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COVERING SYSTEMS AND PERIODIC ARITHMETICAL FUNCTIONS

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Abstract. If the ring $\mathbb{Z}$ of integers is the union of finitely many residue classes $a_1(\operatorname{mod} n_1), \ldots, a_k(\operatorname{mod} n_k)$, then the system $A = \{a_s(\operatorname{mod} n_s)\}_{s=1}^k$ is called a cover of $\mathbb{Z}$ or a covering system. There are many problems and results on this topic initiated by Paul Erdös. In this talk we introduce progress on the main open problems in this field, as well as the algebraic theory of periodic arithmetical maps motivated by the speaker's research on covering systems. The talk is related to number theory, combinatorics, algebraic structures and special functions.

Perhaps my favorite problem of all concerns covering systems.

—Paul Erdős (1995)

1. Introduction to covers of $\mathbb{Z}$

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we let

$$a(n) = a + n\mathbb{Z} = \{x \in \mathbb{Z} : \ x \equiv a \pmod{n}\}.$$

This is called a residue class (with *modulus* n) or an arithmetic sequence (with *common difference* n). In particular, we regard $\mathbb{Z} = 0(1)$ as a residue class with modulus 1.

A finite collection

$$A = \{a_s(n_s)\}_{s=1}^k = \{a_1(n_1), \ldots, a_k(n_k)\}$$

of residue classes is said to be a *covering system* (or a *cover* of $\mathbb{Z}$ in short) if each integer lies in at least one of the members in A.

The concept of cover of $\mathbb{Z}$ was first introduced by P. Erdős in his solution to a question of Romanoff given in 1934. Since the Chinese Remaider Theorem tells us when $\bigcap_{s=1}^k a_s(n_s) \neq \emptyset$, it is fundamental to study when we have $\bigcup_{s=1}^k a_s(n_s) = \mathbb{Z}$.

Clearly $A = \{a_s(n_s)\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ if and only if it covers $0, 1, \cdots, N_A - 1$, where N_A is the least common multiple of the moduli $n_1, \cdots, n_k$.

The first nontrivial cover of $\mathbb{Z}$ with distinct moduli is the following one discovered by P. Erdős:

$$B = \{0(2),\ 0(3),\ 1(4),\ 5(6),\ 7(12)\}.$$

Note that $N_B = 12$ and B covers $0, 1, \ldots, 11$. P. Erdős also constructed the following cover of $\mathbb{Z}$ whose moduli are all proper divisors of $2 \times 3 \times 5 \times 7 = 210$:

$$\begin{aligned} C = \{&0(2), 0(3),\ 0(5),\ 1(6),\ 0(7),\ 1(10),\ 1(14),\ 2(15), \\ &2(21),\ 23(30),\ 4(35),\ 5(42),\ 59(70),\ 104(105)\}. \end{aligned}$$

If A covers every integer exactly once, then we call A an *exact cover* of $\mathbb{Z}$ or a *disjoint cover* of $\mathbb{Z}$.

As any integer can be written uniquely in the form $nq + r$ with $q \in \mathbb{Z}$ and $r \in [0, n-1] = \{0, 1, \dots, n-1\}$, the finite system $\{r(n)\}_{r=0}^{n-1}$ is a disjoint cover of $\mathbb{Z}$. Since $0(2^n)$ is a disjoint union of the residue classes $2^n(2^{n+1})$ and $0(2^{n+1})$, the systems

$$A_1 = \{1(2), 0(2)\},\ A_2 = \{1(2), 2(4), 0(4)\},\ A_3 = \{1(2), 2(4), 4(8), 0(8)\},$$
$$\cdots\cdots,\ A_k = \{1(2), 2(2^2), \dots, 2^{k-1}(2^k), 0(2^k)\},\ \cdots\cdots$$

are disjoint covers of $\mathbb{Z}$.

Clearly each residue class $a_s(n_s)$ in system $A = \{a_s(n_s)\}_{s=1}^k$ covers exactly N_A/n_s integers in $[0, N_A - 1]$. Thus, if A is a cover of $\mathbb{Z}$ then

$$|[0, N_A - 1]| \leqslant \sum_{s=1}^{k} \frac{N_A}{n_s} \quad \text{and hence} \quad \sum_{s=1}^{k} \frac{1}{n_s} \geqslant 1;$$

if A is a disjoint cover of $\mathbb{Z}$ then $\sum_{s=1}^{k} 1/n_s = 1$.

Soon after his invention of the concept of covering system, Erdős made the following conjecture: *If $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$, then $\sum_{s=1}^{k} 1/n_s > 1$, i.e., A covers some integer more than once.*

This conjecture of Erdős was soon confirmed independently by H. Davenport, L. Mirsky, D. Newman and R. Rado in the 1950s.

The Davenport-Mirsky-Newman-Rado Result. *Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact cover of $\mathbb{Z}$ with $n_1 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$. Then we must have $n_{k-1} = n_k$.*

Proof. Without loss of generality we assume that $0 \leqslant a_s < n_s$ for all $s \in [1,k]$. For $|z|<1$ we have

$$\sum_{s=1}^{k} \frac{z^{a_s}}{1-z^{n_s}} = \sum_{s=1}^{k}\sum_{q=0}^{\infty} z^{a_s+qn_s} = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}.$$

If $n_{k-1} < n_k$ then

$$\infty = \lim_{\substack{z\to e^{2\pi i/n_k}\\ |z|<1}} \frac{z^{a_k}}{1-z^{n_k}} = \lim_{\substack{z\to e^{2\pi i/n_k}\\ |z|<1}} \left(\frac{1}{1-z} - \sum_{s=1}^{k-1}\frac{z^{a_s}}{1-z^{n_s}}\right) < \infty,$$

a contradiction! □

There are many extensions of the Davenport-Mirsky-Newman-Rado result, the reader may consult the speaker's recent paper [J. Number Theory 111(2005)].

Another improvement to the conjecture of Erdős was obtained by the speaker in 1996.

An Inequality due to Sun [Trans. Amer. Math. Soc. 348(1996)]**.** *Let* $A = \{a_s(n_s)\}_{s=1}^{k}$ *be a cover of* $\mathbb{Z}$ *with*

$$n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k.$$

Then we have $\sum_{s=1}^{k-l} 1/n_s \geqslant 1$ *or* $l \geqslant n_k/n_{k-l}$*. In particular, if* $n_{k-1} < n_k$ *then* $\sum_{s=1}^{k-1} 1/n_s \geqslant 1$ *and hence* $\sum_{s=1}^{k} 1/n_s \geqslant 1 + 1/n_k$*.*

By Example 3 of Z. W. Sun [Trans. Amer. Math. Soc. 348(1996)], if $n>1$ is odd then

$$\{1(2), 2(2^2), \ldots, 2^{n-2}(2^{n-1}), 2^{n-1}(n), 2^{n-1}2(2n), \ldots, 2^{n-1}n(2^{n-1}n)\}$$

forms a cover of $\mathbb{Z}$ with distinct moduli, and the sum of reciprocals of the moduli is less than $1+2/n$ which tends to 1 as $n \to +\infty$.

Now we introduce three main open conjectures concerning covers of $\mathbb{Z}$.

Erdős' Conjecture. *For any arbitrarily large $c > 0$, there exists a cover of $\mathbb{Z}$ whose moduli are distinct and greater than c.*

P. Erdős offers \$1000 for a solution of this conjecture. The conjecture implies the well-known fact that $\sum_{n=1}^{\infty} 1/n$ diverges, because we can construct infinitely many covers

$$A^{(i)} = \{a_s^{(i)}(n_s^{(i)})\}_{s=1}^{k_i} \quad (i = 1, 2, 3, \dots)$$

such that

$$n_1^{(1)} < \cdots < n_{k_1}^{(1)} < n_1^{(2)} < \cdots < n_{k_2}^{(2)} < \cdots\cdots$$

and hence

$$\sum_{i=1}^{m} \sum_{s=1}^{k_i} \frac{1}{n_s^{(i)}} \geqslant m \qquad \text{for } m = 1, 2, 3, \dots .$$

In 1981 R. Morikawa [Bull. Fac. Lib. Arts; MR 84j:10064] constructed a cover $\{a_s(n_s)\}_{s=1}^{k}$ of $\mathbb{Z}$ with $n_1 = 24 < n_2 < \cdots < n_k$.

By the Davenport-Mirsky-Newman-Rado result, an exact cover other than $\{0(1)\}$ cannot have distinct moduli. Z. W. Sun [J. Algebra 273(2004)] was able to provide a negative answer to an analogy of Erdős' problem for uniform covers of $\mathbb{Z}$ (or uniform covers of groups by cosets of subnormal subgroups).

A Bounded Result of Sun [J. Algebra 273(2004)]**.** *For any $M > 1$, if $\{a_s(n_s)\}_{s=1}^{k}$ $(n_1 \leqslant \cdots \leqslant n_k)$ covers every integer the same number of times and each of the moduli occurs at most M times, then the smallest modulus n_1 has an upper bound in terms of M; namely,*

$$\log n_1 \leqslant \frac{e^{\gamma}}{\log 2} M \log^2 M + O(M \log M \log\log M),$$

where γ is Euler's constant and the O-constant is absolute.

A recent breakthrough on Erdős' problem was given by M. Filaseta, K. Ford, S, Konyagin, C. Pomerance and G. Yu [`arXiv:math.NT/0507374`] who confirmed some conjectures of P. Erdős, R. L. Graham and J. L. Selfridge.

The Filaseta-Ford-Konyagin-Pomerance-Yu Theorem (2005)**.** *For any $L > 0$ there is a constant $c(L) > 0$ such that $\sum_{s=1}^{k} 1/n_s > L$ for any cover $\{a_s(n_s)\}_{s=1}^{k}$ of $\mathbb{Z}$ with $c(L) < n_1 < \cdots < n_k$.*

The following famous conjecture also remains open.

Erdős-Selfridge Conjecture. *Let $A = \{a_s(n_s)\}_{s=1}^{k}$ be a cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$. Then $n_1, \ldots, n_k$ cannot be all odd.*

P. Erdös offers \$25 for a positive answer, and Selfridge offers \$900 for a counterexample.

Let $A = \{a_s(n_s)\}_{s=1}^{k}$ be a cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$. Recently S. Guo and Z. W. Sun [Adv. Appl. Math. 35(2005)] showed that if $n_1, \ldots, n_k$ are odd and squarefree, then $N_A = [n_1, \ldots, n_k]$ has at least 22 prime divisors.

Schinzel's Conjecture. *If $A = \{a_s(n_s)\}_{s=1}^{k}$ is a cover of $\mathbb{Z}$, then $n_i | n_j$ for some $1 \leqslant i \neq j \leqslant k$.*

A. Schinzel [Acta Arith. 13(1967)]: The Erdős-Selfridge conjecture is stronger than and Schinzel's conjecture is weaker than the following proposition: *For any polynomial $P(x) \in \mathbb{Z}[x]$ with $P(0) \neq 0$, $P(1) \neq -1$*

and $P(x) \neq 1$, there exists an infinite arithmetic progression of positive integers such that $x^n + P(x)$ is irreducible over the rational field $\mathbb{Q}$ for every n in the progression.

A Simple Proof of that the ES Conj. implies Schinzel's Conj. (J. Fabrykowski and T. Smotzer, Math. Mag. 78(2005)). Suppose that $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $n_i \nmid n_j$ for all $1 \leqslant i \neq j \leqslant k$. Write $n_s = 2^{\alpha_s} q_s$ with $\alpha_s \in \mathbb{N} = \{0, 1, \dots\}$ and $q_s \in \{1, 3, 5, \dots\}$. Then $\{a_s(q_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with the moduli distinct and odd. If $q_1, \dots, q_k$ are greater than one, then this shows that the Erdős-Selfridge conjecture is false.

Below we assume that $q_t = 1$ where $1 \leqslant t \leqslant k$. Then those q_s with $s \neq t$ are odd, distinct and greater than one. For $s \neq t$, as $n_t = 2^{\alpha_t}$ does not divide $n_s = 2^{\alpha_s} q_s$ we have $\alpha_s < \alpha_t$ and hence $(n_s, n_t) = 2^{\alpha_s}$. Set $I = \{s \neq t : 2^{\alpha_s} \mid a_s - a_t + 1\}$. For $s \in I$, as $(n_s, n_t) \mid a_s - a_t + 1$ there is an $a_s^* \in \mathbb{Z}$ such that

$$a_s^* n_t + a_t - 1 \equiv a_s \pmod{n_s}.$$

For any $x \in \mathbb{Z}$, as $xn_t + a_t - 1 \not\equiv a_t \pmod{n_t}$, for some $s \in I$ we have $xn_t + a_t - 1 \equiv a_s \pmod{n_s}$ and hence $x \equiv a_s^* \pmod{q_s}$. Thus $\{a_s^*(q_s)\}_{s \in I}$ is a cover of $\mathbb{Z}$, which shows that the Erdős-Selfridge conjecture is false. □

Computational Complexity of Covers (L. J. Stockmeyer and A. R. Meyer, 1973). *The question whether a given $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ is co-NP-complete. Thus, $NP = P$ if and only if we can decide whether $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ in polynomial time.*

Now we mention a few applications of covers.

In 1934, by using the cover

$$\{0(2),\ 0(3),\ 1(4),\ 3(8),\ 7(12),\ 23(24)\},$$

Erdős [Summa Brasil. Math. 1950] proved that there is an infinite arithmetic progression of positive odd integers no term of which is of the form $2^n + p$, where n is a positive integer and p is an odd prime. Later R. Crocker [Pacific J. Math. 1971] proved further that there are infinitely many positive odd integers not of the form $2^a + 2^b + p$ where $a, b \in \mathbb{Z}^+$ and p is an odd prime.

In 2001 Z. W. Sun and M. H. Le [Acta Arith. 99(2001)] improved a result of Schinzel and Crocker by showing that for each $n = 4, 5, \ldots$ the number $2^{2^n} - 1$ cannot be written as the sum of two distinct powers of 2 and a prime power.

On the basis of the work of Corcker, and Sun and Le, P. Z. Yuan [Acta Arith. 2004] confirmed a conjecture of the speaker.

A Result Conjectured by Sun and Proved by Yuan. *For any positive integer c, there are infinitely many positive odd integers not of the form $c(2^a + 2^b) + p^\alpha$ where $a, b, \alpha \in \mathbb{N} = \{0, 1, 2, \ldots\}$ and p is an odd prime.*

With help of covers, F. Cohen and J. L. Selfridge [Math. Comput. 1975] proved that not every positive integer is the sum or difference of two prime powers. By introducing a method to avoid a bunch of extra congruences, the speaker [Proc. Amer. Math. Soc. 128(2000)] refined the Cohen-Selfridge result and established the following explicit one:

A Result due to the Efforts of Cohen, Selfridge and Sun. *Let P be the 26-digit prime* 47867742232066880047611079*, and let M be the 29-digit number given by*

$$\prod_{p\leqslant 19} p \times 31 \times 37 \times 41 \times 61 \times 73 \times 97 \times 109 \times 151 \times 241 \times 257 \times 331$$
$$= 66483084961588510124010691590.$$

If $x \equiv P \pmod{M}$, then x is not of the form $\pm p^a \pm q^b$, where p, q are primes and a, b are nonnegative integers.

2. Two Local-Global Theorems on Periodic Maps

In 1958 S. K. Stein [Math. Ann.] conjectured that if $A = \{a_s(n_s)\}_{s=1}^k$ is disjoint (i.e. the k residue classes in A are pairwise disjoint) with $1 < n_1 < \cdots < n_k$ then there exists an integer $x \notin \bigcup_{s=1}^k a_s(n_s)$ with $1 \leqslant x \leqslant 2^k$. In 1965 P. Erdős offered a prize for a proof of his following stronger conjecture.

Erdős' Conjecture. $A = \{a_s(n_s)\}_{s=1}^k$ *forms a cover of $\mathbb{Z}$ if it covers those integers from* 1 *to* 2^k.

The 2^k in Erdős' conjecture is best possible because $\{2^{s-1}(2^s)\}_{s=1}^k$ covers $1, \ldots, 2^k - 1$ but does not cover any multiple of 2^k.

In 1969–1970 R. B. Crittenden and C. L. Vanden Eynden [Bull. Amer. Math. Soc. 1969; Proc. Amer. Math. Soc. 1970] supplied a long and awkward proof of the Erdős conjecture for $k \geqslant 20$, which involves some deep results concerning the distribution of primes.

By a simple trick involving Vandermonde determinants, the speaker [Acta Arith. 72(1995), Trans. Amer. Math. Soc. 348(1996)] deduced the following result which is stronger than Erdős' conjecture.

The First Local-Global Theorem (Sun, 1995-96). *Let $A=\{a_s(n_s)\}_{s=1}^k$ be a finite system of residue classes, and let $m_1,\ldots,m_k$ be integers relatively prime to $n_1,\ldots,n_k$ respectively. Then A covers all the integers at least m times if it covers $|S|$ consecutive integers at least m times, where*

$$S=\bigg\{\bigg\{\sum_{s\in I}\frac{m_s}{n_s}\bigg\}:\ I\subseteq\{1,\ldots,k\}\bigg\}.$$

(*As usual the fractional part of a real number x is denoted by $\{x\}$.*)

`A proof in the case` $m=1$. For any integer x, clearly

$$\begin{aligned}
&x\text{ is covered by }A\\
\iff &e^{2\pi i(a_s-x)m_s/n_s}=1\text{ for some }s=1,\ldots,k\\
\iff &\prod_{s=1}^k\Big(1-e^{2\pi i(a_s-x)m_s/n_s}\Big)=0\\
\iff &\sum_{I\subseteq\{1,\ldots,k\}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}a_sm_s/n_s}\cdot e^{-2\pi ix\sum_{s\in I}m_s/n_s}=0\\
\iff &\sum_{\theta\in S}e^{-2\pi ix\theta}z_\theta=0,
\end{aligned}$$

where

$$z_\theta=\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\sum_{s\in I}m_s/n_s\}=\theta}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}a_sm_s/n_s}.$$

Suppose that A covers $|S|$ consecutive integers $a,a+1,\ldots,a+|S|-1$ where $a\in\mathbb{Z}$. By the above,

$$\sum_{\theta\in S}(e^{-2\pi i\theta})^r(e^{-2\pi ia\theta}z_\theta)=0$$

for $r = 0, 1, \ldots, |S| - 1$. As the determinant $\|(e^{-2\pi i\theta})^r\|_{0\leqslant r<|S|,\,\theta\in S}$ is of Vandermonde's type and hence nonzero, by Cramer's rule we have $z_\theta = 0$ for all $\theta \in S$. Therefore $\sum_{\theta\in S} e^{-2\pi ix\theta} z_\theta = 0$ for all $x \in \mathbb{Z}$, i.e., any $x \in \mathbb{Z}$ is covered by A. $\square$

The characteristic function of a residue class is a periodic arithmatical map. Dirichlet characters are also periodic functions. If an element a in an additive abelian group G has order n, then the map $\psi : \mathbb{Z} \to G$ given by $\psi(x) = xa$ is periodic mod n.

In 2005 the speaker [J. Algebra 293(2005)] published another local-global theorem motivated by his study of covers of $\mathbb{Z}$.

The Second Local-Global Theorem (Sun, 2005). *Let G be any additive abelian group, and let $\psi_1, \ldots, \psi_k$ be maps from $\mathbb{Z}$ to G with periods $n_1, \ldots, n_k \in \mathbb{Z}^+$ respectively. Then the function $\psi = \psi_1 + \cdots + \psi_k$ is constant if $\psi(x)$ equals a constant for $|T|$ consecutive integers x, where*

$$T = \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : r = 0, 1, \ldots, n_s - 1 \right\}.$$

Moreover, there are periodic maps $f_0, \ldots, f_{|T|-1} : \mathbb{Z} \to \mathbb{Z}$ only depending on T such that $\psi(x) = \sum_{r=0}^{|T|-1} f_r(x)\psi(r)$ for all $x \in \mathbb{Z}$.

Here is an interesting consequence.

A Criterion for Exact m-covers [Sun, Math. Res. Lett. 11(2004)]**.** $A = \{a_s(n_s)\}_{s=1}^k$ *covers every integer exactly m times if it covers*

$$\left|\left\{ \frac{r}{n_s} : r = 0, 1, \ldots, n_s - 1;\ s = 1, \ldots, k \right\}\right| \leqslant n_1 + \cdots + n_k - k + 1$$

consecutive integers exactly m times.

In 1965 N. J. Fine and H. S. Wilf [Proc. Amer. Math. Soc. 1965] obtained the following result which follows from the Second Local-Global Theorem in the case $k = 2$ and $G = \mathbb{R}$.

The Fine-Wilf Theorem. *Let $\{f_n\}_{n\geqslant 0}$ and $\{g_n\}_{n\geqslant 0}$ be real sequences with respective periods h_1 and h_2. If $f_n = g_n$ for $0 \leqslant n < h_1+h_2-(h_1, h_2)$, then $f_n = g_n$ for every $n = 0, 1, 2, \ldots$.*

3. The Algebraic Theory of Periodic Arithmetical Functions

For a system $A = \{a_s(n_s)\}_{s=1}^k$, we define its *covering function* $w_A : \mathbb{Z} \to \mathbb{N} = \{0, 1, \ldots\}$ by

$$w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\}|.$$

Clearly $w_A(x)$ is periodic modulo the least common multiple $N_A = [n_1, \ldots, n_k]$. It is easy to check that

$$\frac{1}{N_A}\sum_{x=0}^{N_A-1} w_A(x) = \sum_{s=1}^{k}\frac{1}{n_s}.$$

Two systems $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$ of residue classes are said to be *covering equivalent* ($A \sim B$) if they have the same covering function (i.e., $w_A = w_B$). For example,

$$\{a + jd(nd)\}_{j=0}^{d-1} \sim \{a(d)\} \qquad \text{for any } a \in \mathbb{Z} \text{ and } d, n \in \mathbb{Z}^+.$$

Note that $A = \{a_s(n_s)\}_{s=1}^k$ is an exact cover if and only if $A \sim \{0(1)\}$.

By the proof of the Davenport-Mirsky-Newman-Rado result, if $A = \{a_s(n_s)\}_{s=1}^k \sim \{0(1)\}$ and $0 \leqslant a_s < n_s$ for $s = 1, \ldots, k$ then

$$\sum_{s=1}^{k} \frac{z^{a_s}}{1 - z^{n_s}} = \frac{1}{1-z} = \frac{z^0}{1-z^1} \quad \text{for all } z \text{ with } |z| < 1.$$

In 1985, as an undergraduate student, the speaker posed the following problem to himself.

A Problem of Sun on Covering Equivalence. *Determine all those functions* $f : \bigcup_{n \in \mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ *such that*

$$\sum_{s=1}^{k} f(a_s + n_s\mathbb{Z}) = \sum_{t=1}^{l} f(b_t + m_t\mathbb{Z})$$

whenever $A = \{a_s(n_s)\}_{s=1}^k \sim B = \{b_t(m_t)\}_{t=1}^l$*, i.e.,* $\sum_{s=1}^k f(a_s + n_s\mathbb{Z})$ *only depends on the covering function of* $A = \{a_s(n_s)\}_{s=1}^k$.

Clearly the desired function f should satisfy the functional equation

$$\sum_{j=0}^{n-1} f(a + jd + nd\mathbb{Z}) = f(a + d\mathbb{Z})$$

for any $a \in \mathbb{Z}$ and $d, n \in \mathbb{Z}^+$, since $\{a + jd(nd)\}_{j=0}^{d-1} \sim \{a(d)\}$. The speaker showed in 1989 that **this necessary condition is also sufficient!** (See, Z. W. Sun, Adv. in Math. (China), 1989; J. Algebra, 2001.)

Let M be an abelian group, and let $F(M)$ denote the set of all maps from $\bigcup_{n \in \mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z}$ to an abelian group M. The speaker calls a map $f \in F(M)$ *equivalent* if

$$\sum_{j=0}^{n-1} f(a + jd + nd\mathbb{Z}) = f(a + d\mathbb{Z}) \quad \text{for all } a \in \mathbb{Z} \text{ and } d, n \in \mathbb{Z}^+.$$

Those equivalent $f \in F(M)$ forms a subgroup $E(M)$ of the abelian group $F(M)$ under the functional addition.

Let R be a ring. Then $F(R)$ forms a ring with subring $E(R)$ under the addition and the convolution $*$ defined below:

$$f * g(a + n\mathbb{Z}) = \sum_{r=0}^{n-1} f(r + n\mathbb{Z})g(a - r + n\mathbb{Z}) \ \ (a \in \mathbb{Z},\ n \in \mathbb{Z}^+).$$

If R has the identity 1, then $F(R)$ and $E(R)$ have the identity

$$e(a + n\mathbb{Z}) = \begin{cases} 1 & \text{if } n \mid a, \\ 0 & \text{otherwise.} \end{cases}$$

Equivalent maps are closely related to uniform functions introduced by the speaker [Nanjing Univ. J. Math. Biquarterly 1989].

Definition of Uniform Functions. Let M be an additive abelian group, and F be a map from a subset of $\mathbb{C} \times \mathbb{C}$ into M. If for any ordered pair $\langle x, y \rangle$ in the domain $\mathrm{Dom}(F)$ of F and each positive integer n, we have

$$\left\{ \left\langle \frac{x + r}{n}, ny \right\rangle : r = 0, \ldots, n - 1 \right\} \subseteq \mathrm{Dom}(F)$$

and

$$\sum_{r=0}^{n-1} F\left(\frac{x + r}{n}, ny \right) = F(x, y),$$

then we call F a *uniform function* (to M).

If F is a uniform function to an abelian group M, then for any $\langle x, y \rangle \in \mathrm{Dom}(F)$, the function $f(a + n\mathbb{Z}) = F(\frac{x+a}{n}, ny)$ $(a, n \in \mathbb{Z}$ and $0 \leq a < n)$ is an equivalent map to M. Conversely, if $f \in E(M)$ then the function $F(x, y) = f(xy + y\mathbb{Z})$ $(y \in \mathbb{Z}^+$ and $xy \in \mathbb{Z})$ is a uniform function to M.

If $F(x,y)$ is a uniform function, then so is $F^-(x,y) = F(\{x\},y)$. Uniform functions are rich in examples.

An identity of Hermite is as follows:

$$\sum_{r=0}^{n-1} \left\lfloor x + \frac{r}{n} \right\rfloor = \lfloor nx \rfloor \qquad \text{for } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}^+.$$

This shows that $[\ ](x,y) = \lfloor x \rfloor$ is a uniform function.

Raabe's multiplication formula for Bernoulli polynomials

$$\sum_{r=0}^{n-1} B_m\left(x + \frac{r}{n}\right) = n^{1-m} B_m(nx) \quad (n = 1, 2, 3, \dots)$$

indicates that the function $b_m(x,y) = B_m(x)y^{m-1}$ is a uniform map to $\mathbb{C}$. Note that $b_0(x,y) = 1/y,\ b_1(x,y) = x - 1/2$ and $b_1(x,y) - b_1^-(x,y) = \lfloor x \rfloor$.

For $s \in \mathbb{C}$ with $Re(s) > 1$, we can easily verify that $\zeta_s(x,y) = y^{-s}\zeta(s,x)$ is a uniform function, where $x, y > 0$ and $\zeta(s,x) = \sum_{n=0}^{\infty} 1/(n+x)^s$ (Hurwitz zeta function).

In 1989 the speaker observed that Gauss' multiplication formula

$$\prod_{r=0}^{n-1} \Gamma\left(z + \frac{r}{n}\right) = (2\pi)^{(n-1)/2} n^{1/2-nz} \Gamma(nz) \quad (n \in \mathbb{Z}^+ \text{ and } -nz \notin \mathbb{N})$$

is actually equivalent to the following statement: $\gamma(x,y)$ is a uniform function to the multiplicative group $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, where

$$\gamma(x,y) = \Gamma(x)y^{x-1/2}/\sqrt{2\pi} \qquad \text{for } x \neq 0, -1, -2, \dots \text{ and } y > 0.$$

The speaker also noted many other examples of uniform functions such as $y^{-1}\cot \pi x$ with $x \notin \mathbb{Z}$ and $y \neq 0$.

In view of various different kinds of uniform functions, it seems that we can not give a unified form for equivalent functions. Despite of this, the speaker proved [Adv. Math. China, 1989; J. Algebra, 240(2001)] the following result.

Universal Representation of Equivalent Maps to $\mathbb{C}$. *A function $f : \bigcup_{n\in\mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}$ is equivalent (i.e., $f \in E(\mathbb{C})$) if and only if f has the following form:*

$$f(a+n\mathbb{Z}) = \frac{1}{n}\sum_{m=0}^{n-1}\psi\left(\frac{m}{n}\right)e^{2\pi i\frac{m}{n}a} = \sum_{\alpha\in[0,1)\cap\mathbb{Q}}\psi(\alpha)\rho_\alpha(a+n\mathbb{Z}),$$

where ψ is a function from $\mathbb{Q}\cap[0,1)$ to $\mathbb{C}$ and

$$\rho_\alpha(a+n\mathbb{Z}) = \begin{cases} e^{2\pi i\alpha a}/n & \textit{if } \alpha n \in \mathbb{Z}, \\ 0 & \textit{otherwise.} \end{cases}$$

For each $f \in E(\mathbb{C})$, there is a unique ψ with this property. Moreover, the ring $E(\mathbb{C})$ is isomorphic to the ring of functions from $\mathbb{Q}\cap[0,1)$ to C with the usual addition and multiplication.

From this and the well-known Ramanujan sum, we have the following example of equivalent functions:

$$\Phi(a+n\mathbb{Z}) = \begin{cases} 1/\varphi(n) & \text{if } (a,n)=1, \\ 0 & \text{otherwise,} \end{cases}$$

where φ is Euler's totient function.

Let M be an additive abelian group. A triple $\langle\lambda, a, n\rangle$ with $\lambda \in M$, $n \in \mathbb{Z}^+$ and $a \in \{0,\dots,n-1\}$, can be viewed as the residue class $a(n)$ associated with *weight* λ. For

$$\mathcal{A} = \{\langle\lambda_s, a_s, n_s\rangle\}_{s=1}^{k} \text{ with } \lambda_s \in M,$$

the *covering map*

$$w_{\mathcal{A}}(x) = \sum_{\substack{s=1 \\ x \in a_s(n_s)}}^{k} \lambda_s \quad (x \in \mathbb{Z}),$$

is periodic modulo the least common multiple $N_{\mathcal{A}} = [n_1, \ldots, n_k]$. Two such systems $\mathcal{A}_1$ and $\mathcal{A}_2$ are said to be equivalent (we denote this by $\mathcal{A}_1 \sim \mathcal{A}_2$) if they have the same covering map. The set

$$P(M) = \{\text{periodic maps from } \mathbb{Z} \text{ to } M\}$$

just consists of such covering maps, because any $\psi \in P(M)$ is the covering map of the system $\{\langle \psi(r), r, n \rangle\}_{r=0}^{n-1}$ where n is a positive period of ψ. Thus the concepts of cover and covering function help us understand periodic arithmetical functions.

Let R be a commutative ring with identity and M, N be R-modules. The set of all R-module homomorphisms from M to N forms an R-module in a natural way and we denote it by $\mathrm{Hom}_R(M, N)$. Those $T \in \mathrm{Hom}_R(P(M), P(N))$ commute with the shift operator E (i.e. $T(E\psi) = E(T(\psi))$ where $E\psi(x) = \psi(x+1)$) form a submodule $\mathrm{Hom}'_R(P(M), P(N))$ of $\mathrm{Hom}_R(P(M), P(N))$.

The importance of equivalent maps can be seen from the following theorem due to the speaker [J. Algebra, 240(2001)].

Algebraic Roles of Equivalent Maps. *Let R be a ring with identity, and let M be a left R-module.*

(i) *If R is commutative, then the R-module* $\mathrm{Hom}'_R(P(R), P(M))$ *is isomorphic to the R-module $E(M)$.*

(ii) *$P(M)$ forms an $E(R)$-module with respect to the natural addition and the scalar multiplication $\circ$ defined below:*

$$f \circ \psi(x) = \sum_{r=0}^{n-1} f(r+n\mathbb{Z})\psi(x-r) \quad \text{where } n \in \mathbb{Z}^+ \text{ is a period of } \psi.$$

(For $f \in E(R)$ the last sum does not depend on the choice of a period n of ψ.)

(iii) *If R is commutative, then $\mathrm{Hom}'_R(P(R), P(R))$ consists of those $T_f : \psi \mapsto f \circ \psi$ with $f \in E(R)$, and the ring $\mathrm{Hom}'_R(P(R), P(R))$ is isomorphic to $E(R)$.*

Here is a basic theorem on covering equivalence [Z. W. Sun, Nanjing Univ. J. Math. Biquarterly, 1989].

Fundamental Theorem on Covering Equivalence. *Let M be a left R-module where R is a ring with identity. Let F be a map to M with* $\mathrm{Dom}(F) \subseteq \mathbb{C} \times \mathbb{C}$ *such that*

$$\left\{\left\langle \frac{x+r}{n}, ny\right\rangle : r = 0, 1, \cdots, n-1\right\} \subseteq \mathrm{Dom}(F)$$

for any $\langle x, y\rangle \in \mathrm{Dom}(F)$ and $n \in \mathbb{Z}^+$. Then the following two statements are equivalent:

(a) *F is a uniform function to M.*

(b) *Whenever*

$$\{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^k \sim \{\langle \mu_t, b_t, m_t\rangle\}_{t=1}^l, \text{ i.e., } \sum_{\substack{1\leqslant s\leqslant k \\ x\in a_s(n_s)}} \lambda_s = \sum_{\substack{1\leqslant t\leqslant l \\ x\in b_t(m_t)}} \mu_t \text{ for } x \in \mathbb{Z}$$

(with $\lambda_s, \mu_t \in R$, $0 \leqslant a_s < n_s$ and $0 \leqslant b_t < m_t$), we have

$$\sum_{s=1}^{k} \lambda_s F\left(\frac{x+a_s}{n_s}, n_s y\right) = \sum_{t=1}^{l} \mu_t F\left(\frac{x+b_t}{m_t}, m_t y\right) \quad \textit{for } \langle x, y\rangle \in \mathrm{Dom}(F).$$

Proof. (b) implies (a) because $\{r(n)\}_{r=0}^{n-1} \sim \{0(1)\}$.

(a)⇒(b). Suppose that $\{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^{k} \sim \{\langle \mu_t, b_t, m_t\rangle\}_{t=1}^{l}$. Then $\sum_{s=1}^{k} \lambda_s \psi_s(z) = \sum_{t=1}^{l} \mu_t \chi_t(z)$ where $\psi_s(z) = 1$ or 0 according to whether $n_s \mid z + a_s$ or not, and $\chi_t(z) = 1$ or 0 according to whether $m_t \mid z + b_t$ or not. Fix $\langle x, y\rangle \in \mathrm{Dom}(F)$ and let $f(a + n\mathbb{Z}) = F(\frac{x+a}{n}, ny)$ $(0 \leq a < n)$. Note that

$$T_f(\psi_s)(z) = f \circ \psi_s(z) = \sum_{r=0}^{n_s-1} f(r + n_s\mathbb{Z})\psi_s(z - r) = f(z + a_s + n_s\mathbb{Z}).$$

Similarly, $T_f(\chi_t)(z) = f(z + b_t + m_t\mathbb{Z})$. So

$$\sum_{s=1}^{k} \lambda_s f(a_s + n_s\mathbb{Z}) = \sum_{t=1}^{l} \mu_t f(b_t + m_t\mathbb{Z}).$$

We are done. □

The speaker's first proof (1989) of the Fundamental Theorem is by induction. A direct proof of a slightly extended version was given by him in [Acta Arith. 97(2001)].

In view of the Fundamental Theorem on Covering Equivalence, the functional equation satisfied by uniform functions can be extended via covering equivalence. Here is an example given in [Z. W. Sun, Nanjing Univ. J. Math. Biquarterly, 1989] as a consequence of the Fundamental Theorem on Covering Equivalence.

An Extension of Gauss' Multiplication Formula. *Whenever* $A = \{a_s(n_s)\}_{s=1}^k \sim B = \{b_t(m_t)\}_{t=1}^l$ $(0 \le a_s < n_s$ *and* $0 \le b_t < m_t)$, *we have*

$$\prod_{s=1}^k \Gamma\left(\frac{x+a_s}{n_s}\right) n_s^{(x+a_s)/n_s-1/2} = (2\pi)^{(k-l)/2} \prod_{t=1}^l \Gamma\left(\frac{x+b_t}{m_t}\right) m_t^{(x+b_t)/m_t-1/2}$$

for $x \ne 0, -1, -2, \cdots$.

Proof. This is because $\gamma(x,y) = \Gamma(x)y^{x-1/2}/\sqrt{2\pi})$ is a uniform function to the $\mathbb{Z}$-module $\mathbb{C}^*$ with the scalar multiplication $m \circ z = z^m$. $\square$