On Covers of Groups by Cosets

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Jan. 12, 2018
Abstract

If a group $G$ is the union of finitely many left cosets $a_1G_1, \ldots, a_kG_k$ of subgroups $G_1, \ldots, G_k$, then the system $\{a_sG_s\}_{s=1}^k$ is said to be a cover of $G$. We will give a survey of problems and results on extremal problems concerning covers of groups, and introduce progress on the famous Herzog-Schönheim conjecture which states that if $\{a_sG_s\}_{s=1}^k (k > 1)$ is a partition of a group $G$ into finitely many left cosets then the (finite) indices $[G : G_1], \ldots, [G : G_k]$ cannot be distinct. We will also mention some new challenging conjectures in the field.
Part I. Basic Results on Covers of Groups
Disjoint covers of a group by left or right cosets

Let $H$ be a subgroup of a group $G$ with $[G : H] = k < \infty$. Then we can partition $G$ into $k$ left cosets $g_1H, \ldots, g_kH$, and $\{g_iH\}_{i=1}^k$ forms a disjoint cover of $G$ by left cosets.

Let $\{Ha_i\}_{i=1}^k$ be a right coset decomposition of $G$. Then $\{a_iG_i\}_{i=1}^k$ is a disjoint cover of $G$ where $G_i = a_i^{-1}Ha_i$. Observe that

$$\bigcap_{i=1}^k G_i = \bigcap_{i=1}^k \bigcap_{h \in H} a_i^{-1}h^{-1}Hha_i = \bigcap_{g \in G} g^{-1}Hg$$

is the normal core $H_G$ of $H$ in $G$ which is the largest normal subgroup of $G$ contained in $H$.

In group theory, it is known that $G/H_G$ can be embedded into the symmetric group $S_{[G:H]} = S_k$ and thus

$$\left[ G : \bigcap_{i=1}^k G_i \right] = |G/H_G| \leq k!.$$
A basic theorem on covers of groups

**An Example of M. J. Tomkinson.** Let \( k > 1 \) be a positive integer, and let \( G \) be the symmetric group \( S_k \) and \( H \) be the stabilizer of 1. Then \( G_i = (1i)^{-1}H(1i) \) is the stabilizer of \( i \) for each \( i = 1, \ldots, k \). Clearly,

\[
\{ G_1, (12)G_2, \ldots, (1k)G_k \} = \{ H, H(12), \ldots, H(1k) \}
\]

forms a disjoint cover of \( G \) with \( \bigcap_{i=1}^{k} G_i = H_G = \{ e \} \). Note that \([ G : \bigcap_{i=1}^{k} G_i ] = |G| = k! \).

**A Basic Theorem on Covers of Groups.** Let \( \mathcal{A} = \{ a_i G_i \}_{i=1}^{k} \) be a finite system of left cosets in a group \( G \) where \( G_1, \ldots, G_k \) are subgroups of \( G \). Suppose that \( \mathcal{A} \) forms a minimal cover of \( G \) (i.e. \( \mathcal{A} \) covers all the elements of \( G \) but none of its proper systems does).

(i) (B. H. Neumann, 1954) There is a constant \( c_k \) depending only on \( k \) such that \([ G : G_i ] \leq c_k \) for all \( i = 1, \ldots, k \).

(ii) (M. J. Tomkinson, 1987) We have \([ G : \bigcap_{i=1}^{k} G_i ] \leq k! \), where the upper bound \( k! \) is best possible.
Tomkinson’s proof of the second part

We show by induction that

\[
\left[ \bigcap_{i \in l} G_i : \bigcap_{i=1}^k G_i \right] \leq (k - |l|)! \]  

(*l)

for all \( l \subseteq \{1, \ldots, k\} \), where \( \bigcap_{i \in \emptyset} G_i \) is regarded as \( G \).

Clearly (*l) holds for \( l = \{1, \ldots, k\} \). Now let \( l \subset \{1, \ldots, k\} \) and assume (*j) for all \( J \subseteq \{1, \ldots, k\} \) with \(|J| > |l|\). Since \( \{a_i G_i\}_{i \in l} \) is not a cover of \( G \), there is an \( a \in G \) not covered by \( \{a_i G_i\}_{i \in l} \). Clearly \( a(\bigcap_{i \in l} G_i) \) is disjoint from the union \( \bigcup_{i \in l} a_i G_i \) and hence contained in \( \bigcup_{j \not\in l} a_j G_j \). Thus

\[
a\left(\bigcap_{i \in l} G_i\right) = \bigcup_{a_j G_j \cap a\left(\bigcap_{i \in l} G_i\right) \neq \emptyset} \left(a_j G_j \cap a\left(\bigcap_{i \in l} G_i\right)\right),
\]

\[
\left[ \bigcap_{i \in l} G_i : H \right] \leq \sum_{j \not\in l} \left[ G_j \cap \bigcap_{i \in l} G_i : H \right] \leq \sum_{j \not\in l} (k - (|l| + 1))! = (k - |l|)!
\]

where \( H = \bigcap_{i=1}^k G_i \). This concludes the induction proof.
m-covers and exactly m-covers

A right coset $Ha$ in a group $G$ is also a left coset $a(a^{-1}Ha)$. So we only consider left cosets.

Let $m \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, and let $A = \{a_i G_i\}_{i=1}^k$ be a finite system of left cosets in a group $G$. If each element of $G$ is covered by $A$ at least (resp., exactly) $m$ times, then we call $A$ an $m$-cover (resp., exact $m$-cover) of $G$. If $A$ is an $m$-cover of $G$ but none of its proper subsystems does, then $A$ is said to be a minimal $m$-cover of $G$.

Part II. Extremal Problems for $m$-Covers
The Mycielski function $f : \mathbb{Z}^+ \to \mathbb{N} = \{0, 1, 2, \ldots\}$ is given by $f(1) = 0$ and $f(mn) = f(m) + f(n)$ for $m, n \in \mathbb{Z}^+$.  

**An Example of Š. Znám.** Let $n > 1$ be an integer with the factorization $\prod_{t=1}^{r} p_t^{\alpha_t}$, where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$. Then $0(\text{mod} \ n)$ and the following $f(n) = \sum_{s=1}^{r} \alpha_s (p_s - 1)$ residue classes

$$jp_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s^{\alpha-1}(\text{mod} \ p_1^{\alpha_1} \cdots p_{s-1}^{\alpha_{s-1}} p_s)$$

$$(\alpha = 1, \ldots, \alpha_s; \ j = 1, \ldots, p_s - 1; \ s = 1, \ldots, r)$$

form a disjoint cover of $\mathbb{Z}$ whose moduli have the least common multiple $n$.  

"My"cielski’s function and an example of Zná"m"
An extremal problem for exact $m$-covers of groups

Let $A = \{a_iG_i\}_{i=1}^k$ be an exact $m$-cover of a group $G$ with $\bigcap_{i=1}^k G_i = H$. By the Neumann-Tomkinson theorem, $[G : H] \leq k!$. How to provide a sharp lower bound of $k$ in terms of $G$ and $H$?

**An Example of Z.-W. Sun** [Fund. Math. 134(1990)]. Let $H$ be a subnormal subgroup of a group $G$ with finite index. Let

$$H_0 = H \subset H_1 \subset \cdots \subset H_n = G$$

be a composition series from $H$ to $G$. For $i = 0, \ldots, n-1$, write

$$H_{i+1} \setminus H_i = \bigcup_{j=1}^{[H_{i+1}:H_i]-1} b_j^{(i)} H_i.$$

Then the following $d(G, H) = \sum_{i=0}^{n-1} ([H_{i+1} : H_i] - 1)$ left cosets $b_j^{(i)} H_i$ ($0 \leq i < n; 1 \leq j < [H_{i+1} : H_i]$),

together with $H$ and $m - 1$ copies of $G$, form an exact $m$-cover of $G$ by $m + d(G, H)$ left cosets of subgroups whose intersection is $H$. (In the case $H = G$ we define $d(G, H) = 0$.)

$$d(G, H) \geq f([G : H]) \geq \log_2[G : H].$$

Also, $d(G, H) = f([G : H])$ if and only if $G/H_G$ is solvable.
Mycielski’s conjecture and further extensions

**Mycielski’s Conjecture** (J. Mycielski, 1966). If \( \{a_i G_i\}_{i=1}^k \) is a disjoint cover of an abelian group \( G \), then \( k \geq 1 + f([G : G_i]) \) for all \( i = 1, \ldots, k \).

**Related Results on Exact \( m \)-covers.** Let \( A = \{a_i G_i\}_{i=1}^k \) be an exact \( m \)-cover of a group \( G \) with \( \bigcap_{i=1}^k G_i = H \).

(i) (I. Korec [Fund. Math., 1974]) If \( m = 1 \) and \( G_1, \ldots, G_k \) are normal in \( G \), then \( k \geq 1 + f([G : H]) \).

(ii) (Z.-W. Sun [European J. Combin., 2001]) If \( G_1, \ldots, G_k \) are subnormal in \( G \), then \( k \geq m + d(G, H) \), with the lower bound best possible.

The proof is by induction, on the basis of the following key lemma.

**A Lemma** (Z.-W. Sun [European J. Combin., 2001]). Let \( A = \{a_i G_i\}_{i=1}^k \) be an exact \( m \)-cover of a group \( G \) by left cosets of subnormal subgroups \( G_1, \ldots, G_k \). For any maximal normal subgroup \( H \) of \( G \), we have

\[ \{ C \in G/H : C \supseteq a_i G_i \text{ for some } i = 1, \ldots, k \} = \emptyset \text{ or } G/H. \]
On the core of a subnormal subgroup

Note that Korec’s result is stronger than Mycielski’s conjecture, and also Sun’s result has the following consequence.

**Corollary** (Sun [Fund. Math., 1990]). Let $H$ be a subnormal subgroup of a group $G$ with $[G : H] < \infty$. Then

$$[G : H] \geq 1 + d(G, H_G) \geq 1 + f([G : H_G])$$

and hence

$$|G/H_G| \leq 2^{[G:H]^{-1}}.$$

**Proof.** Let $\{Ha_i\}_{i=1}^k$ be a right coset decomposition of $G$ where $k = [G : H]$. Then $\{a_i G_i\}_{i=1}^k$ is a disjoint cover of $G$ where all the $G_i = a_i^{-1} H a_i$ are subnormal in $G$ and $\bigcap_{i=1}^k G_i = H_G$. So the desired result follows.
On minimal $m$-covers

Korec's and Sun's results on exact $m$-covers can be extended to minimal $m$-covers of $\mathbb{Z}$, see R. J. Simpson [Acta Arith., 1985] for the case $m = 1$ and Z. W. Sun [Internat. J. Math. 17(2006)] for general $m \geq 1$. However, they cannot be extended to minimal $m$-covers of abelian groups as illustrated by the following example.

**Example** (G. Lettl and Z.-W. Sun [Acta Arith., 2008]). Let $G$ be the abelian group $C_p \times C_p$ where $p$ is a prime and $C_p$ is the cyclic group of order $p$. Then any element $a \neq e$ of $G$ has order $p$. Let $G_1, \ldots, G_k$ be all the distinct subgroups of $G$ with order $p$. If $1 \leq i < j \leq k$, then $G_i \cap G_j = \{e\}$. Thus $\{G_s\}_{s=1}^k$ forms a minimal cover of $G$ with $\bigcap_{s=1}^k G_s = \{e\}$. As $1 + k(p - 1) = |\bigcup_{s=1}^k G_s| = |G| = p^2$, we have

$$k = p + 1 \geq 1 + f([G : G_s]) = 1 + f(p) = p.$$  

However, if $p > 2$ then

$$k = p + 1 < 2p - 1 = 1 + f([G : \{e\}]) = 1 + d\left(G, \bigcap_{s=1}^k G_s\right).$$
On \( m \)-covers of abelian groups

Let \( m \) and \( n \) be positive integers. Is \( m + f(n) \) the smallest positive integer \( k \) such that for any abelian group having a subgroup of index \( n \) there is a minimal \( m \)-cover of \( G \) by \( k \) cosets of subgroups one of which has index \( n \)?

**Theorem** (G. Lettl and Z.-W. Sun [Acta Arith. 131(2008)]). Let \( A = \{a_sG_s\}_{s=1}^k \) be a minimal \( m \)-cover of an abelian group \( G \) by left cosets. Then

\[
k \geq m + f([G : G_t]) \quad \text{for any} \quad t = 1, \ldots, k.
\]

This theorem implies the following conjecture of W. D. Gao and A. Geroldinger [European J. Combin. 2003].

**Gao-Geroldinger Conjecture.** Let \( G \) be a finite abelian group with identity \( e \). If \( G \setminus \{e\} \) is a union of \( k \) cosets \( a_1G_1, \ldots, a_kG_k \), then we have \( k \geq f(|G|) \).

In fact, if we set \( a_0 = e \) and \( G_0 = \{e\} \) then \( \{a_sG_s\}_{s=0}^k \) forms a cover of \( G \) with \( a_0G_0 \) irredundant, hence \( k + 1 \geq 1 + f([G : G_0]) \) and thus \( k \geq f(|G|) \).
The Lettl-Sun result cannot be shown in the way that we prove Korec’s or Sun’s result because we don’t have the corresponding lemma for minimal $m$-covers of abelian groups. Thus, new ideas are needed!

The proof of the Lettl-Sun result was obtained via characters of abelian groups and algebraic number theory; below is a key lemma used in the proof.

**Lemma** (G. Lettl and Z.-W. Sun [Acta Arith. 131(2008)]). Let $n > 1$ be an integer. Then $f(n)$ is the smallest positive integer $k$ such that there are roots of unity $\zeta_1, \ldots, \zeta_k$ different from 1 for which $\prod_{s=1}^{k}(1 - \zeta_s) \equiv 0 \pmod{n}$ in the ring $\overline{\mathbb{Z}}$ of algebraic integers.
On cyclotomic polynomials

The $n$th cyclotomic polynomial is given by

$$\Phi_n(x) = \prod_{m=1}^{n} \left( x - e^{2\pi i m/n} \right).$$

$$x^n - 1 = \prod_{m=1}^{n} \left( x - e^{2\pi i m/n} \right) = \prod_{d | n} \prod_{c=1 \atop (c,d)=1}^{d} \left( x - e^{2\pi i c/d} \right) = \prod_{d | n} \Phi_d(x).$$

Applying the Möbius inversion we obtain

$$\Phi_n(x) = \prod_{d | n} (x^d - 1)^{\mu(n/d)} \in \mathbb{Z}[x].$$

If $n > 1$, then $\sum_{d | n} \mu(n/d) = \sum_{d | n} \mu(d) = 0$, thus

$$\Phi_n(x) = \prod_{d | n} \left( \frac{x^d - 1}{x - 1} \right)^{\mu(n/d)} = \prod_{d | n} (1 + x + \cdots + x^{d-1})^{\mu(n/d)}$$

and hence $\Phi_n(1) = \prod_{d | n} d^{\mu(n/d)}$. 
Using the Mangoldt function

Recall that the Mangoldt function is given by

\[ \Lambda(n) = \begin{cases} \log p & \text{if } n = p^a \text{ for some prime } p \text{ and } a \in \mathbb{Z}^+, \\ 0 & \text{otherwise}. \end{cases} \]

If \( n \) has the primary factorization \( \prod_{i=1}^{r} p_i^{\alpha_i} \) where \( p_1, \ldots, p_r \) are distinct primes and \( \alpha_1, \ldots, \alpha_r \in \mathbb{N} \), then

\[ \sum_{d|n} \Lambda(d) = \sum_{i=1}^{r} \sum_{\beta_i=0}^{\alpha_i} \Lambda(p_i^{\beta_i}) = \sum_{i=1}^{r} \alpha_i \log p_i = \log \prod_{i=1}^{r} p_i^{\alpha_i} = \log n. \]

Applying the Möbius inversion formula we get

\[ \sum_{d|n} \mu\left(\frac{n}{d}\right) \log d = \Lambda(n). \]

Therefore

\[ \log \Phi_n(1) = \sum_{d|n} \mu\left(\frac{n}{d}\right) \log d = \begin{cases} \log p & \text{if } n \text{ is a power of some prime } p, \\ 0 & \text{otherwise}. \end{cases} \]
Find roots $\zeta_1, \ldots, \zeta_{f(n)} \neq 1$ of unity with $n \mid \prod_{s=1}^{f(n)} (1 - \zeta_s)$

So we have

$$\prod_{m=1}^{n} (1 - e^{2\pi i m/n}) = \Phi_n(1) = \begin{cases} p & \text{if } n \text{ is a power of some prime } p, \\ 1 & \text{otherwise.} \end{cases}$$

In particular,

$$\prod_{m=1}^{p-1} (1 - e^{2\pi i m/p}) = p \quad \text{for any prime } p.$$

Note that

$$n = \prod_{p \mid n} p^{\operatorname{ord}_p(n)} = \prod_{p \mid n} \prod_{m=1}^{p-1} (1 - e^{2\pi i m/p})^{\operatorname{ord}_p(n)}.$$

So there exist $f(n) = \sum_{p \mid n} \operatorname{ord}_p(n)(p - 1)$ roots of unity $\zeta_1, \ldots, \zeta_{f(n)} \neq 1$ such that $n$ divides $(1 - \zeta_1) \cdots (1 - \zeta_{f(n)})$ in the ring $\mathbb{Z}$ of all algebraic integers.
Proof of the lemma of Lettl and Sun

Now suppose that \( \prod_{s=1}^{k}(1 - \zeta_s) \equiv 0 \pmod{n} \), where \( \zeta_s \) is a primitive \( n_s \)th root of unity with \( n_s > 1 \). Recall that

\[
\prod_{\substack{r=1 \atop (r,n_s)=1}}^{n_s} (1 - \zeta_r^r) = \prod_{\substack{m=1 \atop (m,n_s)=1}}^{n_s} (1 - e^{2\pi im/n_s})
\]

\[
eq \begin{cases} 
  p & \text{if } n_s \text{ is a power of some prime } p, \\
  1 & \text{otherwise}.
\end{cases}
\]

Let \( N \) be the least common multiple of \( n_1, \ldots, n_k \). Then

\[
n^{\varphi(N)} \Big| \prod_{s=1}^{k} \left( (1 - \zeta_s)^{\varphi(n_s)} \right)^{\varphi(N)/\varphi(n_s)} \Big| \prod_{s=1}^{k} p(n_s)^{\varphi(N)/\varphi(n_s)},
\]

where \( p(n_s) \) is the least prime divisor of \( n_s \). So, for any prime \( p \),

\[
\text{ord}_p(n)^{\varphi(N)} \leq \sum_{\substack{s=1 \atop n_s \text{ is a power of } p}}^{k} \frac{\varphi(N)}{\varphi(n_s)}
\]
Proof of the lemma of Lettl and Sun

and hence

$$|\{1 \leq s \leq k : n_s \text{ is a power of } p\}| \geq \text{ord}_p(n)(p - 1).$$

It follows that

$$k \geq \sum_{p | n} |\{1 \leq s \leq k : n_s \text{ is a power of } p\}|$$

$$\geq \sum_{p | n} \text{ord}_p(n)(p - 1) = f(n).$$

This concludes the proof of the lemma.
Proof of the Lettl-Sun result

For a finite abelian group $G$, let $\hat{G}$ denote the group of all complex-valued characters of $G$. One has $\hat{G} \cong G$. For any subgroup $H$ of $G$ let $H^\perp$ denote the group of those characters $\chi \in \hat{G}$ with $\ker(\chi) = \{x \in G : \chi(x) = 1\}$ containing $H$. Then there is a canonical isomorphism $H^\perp \cong \hat{G}/H$ by putting $\chi(aH) = \chi(a)$ for any $a \in G$ and any $\chi \in H^\perp$. Furthermore, for each $a \in G \setminus H$ there exists some $\chi \in H^\perp$ with $\chi(a) \neq 1$.

**Proof of the Lettl-Sun Result.** As $H = \bigcap_{s=1}^{k} G_s$ is of finite index in $G$. Instead of the minimal $m$-cover $A = \{a_s G_s\}_{s=1}^{k}$ of $G$, we may consider the minimal $m$-cover $\tilde{A} = \{\tilde{a}_s \tilde{G}_s\}_{s=1}^{k}$ of the finite abelian group $\tilde{G} = G/H$, where $\tilde{a}_s = a_s H$ and $\tilde{G}_s = G_s/H$ (hence $[\tilde{G} : \tilde{G}_s] = [G : G_s]$). Therefore, without any loss of generality, we can assume that $G$ is finite.
Proof of the Lettl-Sun result (continued)

Fix $1 \leq t \leq k$. As $\{a_s G_s\}_{s \neq t}$ is not an $m$-cover of $G$, there is an $a \in a_t G_t$ such that it is covered by exactly $m$ cosets in $A$ and hence $J = \{1 \leq j \leq k : a \not\in a_j G_j\}$ has cardinality $k - m$. For each $j \in J$ we may choose a $\chi_j \in G_j^\perp$ with $\zeta_j := \chi_j(a^{-1}a_j) \neq 1$. For any $x \in G \setminus G_t$ we have $ax \not\in a G_t = a_t G_t$. Since $A$ is an $m$-cover of $G$, there exists some $j \in J$ such that $ax \in a_j G_j$, and therefore $\chi_j(x) = \chi_j(a^{-1}a_j) = \zeta_j$ by the choice of $\chi_j$ and the definition of $\zeta_j$.

For $x \in G$ we define

$$\Psi(x) = \prod_{j \in J} (\chi_j(x) - \zeta_j).$$

If $\chi \in G_t^\perp$ and $\chi(x) \neq 1$, then $x \not\in G_t$ and hence $\Psi(x) = 0$ by the above. Thus $\Psi \chi = \Psi$ for all $\chi \in G_t^\perp$. 
Observe that
\[ \psi(x) = \sum_{I \subseteq J} \left( \prod_{j \in I} \chi_j(x) \right) \prod_{j \in J \setminus I} (-\zeta_j) = \sum_{\psi \in \hat{G}} c(\psi) \psi(x), \]
where
\[ c(\psi) = \sum_{I \subseteq J} \prod_{j \in J \setminus I} (-\zeta_j) \in \mathbb{Z}. \]

Let \( \mathbb{C} \) be the complex field. As the set \( \hat{G} \) is a basis of the \( \mathbb{C} \)-vector space
\[ \mathbb{C}^G = \{ g : g \text{ is a function from } G \text{ to } \mathbb{C} \}, \]
for any \( \chi \in G^\perp \) we have \( c(\psi \chi) = c(\psi) \) for all \( \psi \in \hat{G} \) because \( \psi \chi^{-1} = \psi \).
Proof of the Lettl-Sun result (continued)

Clearly

$$\prod_{j \in J} (1 - \zeta_j) = \Psi(e) = \sum_{\psi \in \hat{G}} c(\psi)\psi(e) = \sum_{\psi \in \hat{G}} c(\psi).$$

Let $$\psi_1 G_t \perp \cdots \perp \psi_l G_t \perp$$ be a coset decomposition of $$\hat{G}$$ where $$l = [\hat{G} : G_t \perp]$$. Then

$$\sum_{\psi \in \hat{G}} c(\psi) = \sum_{r=1}^{l} \sum_{\chi \in G_t \perp} c(\psi_r \chi) = \sum_{r=1}^{l} |G_t \perp| c(\psi_r) = [G : G_t] \sum_{r=1}^{l} c(\psi_r).$$

Therefore $$[G : G_t]$$ divides $$\prod_{j \in J} (1 - \zeta_j)$$ in the ring $$\overline{\mathbb{Z}}$$ of all algebraic integers, and the lemma of Lettl and Sun gives

$$k - m = |J| \geq f([G : G_t]).$$
Covering a group by subnormal subgroups

Recall that a group $G$ is said to be **perfect** if it coincides with its derived group $G' = \langle x^{-1}y^{-1}xy : x, y \in G \rangle$.

**Theorem** (Z.-W. Sun [Internat. J. Math. 17(2006)]). Suppose that $\{G_i\}_{i=1}^k$ is a minimal $m$-cover of a group $G$ by subnormal subgroups. Then there is a composition series from $\bigcap_{i=1}^k G_i$ to $G$ whose factors are of prime orders, and all the $G_i$ contain every perfect subgroup of $G$.

This extends the following result of M. A. Brodie, R. F. Chamberlain and L.-C. Kappe [Proc. Amer. Math. Soc. 104(1988)]: If $\{G_i\}_{i=1}^k$ is a minimal cover of a group $G$ by finitely many normal subgroups, then $G/\bigcap_{i=1}^k G_i$ is solvable and all perfect normal subgroups of $G$ are contained in each of $G_1, \ldots, G_k$. 
Two conjectures

**Conjecture** (Z.-W. Sun) (i) (2008) Whenever $A = \{a_i G_i\}_{i=1}^k$ forms an $m$-cover of a group $G$ by left cosets with $a_t G_t$ irredundant, we have the inequality $k \geq m + f([G : G_t])$ and hence $[G : G_t] \leq 2^{k-m}$.

(ii) (2004) If $A = \{a_i G_i\}_{i=1}^k$ forms a minimal $m$-cover of an abelian group $G$ by left cosets or an exact $m$-cover of a solvable group $G$ by left cosets, then we have $k \geq m + f(N)$, where $N$ is the least common multiple of the indices $[G : G_1], \ldots, [G : G_k]$.

When $\{a_i G_i\}_{i=1}^k$ forms an exact $m$-cover of a solvable group $G$, the inequality $k \geq m + f([G : G_t])$ was shown by Berger, Felzenbaum and Fraenkel [Colloq. Math. 1988] in the case $m = 1$ and proved by the speaker [European J. Combin. 2003] for general $m$.

**Conjecture** (S. Guo and Z.-W. Sun, 2004). If $\{G_i\}_{i=1}^k$ forms a minimal $m$-cover of an abelian group $G$ with $[G : \bigcap_{i=1}^k G_i] = \prod_{t=1}^r p_t^{\alpha_t}$, where $p_1, \ldots, p_r$ are distinct primes and $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}^+$, then $k > m + \sum_{t=1}^r (\alpha_t - 1)(p_t - 1)$. 
Part III. On the Herzog-Schönheim conjecture
The Davenport-Mirsky-Newman-Rado result

Soon after his invention of covers of $\mathbb{Z}$, P. Erdős conjectured that if \( \{a_s(\text{mod } n_s)\}_{s=1}^{k} \) \((k > 1)\) is a system of residue classes with the moduli \(n_1, \ldots, n_k\) distinct, then it cannot be a disjoint cover of $\mathbb{Z}$.

**Theorem** (H. Davenport, L. Mirsky, D. Newman and R. Rado, 1960s). If \( A = \{a_s(\text{mod } n_s)\}_{s=1}^{k} \) is a disjoint cover of $\mathbb{Z}$ with \(1 < n_1 \leq n_2 \leq \cdots \leq n_{k-1} \leq n_k\), then we must have \(n_{k-1} = n_k\).

**Proof.** Without loss of generality we assume \(0 \leq a_s < n_s\) \((1 \leq s \leq k)\). For \(|z| < 1\) we have

\[
\sum_{s=1}^{k} \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{s=1}^{k} \sum_{q=0}^{\infty} z^{a_s + qn_s} = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.
\]

If \(n_{k-1} < n_k\), then

\[
\infty = \lim_{z \to e^{2\pi i/n_k}}_{|z| < 1} \frac{z^{a_k}}{1 - z^{n_k}} = \lim_{z \to e^{2\pi i/n_k}}_{|z| < 1} \left( \frac{1}{1 - z} - \sum_{s=1}^{k-1} \frac{z^{a_s}}{1 - z^{n_s}} \right) < \infty,
\]

which leads a contradiction!
Burshtein’s conjecture

Let \( A = \{ a_s(n_s) \}^{k}_{s=1} \) be a disjoint cover of \( \mathbb{Z} \) with each modulus occurring at most \( M \) times. Write \( [n_1, \ldots, n_k] = \prod_{t=1}^{r} p_t^{\alpha_t} \), where \( p_1 < \cdots < p_r \) are distinct primes and \( \alpha_1, \ldots, \alpha_r \) are positive integers. N. Burshtein [Discrete Math. 14(1976)] conjectured that

\[
p_r \leq M \prod_{p \leq p_r} \frac{p}{p-1}.
\]

R. J. Simpson [Discrete Math. 59(1986)] proved further that

\[
p_r \leq M \prod_{t=1}^{r-1} \frac{p_t}{p_t - 1}.
\]

The last inequality implies that \( M \geq p_1 > 1 \); in fact, if \( r \geq 2 \) then

\[
M > p_r \prod_{t=1}^{r-1} \frac{p_t - 1}{p_t} \geq p_{r-1} \prod_{t=1}^{r-2} \frac{p_t - 1}{p_t} \geq \cdots \geq p_2 \frac{p_1 - 1}{p_1} > p_1 - 1.
\]

This gives a combinatorial approach to the Erdős conjecture.
The Herzog-Schönheim Conjecture

The following conjecture extends the conjecture of P. Erdős to disjoint covers of groups.


Let \( \{a_i G_i\}_{i=1}^k \) be a partition of a group \( G \) into left cosets of subgroups \( G_1, \ldots, G_k \). Then the (finite) indices

\[
n_1 = [G : G_1], \ldots, n_k = [G : G_k]
\]

cannot be distinct.

It is known that any finite nilpotent group is the direct product of its Sylow subgroups. Using this fact and lattice parallelotopes, Berger, Felzenbaum and Fraenkel [Canad. Bull. Math. 1986] confirmed the above conjecture for **finite nilpotent groups**.
A result of Z.-W. Sun

**Theorem** (Z.-W. Sun [J. Algebra 273(2004)]). Let \( \mathcal{A} = \{a_iG_i\}_{i=1}^k \) be a finite system of left cosets in a group \( G \) with not all the \( G_i \) equal to \( G \). Suppose that \( \mathcal{A} \) covers all the elements of \( G \) the same number of times, and that among the indices

\[
 n_1 = [G : G_1] \leq \ldots \leq n_k = [G : G_k].
\]

each occurs at most \( M \in \mathbb{Z}^+ \) times. Let \( p_* \) and \( p^* \) be the smallest and the largest prime divisors of \( N = [n_1, \ldots, n_k] \) respectively. Suppose that all the \( G_i \) with \( n_i \geq p^* \) are subnormal in \( G \), or \( G/H \) is a solvable group having a normal Sylow \( p' \)-subgroup where \( H \) is the core \( \bigcap_{i=1}^k G_i \) and \( p' \) is the greatest prime divisor of \( |G/H| \). Then we have the following (i)–(iv) with the \( O \)-constants absolute.

(i) \( M \geq p_* \), moreover among the \( k \) indices \( n_1, \ldots, n_k \) there exists a multiple of \( p^* \) occurring at least \( 1 + \left\lfloor p^* \prod_{p | N} (p - 1)/p \right\rfloor \geq p_* \) times.
(ii) All prime divisors of $n_1, \ldots, n_k$ are smaller than $e^{-\gamma} M \log M + O(M \log \log M)$.

(iii) The number of distinct prime divisors of $n_1, \ldots, n_k$ does not exceed $e^{-\gamma} M + O(M / \log M)$.

(iv) For the least index $n_1$, we have $\log n_1 \leq \frac{e^{-\gamma}}{\log 2} M \log^2 M + O(M \log M \log \log M)$.

The above theorem was established by a combined use of tools from combinatorics, group theory and number theory. The basic idea is to extend Burshtein’s conjecture to exact $m$-covers of groups by cosets.
A lemma on subnormal subgroups

One of the key lemmas is the following one which is the main reason why covers involving subnormal subgroups are better behaved than general covers.

A Lemma on Indices of Subnormal Subgroups (Z. W. Sun).
Let $G$ be a group, and let $P(n)$ denote the set of prime divisors of a positive integer $n$.

(i) [European J. Combin. 2001] If $G_1, \ldots, G_k$ are subnormal subgroups of $G$ with finite index, then

$$
\left[ \frac{G}{\bigcap_{i=1}^{k} G_i} \right] \Bigg| \prod_{i=1}^{k} \left[ \frac{G}{G_i} \right] \quad \text{and} \quad P\left( \left[ \frac{G}{\bigcap_{i=1}^{k} G_i} \right] \right) = \bigcup_{i=1}^{k} P([G : G_i]).
$$

(ii) [J. Algebra, 2004] Let $H$ be a subnormal subgroup of $G$ with finite index. Then

$$
P\left( \left| \frac{G}{H_G} \right| \right) = P([G : H]).$$
A lemma on unions of cosets

Here is another useful lemma of combinatorial nature.

**A Lemma on Unions of Cosets** (Z. W. Sun [J. Algebra, 2004]). Let $G$ be a group and $H$ its subgroup with finite index $N$. Let $a_1, \ldots, a_k \in G$, and let $G_1, \ldots, G_k$ be subnormal subgroups of $G$ containing $H$. Then the union $\bigcup_{i=1}^{k} a_i G_i$ contains at least $|\bigcup_{i=1}^{k} a_i G_i \cap \{0,1,\ldots,N-1\}|$ left cosets of $H$, where $n_i = [G : G_i]$.

This lemma implies the following result of Z. W. Sun [Internat. J. Math. 2006]: If $G_1, \ldots, G_k$ are normal Hall subgroups of a finite group $G$, then

$$\left| \bigcup_{i=1}^{k} a_i G_i \right| \geq \left| \bigcup_{i=1}^{k} G_i \right|.$$ 

(A subgroup $H$ of a finite group $G$ is called a *Hall subgroup* of $G$ if $|H|$ is relatively prime to $[G : H]$.)
Tools from analytic number theory

We also need the following theorems in analytic number theory.

**The Prime Number Theorem with Error Terms** For $x \geq 2$ we have

$$\pi(x) = \frac{x}{\log x} + O \left( \frac{x}{\log^2 x} \right),$$

where $\pi(x) = \sum_{p \leq x} 1$ is the number of primes not exceeding $x$.

**Mertens’ Theorem.** For $x \geq 2$ we have

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O \left( \frac{1}{\log^2 x} \right).$$

Using the above two theorems we can deduce the following lemma.

**An Analytic Lemma** (Z. W. Sun [J. Algebra, 2004]). For $M \geq 2$, if $q > 1$ is an integer with $q < M \prod_{p \leq q} \frac{p}{(p - 1)}$ then

$q < e^{\gamma} M \log M + O(M \log \log M)$ and $\pi(q) \leq e^{\gamma} M + O(M / \log M)$, where the $O$-constants are absolute.
Finally we mention a challenging conjecture arising from the speaker’s study of Huhn-Megyesi problems and covers of groups.

**A Conjecture on Disjoint Cosets** (Z.-W. Sun, [Internat. J. Math., 2006]). Let $G$ be a group, and $a_1 G_1, \ldots, a_k G_k$ $(k > 1)$ be pairwise disjoint left cosets of $G$ with all the indices $[G : G_i]$ finite. Then, for some $1 \leq i < j \leq k$ we have $\gcd([G : G_i], [G : G_j]) \geq k$.

Z.-W. Sun [Internat. J. Math. 2006] noted that this conjecture holds for $p$-groups as well as the special case $k = 2$. If $G_1$ and $G_2$ are subgroups of $G$ with $[G : G_1]$ and $[G : G_2]$ finite and relatively prime, then $G_1 G_2 = G$ and $a_1 G_1 \cap a_2 G_2 \neq \emptyset$ for all $a_1, a_2 \in G$.

W.-J. Zhu [Int. J. Mod. Math. 3(2008)] proved the conjecture for $k = 3, 4$ via several sophisticated lemmas. K. O’Bryant [Integers 2007] confirmed the conjecture for $G = \mathbb{Z}$ in the case $k \leq 20$. 

A challenging conjecture on disjoint cosets
Thank you!