we define

$$\Psi(x) = \prod_{j \in J} (\chi_j(x) - \zeta_j) \,.$$

If $\chi \in G_t^\perp$ and $\chi(x) \neq 1$, then $x \notin G_t$ and hence $\Psi(x) = 0$ by the above. Thus $\Psi\chi = \Psi$ for all $\chi \in G_t^\perp$.

Proof of the Lettl-Sun result (continued)

Observe that

$$\Psi(x) = \sum_{I \subseteq J} \left(\prod_{j \in I} \chi_j(x) \right) \prod_{j \in J \setminus I} (-\zeta_j) = \sum_{\psi \in \widehat{G}} c(\psi)\psi(x),$$

where

$$c(\psi) = \sum_{\substack{I \subseteq J \\ \prod_{j \in I} \chi_j = \psi}} \prod_{j \in J \setminus I} (-\zeta_j) \in \overline{\mathbb{Z}}.$$

Let $\mathbb{C}$ be the complex field. As the set $\widehat{G}$ is a basis of the $\mathbb{C}$-vector space

$$\mathbb{C}^G = \{g : \ g \text{ is a function from } G \text{ to } \mathbb{C}\},$$

for any $\chi \in G_t^{\perp}$ we have $c(\psi\chi) = c(\psi)$ for all $\psi \in \widehat{G}$ because $\Psi\chi^{-1} = \Psi$.

Proof of the Lettl-Sun result (continued)

Clearly

$$\prod_{j\in J}(1-\zeta_j)=\Psi(e)=\sum_{\psi\in\widehat{G}}c(\psi)\psi(e)=\sum_{\psi\in\widehat{G}}c(\psi).$$

Let $\psi_1 G_t^{\perp}\cup\cdots\cup\psi_l G_t^{\perp}$ be a coset decomposition of $\widehat{G}$ where $l=[\widehat{G}:G_t^{\perp}]$. Then

$$\sum_{\psi\in\widehat{G}}c(\psi)=\sum_{r=1}^{l}\sum_{\chi\in G_t^{\perp}}c(\psi_r\chi)=\sum_{r=1}^{l}|G_t^{\perp}|c(\psi_r)=[G:G_t]\sum_{r=1}^{l}c(\psi_r).$$

Therefore $[G:G_t]$ divides $\prod_{j\in J}(1-\zeta_j)$ in the ring $\overline{\mathbb{Z}}$ of all algebraic integers, and the lemma of Lettl and Sun gives

$$k-m=|J|\geqslant f([G:G_t]).$$

Covering a group by subnormal subgroups

Recall that a group G is said to be *perfect* if it coincides with its derived group $G' = \langle x^{-1}y^{-1}xy :\ x, y \in G\rangle$.

Theorem (Z.-W. Sun [Internat. J. Math. 17(2006)]). Suppose that $\{G_i\}_{i=1}^k$ is a minimal m-cover of a group G by subnormal subgroups. Then there is a composition series from $\bigcap_{i=1}^k G_i$ to G whose factors are of prime orders, and all the G_i contain every perfect subgroup of G.

This extends the following result of M. A. Brodie, R. F. Chamberlain and L.-C. Kappe [Proc. Amer. Math. Soc. 104(1988)]: If $\{G_i\}_{i=1}^k$ is a minimal cover of a group G by finitely many normal subgroups, then $G/\bigcap_{i=1}^k G_i$ is solvable and all perfect normal subgroups of G are contained in each of $G_1, \ldots, G_k$.

Two conjectures

Conjecture (Z.-W. Sun) (i) (2008) Whenever $A = \{a_i G_i\}_{i=1}^k$ forms an m-cover of a group G by left cosets with $a_t G_t$ irredundant, we have the inequality $k \geqslant m + f([G : G_t])$ and hence $[G : G_t] \leqslant 2^{k-m}$.

(ii) (2004) If $A = \{a_i G_i\}_{i=1}^k$ forms a minimal m-cover of an abelian group G by left cosets or an exact m-cover of a solvable group G by left cosets, then we have $k \geqslant m + f(N)$, where N is the least common multiple of the indices $[G : G_1], \ldots, [G : G_k]$.

When $\{a_i G_i\}_{i=1}^k$ forms an exact m-cover of a solvable group G, the inequality $k \geqslant m + f([G : G_t])$ was shown by Berger, Felzenbaum and Fraenkel [Colloq. Math. 1988] in the case $m = 1$ and proved by the speaker [European J. Combin. 2003] for general m.

Conjecture (S. Guo and Z.-W. Sun, 2004). If $\{G_i\}_{i=1}^k$ forms a minimal m-cover of an abelian group G with $[G : \bigcap_{i=1}^k G_i] = \prod_{t=1}^r p_t^{\alpha_t}$, where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$, then $k > m + \sum_{t=1}^r (\alpha_t - 1)(p_t - 1)$.

Part III. On the Herzog-Schönheim conjecture

The Davenport-Mirsky-Newman-Rado result

Soon after his invention of covers of $\mathbb{Z}$, P. Erdős conjectured that if $\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ $(k > 1)$ is a system of residue classes with the moduli $n_1, \ldots, n_k$ distinct, then it cannot be a disjoint cover of $\mathbb{Z}$.

Theorem (H. Davenport, L. Mirsky, D. Newman and R. Rado, 1960s). If $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a disjoint cover of $\mathbb{Z}$ with $1 < n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$, then we must have $n_{k-1} = n_k$.

Proof. Without loss of generality we assume $0 \leqslant a_s < n_s$ $(1 \leqslant s \leqslant k)$. For $|z| < 1$ we have

$$\sum_{s=1}^{k} \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{s=1}^{k} \sum_{q=0}^{\infty} z^{a_s + qn_s} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

If $n_{k-1} < n_k$, then

$$\infty = \lim_{\substack{z \to e^{2\pi i/n_k} \\ |z|<1}} \frac{z^{a_k}}{1 - z^{n_k}} = \lim_{\substack{z \to e^{2\pi i/n_k} \\ |z|<1}} \left(\frac{1}{1-z} - \sum_{s=1}^{k-1} \frac{z^{a_s}}{1 - z^{n_s}} \right) < \infty,$$

which leads a contradiction!

Burshtein's conjecture

Let $A = \{a_s(n_s)\}_{s=1}^k$ be a disjoint cover of $\mathbb{Z}$ with each modulus occurring at most M times. Write $[n_1, \ldots, n_k] = \prod_{t=1}^r p_t^{\alpha_t}$, where $p_1 < \cdots < p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r$ are positive integers. N. Burshtein [Discrete Math. 14(1976)] conjectured that

$$p_r \leqslant M \prod_{p \leqslant p_r} \frac{p}{p-1}.$$

R. J. Simpson [Discrete Math. 59(1986)] proved further that

$$p_r \leqslant M \prod_{t=1}^{r-1} \frac{p_t}{p_t - 1}.$$

The last inequality implies that $M \geqslant p_1 > 1$; in fact, if $r \geqslant 2$ then

$$M > p_r \prod_{t=1}^{r-1} \frac{p_t - 1}{p_t} \geqslant p_{r-1} \prod_{t=1}^{r-2} \frac{p_t - 1}{p_t} \geqslant \cdots \geqslant p_2 \frac{p_1 - 1}{p_1} > p_1 - 1.$$

This gives a combinatorial approach to the Erdös conjecture.

The Herzog-Schönheim Conjecture

The following conjecture extends the conjecture of P. Erdős to disjoint covers of groups.

Herzog-Schönheim Conjecture [Canad. Math. Bull. 17(1974)]. Let $\{a_iG_i\}_{i=1}^k$ ($k > 1$) be a partition of a group G into left cosets of subgroups $G_1, \ldots, G_k$. Then the (finite) indices

$$n_1 = [G : G_1], \ldots, n_k = [G : G_k]$$

cannot be distinct.

It is known that any finite nilpotent group is the direct product of its Sylow subgroups. Using this fact and lattice parallelotopes, Berger, Felzenbaum and Fraenkel [Canad. Bull. Math. 1986] confirmed the above conjecture for **finite nilpotent groups**.

A result of Z.-W. Sun

Theorem (Z.-W. Sun [J. Algebra 273(2004)]). Let $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ be a finite system of left cosets in a group G with not all the G_i equal to G. Suppose that $\mathcal{A}$ covers all the elements of G the same number of times, and that among the indices

$$n_1 = [G : G_1] \leqslant \ldots \leqslant n_k = [G : G_k].$$

each occurs at most $M \in \mathbb{Z}^+$ times. Let p_* and p^* be the smallest and the largest prime divisors of $N = [n_1, \ldots, n_k]$ respectively. Suppose that all the G_i with $n_i \geqslant p^*$ are subnormal in G, or G/H is a solvable group having a normal Sylow p'-subgroup where H is the core $(\bigcap_{i=1}^k G_i)_G$ and p' is the greatest prime divisor of $|G/H|$. Then we have the following (i)–(iv) with the O-constants absolute.

(i) $M \geqslant p_*$, moreover among the k indices $n_1, \ldots, n_k$ there exists a multiple of p^* occurring at least $1 + \lfloor p^* \prod_{p|N}(p-1)/p \rfloor \geqslant p_*$ times.

A result of Z.-W. Sun (continued)

(ii) All prime divisors of $n_1, \ldots, n_k$ are smaller than $e^\gamma M \log M + O(M \log\log M)$.

(iii) The number of distinct prime divisors of $n_1, \ldots, n_k$ does not exceed $e^\gamma M + O(M/\log M)$.

(iv) For the least index n_1, we have $\log n_1 \leqslant \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log\log M)$.

The above theorem was established by a combined use of tools from combinatorics, group theory and number theory. The basic idea is to extend Burshtein's conjecture to exact m-covers of groups by cosets.

A lemma on subnormal subgroups

One of the key lemmas is the following one which is the main reason why covers involving subnormal subgroups are better behaved than general covers.

A Lemma on Indices of Subnormal Subgroups (Z. W. Sun). Let G be a group, and let $P(n)$ denote the set of prime divisors of a positive integer n.

(i) [European J. Combin. 2001] If $G_1, \ldots, G_k$ are subnormal subgroups of G with finite index, then

$$\left[G : \bigcap_{i=1}^{k} G_i\right] \,\Big|\, \prod_{i=1}^{k}[G : G_i] \text{ and } P\left(\left[G : \bigcap_{i=1}^{k} G_i\right]\right) = \bigcup_{i=1}^{k} P([G : G_i]).$$

(ii) [J. Algebra, 2004] Let H be a subnormal subgroup of G with finite index. Then

$$P(|G/H_G|) = P([G : H]).$$

A lemma on unions of cosets

Here is another useful lemma of combinatorial nature.

A Lemma on Unions of Cosets (Z. W. Sun [J. Algebra, 2004]). Let G be a group and H its subgroup with finite index N. Let $a_1, \ldots, a_k \in G$, and let $G_1, \ldots, G_k$ be subnormal subgroups of G containing H. Then the union $\bigcup_{i=1}^k a_i G_i$ contains at least $|\bigcup_{i=1}^k 0 (\mathrm{mod}\ n_i) \cap \{0, 1, \ldots, N-1\}|$ left cosets of H, where $n_i = [G : G_i]$.

This lemma implies the following result of Z. W. Sun [Internat. J. Math. 2006]: *If $G_1, \ldots, G_k$ are normal Hall subgroups of a finite group G, then*

$$\left| \bigcup_{i=1}^k a_i G_i \right| \geqslant \left| \bigcup_{i=1}^k G_i \right|.$$

(A subgroup H of a finite group G is called a *Hall subgroup* of G if $|H|$ is relatively prime to $[G : H]$.)

Tools from analytic number theory

We also need the following theorems in analytic number theory.

The Prime Number Theorem with Error Terms For $x \geqslant 2$ we have

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

where $\pi(x) = \sum_{p \leqslant x} 1$ is the number of primes not exceeding x.

Mertens' Theorem. For $x \geqslant 2$ we have

$$\prod_{p \leqslant x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right).$$

Using the above two theorems we can deduce the following lemma.

An Analytic Lemma (Z. W. Sun [J. Algebra, 2004]). For $M \geqslant 2$, if $q > 1$ is an integer with $q < M \prod_{p \leqslant q} p/(p-1)$ then

$q < e^{\gamma} M \log M + O(M \log\log M)$ and $\pi(q) \leqslant e^{\gamma} M + O(M/\log M)$,

where the O-constants are absolute.

A challenging conjecture on disjoint cosets

Finally we mention a challenging conjecture arising from the speaker's study of Huhn-Megyesi problems and covers of groups.

A Conjecture on Disjoint Cosets (Z.-W. Sun, [Internat. J. Math., 2006]). Let G be a group, and $a_1G_1, \ldots, a_kG_k$ $(k > 1)$ be pairwise disjoint left cosets of G with all the indices $[G : G_i]$ finite. Then, for some $1 \leqslant i < j \leqslant k$ we have $\gcd([G : G_i], [G : G_j]) \geqslant k$.

Z.-W. Sun [Internat. J. Math. 2006] noted that this conjecture holds for p-groups as well as the special case $k = 2$. If G_1 and G_2 are subgroups of G with $[G : G_1]$ and $[G : G_2]$ finite and relatively prime, then $G_1G_2 = G$ and $a_1G_1 \cap a_2G_2 \neq \emptyset$ for all $a_1, a_2 \in G$.

W.-J. Zhu [Int. J. Mod. Math. 3(2008)] proved the conjecture for $k = 3, 4$ via several sophisticated lemmas. K. O'Bryant [Integers 2007] confirmed the conjecture for $G = \mathbb{Z}$ in the case $k \leqslant 20$.

Thank you!