A talk given at the Confer. on *Diophantine Analysis and Related Topics*
(Wuhan, March 10-13, 2016)

Some New Diophantine Problems

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March 11, 2016

Abstract

In this talk we introduce various new diophantine problems and related conjectures made by the speaker. It consists of four parts:

(I) Exponential Diophantine Equations,

(II) Universal Representations involving Mixed Powers,

(III) Problems with Related Numbers Distinct,

(IV) On Representations of Positive Rational Numbers.

The problems here might interest number theorists and stimulate further research.

Hilbert's Tenth Problem

In 1900 D. Hilbert asked for an effective algorithm to test whether an arbitrary polynomial equation

$$P(x_1, \ldots, x_n) = 0$$

(with integer coefficients) has solutions over the ring $\mathbb{Z}$ of the integers. At that time the exact meaning of algorithm was not known.

The theory of computability was born in the 1930's.

On the basis of the important work of M. Davis, H. Putnam and J. Robinson [Ann. Math., 1961], in 1970 Matijasevič finally solved Hilbert's Tenth Problem negatively.

In the speaker's PhD thesis *Further Results on Hilbert's Tenth Problem*, he established the following theorem.

11 Unkownns Theorem (S., 1992) There is no effective algorithm to decide whether an arbitrary polynomial equation $P(x_1, \ldots, x_{11}) = 0$ (with integer coefficients) in 11 unknowns has integral solutions.

Part I. Exponential Diophantine Equations

A conjecture related to Fermat's Last Theorem

In 1936 K. Mahler discovered that

$$(9t^3+1)^3+(9t^4)^3-(9t^4+3t)^3=1.$$

Clearly, $|1^n+1^n-2^n|=2^n-2$ for $n=4,5,6,\ldots$, and

$$13^5+16^5-17^5=371293+1048576-1419857=12<2^5-2.$$

Based on our computation via `Mathematica`, the speaker posed the following refinement of Fermat's Last Theorem.

Conjecture (S., Sept. 24-25, 2015). (i) For any integers $n>3$ and $x,y,z>0$ with $\{x,y\}\neq\{1,z\}$, we have $|x^n+y^n-z^n|\geqslant 2^n-2$, unless $n=5$, $\{x,y\}=\{13,16\}$ and $z=17$.

(ii) For any integers $n>3$ and $x,y,z>0$ with $z\notin\{x,y\}$, there is a prime p with $x^n+y^n<p<z^n$ or $z^n<p<x^n+y^n$, unless $n=5$, $\{x,y\}=\{13,16\}$ and $z=17$.

(iii) For any integers $n>3$, $x>y\geqslant 0$ and $z>0$ with $x\neq z$, there always exists a prime p with $x^n-y^n<p<z^n$ or $z^n<p<x^n-y^n$.

On the equations $x^n + n = y^m$ and $x^n - n = y^m$

Conjecture (S., Dec. 2, 2013). The only solutions to the diophantine equation

$$x^n + n = y^m \quad \text{with } m, n, x, y > 1$$

are

$$5^2 + 2 = 3^3 \quad \text{and} \quad 5^3 + 3 = 2^7.$$

Also, the only solutions to the diophantine equation

$$x^n - n = y^m \quad \text{with } m, n, x, y > 1$$

are

$$2^5 - 5 = 3^3 \quad \text{and} \quad 2^7 - 7 = 11^2.$$

This was motivated by the speaker's following result.

Theorem (S., `arXiv:1312.1166`) Let a and $m > 0$ be integers. For any integer b relatively prime to m , the set

$$\{a^n + bn : \ n = 1, \ldots, m^2\}$$

contains a complete system of residues modulo m.

Write $n = k + m$ with $2^k + m$ prime

Conjecture (S., Nov. 10, 2013). Any integer $n > 1$ can be written as $k + m$ with $2^k + m = 2^k - k + n$ prime, where k and m are positive integers.

Examples: $8 = 3 + 5$ with $2^3 + 5 = 13$ prime, and $53 = 20 + 33$ with $2^{20} + 33 = 1048609$ prime.

In 2013, we verified the conjecture for n up to 2×10^6 except for $n = 1657977$.

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On August 30, 2015 we found that $1657977 = 205494 + 1452483$ with $2^{205494} + 1452483$ a prime of 61860 decimal digits.

Until today, we have verified the conjecture for all $n \leqslant 10^7$. On Nov. 16, 2015, we found that $5120132 = 250851 + 4869281$ with

$$2^{250851} + 4869281$$

a prime of 75514 decimal digits!

Remark. After Prof. B. Poonen at MIT saw the conjecture, he provided certain heuristic arguments to support it.

Write $n = p + (2^k - k) + (2^m - m)$

In 1971 R. Crocker [Pacific J. Math. 36(1971)] proved that there are infinitely many positive odd integers not of the form $p + 2^k + 2^m$ with p prime and $k, m \in \mathbb{Z}^+$.

Conjecture (S., Nov. 23, 2013). Any integer $n > 3$ can be written in the form $p + (2^k - k) + (2^m - m)$, where p is a prime, and k and m are positive integers.

Remark. The numbers $2^k - k$ $(k = 1, 2, 3, \ldots)$ are very sparse. The conjecture has been verified for n up to 10^{10}.

Examples. $28 = 11 + (2^3 - 3) + (2^4 - 4)$ with 11 prime, and $94 = 31 + (2^3 - 3) + (2^6 - 6)$ with 31 prime.

Conjecture (S., May 10, 2014). For any integer $n > 1$ with $n \neq 5, 16$, the number $2^n - n$ has a prime divisor p not dividing any $2^k - k$ with $0 < k < n$.

Remark. A classical theorem of A. S. Bang (1886) asserts that if $n > 1$ is different from 6 then $2^n - 1$ has a prime divisor which does not divide any $2^k - 1$ with $0 < k < n$.

Perfect powers in some familiar sequences

An integer $n > 1$ is called a *perfect power* if $n = x^m$ for some integers $x > 1$ and $m > 1$.

Theorem (Y. Bugeaud, M. Mignotte & S. Siksek [Annals of Math., 2006]). The only perfect powers in the Fibonacci sequence are $F_6 = 2^3$ and $F_{12} = 12^2$.

Conjecture (S., Dec. 3, 2013). (i) For the partition function, $p(n)$ is never a perfect power. (We have verified this for $n \leqslant 15000$.)

(ii) No Bell number B_n is a perfect power. (We have verified this for $n \leqslant 600$.)

(iii) No Franel number $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ is a perfect power.

(iv) No Apéry number $A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$ is a perfect power.

Conjecture (S. Sept. 30, 2015). $\pi(2^3) = 4$ is the only perfect power in the sequence $\pi(2^n)$ $(n = 1, 2, 3, \ldots)$, where $\pi(x)$ denotes the number of primes not exceeding x

The Diophantine equation $\pi(x^m) = y^n$

As

$$\int_1^N \frac{1}{n} x^{1/n-1} dx = N^{1/n} - 1 \sim |\{1 \leqslant m \leqslant N : \ m \text{ is an } n\text{-th power}\}|,$$

roughly speaking we may think that an arbitrarily given positive integer x is an n-th power with *'probability'* $1/(nx^{1-1/n})$. In view of this, the heuristic arguments led the speaker to pose the following conjecture.

Conjecture (S., Sept. 24, 2015). Consider the diophantine equation

$$\pi(x^m) = y^n \ \text{ with } m > 1, \ n > 1, \ x > 0 \text{ and } y > 0. \qquad (*)$$

(i) In the case $m = n = 2$, $(*)$ has infinitely many solutions.

(ii) When $\{m, n\} = \{2, 3\}$, $(*)$ only has the following solutions:

$$\pi(89^2) = 10^3, \ \pi(2^3) = 2^2, \ \pi(3^3) = 3^2,$$
$$\pi(14^3) = 20^2, \ \pi(1122^3) = 8401^2.$$

(iii) $(*)$ has no solution with $m + n > 5$.

On $\pi(x^n)+\pi(y^n)=\pi(z^n)$ and $\pi(x^n)\pi(y^n)=\pi(z^n)$

Conjecture 1 (S., Sept. 22, 2015) (i) There are infinitely many distinct primes p, q, r such that $\pi(p^2)=\pi(q^2)+\pi(r^2)$.

(ii) The diophantine equation

$$\pi(x^3)+\pi(y^3)=\pi(z^3) \text{ with } 0<x\leqslant y<z$$

has *infinitely* many solutions.

(iii) For each $n=4,5,6,\ldots$ the diophantine equation

$$\pi(x^n)+\pi(y^n)=\pi(z^n) \ \text{ with } 0<x\leqslant y<z$$

has no solution.

Conjecture 2 (S., Sept. 23, 2015) (i) The diophantine equation

$$\pi(x^2)\pi(y^2)=\pi(z^2) \ \text{ with } 0<x<y<z$$

has infinitely many solutions.

(ii) For each $n=3,4,5,\ldots$ the diophantine equation

$$\pi(x^n)\pi(y^n)=\pi(z^n) \ \text{ with } 0<x\leqslant y<z$$

has no solution.

25 solutions to the equation $\pi(x^3) + \pi(y^3) = \pi(z^3)$

We have found 25 solutions (x, y, z) to the diophantine equation

$$\pi(x^3) + \pi(y^3) = \pi(z^3) \quad \text{with } 0 < x \leqslant y < z.$$

They are

$$(3, 3, 4), (54, 80, 89), (63, 85, 97), (27, 100, 101), (47, 106, 110),$$
$$(80, 190, 196), (122, 223, 237), (229, 335, 372), (151, 401, 410),$$
$$(263, 1453, 1457), (1302, 2382, 2522), (879, 3301, 3327),$$
$$(2190, 4011, 4244), (498, 4434, 4437), (3792, 4991, 5684),$$
$$(4496, 4584, 5777), (3113, 7442, 7647), (5239, 8090, 8827),$$
$$(6904, 8116, 9608), (5659, 8910, 9680), (5323, 9187, 9807),$$
$$(5527, 10168, 10744), (7395, 17050, 17563),$$
$$(11637, 17438, 19146), (4486, 21125, 21208).$$

Part II. Universal Representations involving Mixed Powers

Mixed sums of squares and triangular numbers

Gauss-Legendre Theorem:
$\{x^2 + y^2 + z^2 : \ x, y, z \in \mathbb{N}\} = \mathbb{N} \setminus \{4^k(8l + 7) : \ k, l \in \mathbb{N}\}$.

Euler's Observation:

$$8n + 1 = (2x)^2 + (2y)^2 + (2z + 1)^2 \text{ with } x \equiv y \pmod 2$$

$$\Longrightarrow n = \frac{x^2 + y^2}{2} + \frac{z(z+1)}{2} = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + \frac{z(z+1)}{2}.$$

Lionnet's Assertion (proved by Lebesgue & Réalis in 1872). Any $n \in \mathbb{N}$ is the sum of two triangular numbers and a square.

B. W. Jone and G. Pall [Acta Math. 1939]. Every $n \in \mathbb{N}$ is the sum of a square, an *even* square and a triangular number.

Theorem (i) [S., Acta Arith. 2007] Any $n \in \mathbb{N}$ is the sum of an *even* square and two triangular numbers.

(ii) (Conjectured by S. and proved by B. K. Oh and S. [JNT, 2009]). Any positive integer n can be written as the sum of a square, an *odd* square and a triangular number.

Modify sums of three squares

Conjecture (S., Oct. 2, 2015). Any positive integer n can be written as $x^2 + y^2 + p(p \pm 1)/2$ with p prime and $x, y \in \mathbb{Z}$.

For example, 97 has a unique representation $97 = 1^2 + 9^2 + 5(5+1)/2$ with 5 prime, and 538 has a unique representation $538 = 3^2 + 8^2 + 31(31-1)/2$ with 31 prime.

It is easy to see that all the numbers

$$\varphi(n^2) = n\varphi(n) \ (n = 1, 2, 3, \ldots)$$

are pairwise distinct, where φ is Euler's totient function.

The speaker's following conjecture seems novel and curious.

Conjecture (S., Oct. 1, 2015). Any integer $n > 1$ can be written as $x^2 + y^2 + \varphi(z^2)$, where x, y and z are integers with $0 \leqslant x \leqslant y$ and $z > 0$ such that y or z is prime.

For example, 13 has a unique representation $13 = 1^2 + 2^2 + \varphi(4^2)$ with 2 prime, and 94415 has a unique representation $94415 = 115^2 + 178^2 + \varphi(223^2)$ with 223 prime.

Representations involving mixed powers

In Jan. 2015, using q-series the speaker proved that any positive integer can be represented as the sum of two squares and a *positive* triangular number. Below is a variant of this involving cubes.

Conjecture (S., Oct. 3, 2015). Any positive integer n can be written as the sum of a nonnegative cube, a square and a positive triangular number.

We have verified this for $n \leqslant 10^5$. For example, 306 has a unique representation: $306 = 1^3 + 13^2 + 16 \times 17/2$.

In contrast with Largrange's theorem on sums of four squares, the following conjecture seems difficult.

Conjecture (S., Oct. 3, 2015). Any $n \in \mathbb{N}$ can be written as

$$w^2 + x^3 + y^4 + 2z^4 \text{ with } w, x, y, z \in \mathbb{N}.$$

Exzample. 1248 has a unique required representation

$$1248 = 31^2 + 5^3 + 0^4 + 2 \times 3^4.$$

Write integers as $x^a + y^b - z^c$ with $x, y, z \in \mathbb{Z}^+$

Conjecture (S., Dec. 2015). If $\{a, b, c\}$ is among the multisets

$\{2, 2, p\}$ (p is prime or a product of some primes congruent to 1 mod 4),

$$\{2,3,3\},\ \{2,3,4\} \text{ and } \{2,3,5\},$$

then any integer m can be written as $x^a + y^b - z^c$, where x, y and z are positive integers.

We have verified that $\{x^4 - y^3 + z^2 :\ x, y, z \in \mathbb{Z}^+\}$ contains all integers m with $|m| \leqslant 10^5$. For example,

$$0 = 4^4 - 8^3 + 16^2,\quad -1 = 1^4 - 3^3 + 5^2,$$

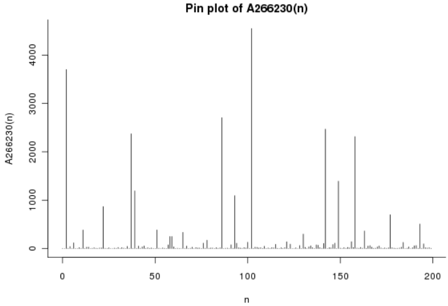
$$-20 = 32^4 - 238^3 + 3526^2,\quad 11019 = 4325^4 - 71383^3 + 3719409^2.$$

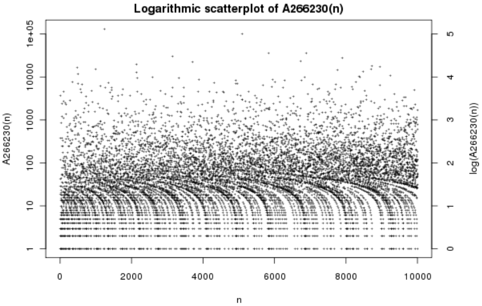
Other Examples.

$$394 = 2283^3 + 128^4 - 110307^2,\quad 570 = 546596^2 + 8595^3 - 983^4,$$

$$445 = 9345^3 + 34^5 - 903402^2,\quad 435 = 475594653^2 + 290845^3 - 3019^5.$$

Fountain Graph for $a(n) = \min\{x \in \mathbb{Z}^+ : \ n + x^2 = y^3 + z^3 \text{ for some } y, z \in \mathbb{Z}^+\}$





Why is the conjecture reasonable?

If $m \equiv 6 \pmod 8$, then for $a = 4, 6, 8, \ldots$ we have $m + x^a \equiv 6, 7 \pmod 8$ and hence $m + x^a$ can never be the sum of two squares.

For any prime $p \equiv 3 \pmod 4$ and odd integer $n > 1$, we have proved that $x^{pn} + (2p)^p$ cannot be the sum of two squares.

Heuristic Arguments (not rigorous): As

$$\{1 \leqslant n \leqslant N : \ n = x^a + y^b \text{ for some } x, y \in \mathbb{Z}^+\}$$

$$\sim C_0 N^{1/a+1/b} \sim C_0 \int_1^N \left(\frac{1}{a} + \frac{1}{b}\right) t^{1/a+1/b-1} dt,$$

we think that $t \in \mathbb{Z}^+$ has the form $x^a + y^b$ $(x, y \in \mathbb{Z}^+)$ with *'probability'* $C_1 t^{1/a+1/b-1}$ (where C_0 and C_1 are positive constants). Note that the series $\sum_{z=1}^{\infty} (m + z^c)^{1/a+1/b-1}$ diverges if $c(1 - 1/a - 1/b) < 1$. Thus, when $1/a + 1/b + 1/c > 1$, we might expect that there are infinitely many triples (x, y, z) of positive integers with $m = x^a + y^b - z^c$. If $2 \leqslant a < b \leqslant c$, then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} > 1 \iff a = 2, \ b = 3, \text{ and } c \in \{3, 4, 5\}.$$

Part III. Problems with Related Numbers Distinct

A conjecture involving primes with prime index

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ let p_n denote the n-th prime.

Conjecture (S., Oct. 17, 2015). All the quotients

$$\frac{p_q + 2}{q + 2} \quad \text{with } q \text{ prime}$$

are pairwise distinct!

Remark. We have verified that all the ratios $(p_q + 2)/(q + 2)$ with q prime and $q < 5 \times 10^6$ are indeed pairwise distinct. For $k = 3, 6, 7, 8, 9$, we also conjecture that the numbers $(p_q + k)/(q + k)$ with q prime are pairwise distinct. See `http://oeis.org/A263427` for related data.

A conjecture motivated by perfect numbers

Recall that a positive integer n is a prefect number if and only if

$$\frac{\sigma(n)}{n} = \sum_{d|n} \frac{1}{d} = 2.$$

Conjecture (S., Oct. 14, 2015). (i) For any integer $k > 1$, all the numbers

$$\sum_{d|n} \frac{1}{d^k} \quad (n = 1, 2, 3, \ldots)$$

have pairwise distinct fractional parts.

(ii) For any positive integers k and m, all the numbers

$$\sum_{d|n} \frac{1}{(d+m)^k} \quad (n = 1, 2, 3, \ldots)$$

have pairwise distinct fractional parts, and none of them is an integer!

Remark. We really wonder why $\sum_{d|n} \frac{1}{d+1} \notin \mathbb{Z}$.

A conjecture on $\pi(n^k)/n^k$

As $\pi(x) \sim x/\log x$, we see that

$$\lim_{n\to\infty} \frac{\pi(n^k)}{n^k} = 0 \quad \text{for all } k = 1, 2, 3, \ldots .$$

Conjecture (S., Oct. 14, 2015).

(i) All the numbers $\pi(n^2)/n^2$ $(n = 1, 2, 3, \ldots)$ are pairwise distinct. Moreover, we have

$$\frac{\pi(n^2)}{n^2} > \frac{\pi((n+1)^2)}{(n+1)^2} \quad \text{for all } n > 15646.$$

(ii) For any integer $k > 2$ the sequence $\pi(n^k)/n^k$ $(n = 2, 3, \ldots)$ is strictly decreasing.

A conjecture on $\varphi(n)\pi(n^2)$ and $\sigma(n)\pi(n^2)$

Recall that

$$\varphi(n) = |\{1 \leqslant a \leqslant n : (a, n) = 1\}| \text{ and } \sigma(n) = \sum_{d|n} d.$$

Conjecture (S., Oct. 14, 2015). All the numbers

$$\varphi(n)\pi(n^2) \quad (n = 1, 2, 3, \ldots)$$

are pairwise distinct. Also, all the numbers

$$\sigma(n)\pi(n^2) \quad (n = 1, 2, 3, \ldots)$$

are pairwise distinct.

Remark. We have verified that all the numbers $\varphi(n)\pi(n^2)$ (or $\sigma(n)\pi(n^2)$) ($n = 1, 2, \ldots, 4 \times 10^5$) are indeed pairwise distinct!

A conjecture related to the Erdős-Niven theorem

In 1918 J. Kürschak proved that for any integers $k \geqslant j > 1$ the number $1/j + \ldots + 1/k$ is not an integer. In 1946 P. Erdős and I. Niven [Bull. AMS 52(1946)] proved that all the numbers $1/j + \ldots + 1/k$ with $1 \leqslant j \leqslant k$ are pairwise distinct.

Conjecture (i) (S., Sept. 9, 2015) If $1/j + \ldots + 1/k$ and $1/s + \ldots + 1/t$ have the same fractional part with

$$0 < \min\{2, k\} \leqslant j \leqslant k,\ 0 < \min\{2, t\} \leqslant s \leqslant t \text{ and } j \leqslant s,$$

but the ordered pairs (j, k) and (s, t) are different, then (j, k, s, t) is among the following quadruples:

$$(2, 6, 4, 5),\ (2, 4, 12, 12),\ (2, 11, 5, 12),\ (3, 20, 7, 19).$$

(ii) (S., Sept. 11, 2015) Let $a > b \geqslant 0$ and $m > 0$ be integers with $\gcd(a, b) = 1 < \max\{a, m\}$. Then the numbers

$$\sum_{i=j}^{k} \frac{1}{(ai - b)^m} \quad \text{with } 1 \leqslant j \leqslant k \text{ and } (j > 1 \text{ if } k > a - b = 1)$$

have pairwise distinct fractional parts.

A conjecture involving consecutive primes

Conjecture (S., Sept. 9, 2015). Let m be any positive integer. Then all the rational numbers

$$\sum_{i=j}^{k} \frac{1}{(p_i - 1)^m} \quad \text{with } 1 \leqslant j \leqslant k$$

are pairwise distinct! If

$$\sum_{i=j}^{k} \frac{1}{(p_i - 1)^m} \quad \text{and} \quad \sum_{r=s}^{t} \frac{1}{(p_r - 1)^m}$$

have the same fractional part with

$$0 < \min\{2, k\} \leqslant j \leqslant k,\ 0 < \min\{2, t\} \leqslant s \leqslant t \text{ and } j \leqslant s,$$

but the ordered pairs (j, k) and (s, t) are different, then $m = 1$ and $(j, k, s, t) \in \{(2, 6, 5, 5), (2, 5, 18, 18), (2, 17, 6, 18)\}$.

Part IV. On Representations of Positive Rational Numbers

A conjecture on unit fractions involving primes

It is well known that any positive rational number can be written as the sum of some distinct unit fractions (via the simple fact $1/n = 1/(n+1) + 1/(n(n+1))$). For example,

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3} = \frac{1}{3} + \left(\frac{1}{4} + \frac{1}{3 \times 4}\right) = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}.$$

As Euler proved, the series $\sum_p 1/p$ diverges, where p runs over all the primes.

Conjecture (S., Sept. 9, 2015). Let r be any positive rational number. For $d = \pm 1$, there are finitely many distinct primes $q_1, \ldots, q_k$ such that $r = \sum_{j=1}^{k} 1/(q_j + d)$.

Remark. The speaker has announced a prize of 1000 US dollars for the first correct proof. The conjecture has been verified for all those rational numbers $r \in (0, 1]$ with denominators not exceeding 100. (`http://math.nju.edu.cn/~zwsun/UnitFraction.pdf`.)

Examples:

$$1 = \frac{1}{2-1} = \frac{1}{3-1} + \frac{1}{5-1} + \frac{1}{7-1} + \frac{1}{13-1},$$

$$1 = \frac{1}{2+1} + \frac{1}{3+1} + \frac{1}{5+1} + \frac{1}{7+1} + \frac{1}{11+1} + \frac{1}{23+1},$$

$$\begin{aligned} \frac{1}{19} =& \frac{1}{37-1} + \frac{1}{137-1} + \frac{1}{191-1} + \frac{1}{229-1} \\ & + \frac{1}{331-1} + \frac{1}{397-1} + \frac{1}{761-1} + \frac{1}{1021-1} \\ =& \frac{1}{37+1} + \frac{1}{107+1} + \frac{1}{227+1} + \frac{1}{239+1} \\ & + \frac{1}{311+1} + \frac{1}{359+1} + \frac{1}{701+1} + \frac{1}{911+1} \text{ (S.).} \end{aligned}$$

$$\begin{aligned} \frac{6}{29} =& \frac{1}{7-1} + \frac{1}{29-1} + \frac{1}{281-1} + \frac{1}{2437-1} + \frac{1}{2521-1} + \frac{1}{7309-1} \\ =& \frac{1}{5+1} + \frac{1}{29+1} + \frac{1}{271+1} + \frac{1}{509+1} + \frac{1}{1217+1} \\ & + \frac{1}{4079+1} + \frac{1}{7307+1} + \frac{1}{17747+1} \text{ (Q.-H. Hou, Nov. 6, 2015).} \end{aligned}$$

Representations involving the prime-counting function

Conjecture (S., July 3, 2015) The set

$$\left\{\frac{m}{n} :\ m,n\in\mathbb{Z}^+ \text{ and } p_m+p_n \text{ is a square}\right\}$$

contains any positive rational number r.

Remark. We have verified this for all those rational numbers $r=a/b$ with $a,b\in\{1,\ldots,200\}$. For example, $2=20/10$ with $p_{20}+p_{10}=71+29=10^2$ a square.

Conjecture (S., July 3, 2015) Any positive rational number r can be written as m/n with $m,n\in\mathbb{Z}^+$ such that $\pi(m)\pi(n)$ is a positive square.

Remark. We have verified this for $r=a/b$ with $a,b\in\{1,\ldots,60\}$. For example, $49/58=1076068567/1273713814$ with

$$\pi(1076068567)\pi(1273713814)=54511776\times63975626=59054424^2.$$

Representations involving $p(n)$, $\varphi(n)$ and $\sigma(n)$

Conjecture (S., July 2, 2015). Any positive rational number r can be written as m/n with $m, n \in \mathbb{Z}^+$ such that $p(m)^2 + p(n)^2$ is prime, where $p(\cdot)$ is the partition function.

Remark. We have verified this for those rational numbers $r = a/b$ with $a, b \in \{1, \ldots, 100\}$. For example, $4/5 = 124/155$, and

$$\begin{aligned} p(124)^2 + p(155)^2 =& 2841940500^2 + 66493182097^2 \\ =& 4429419891190341567409 \end{aligned}$$

is prime.

Conjecture (S., July 8, 2015) Any positive rational number r can be written as m/n with $m, n \in \mathbb{Z}^+$ such that $\varphi(m)$ and $\sigma(n)$ are both squares.

Remark. We have verified this conjecture for all those rational numbers $r = a/b$ with $a, b \in \{1, \ldots, 150\}$. For example, $4/5 = 136/170$ with $\varphi(136) = 8^2$ and $\sigma(170) = 18^2$, and $5/4 = 1365/1092$ with $\varphi(1365) = 24^2$ and $\sigma(1092) = 56^2$.

Concluding remarks

(A) For sources of most conjectures mentioned in this talk, see

1. Zhi-Wei Sun, *Conjectures on representations involving primes*, in: Combinatorial and Additive Number Theory (edited by M. B. Nathanson), Springer, to appear. [This paper contains 100 conjectures.]

2. Zhi-Wei Sun, Many integer sequences created for OEIS (On-Line Encyclopedia of Integer Sequences) with related conjectures mentioned, `http://oeis.org/`.

(B) In recent years the speaker has formulated many other conjectures on various topics, for example,

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}=\frac{15}{2}\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

where $(\frac{k}{3})$ denotes the Legendre symbol.

Thank you!

JCNT calls for submissions