A talk given at Vienna University of Technology on June 17, 2004.

ON DISJOINT SYSTEMS OF RESIDUE CLASSES OR COSETS OF SUBGROUPS

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Abstract. A finite system $A = \{a_1 \pmod{n_1}, \ldots, a_k \pmod{n_k}\}$ of residue classes is said to be disjoint if the k residue classes in it are pairwise disjoint. A fascinating topic is to investigate the moduli in a disjoint system; in this field several interesting conjectures remain open. We will also talk about disjoint covers of $\mathbb{Z}$ by residue classes and recent progress on the Herzog-Schönheim conjecture concerning disjoint covers of a group G by finitely many left cosets $a_1G_1, \ldots, a_kG_k$.

1. On disjoint residue classes

For integers a and $n > 0$, we call

$$a(n) = a + n\mathbb{Z} = \{x \in \mathbb{Z}: x \equiv a \pmod{n}\}$$

a residue class with modulus n. For two residue classes $a(n)$ and $b(m)$,

$$a(n) \cap b(m) \neq \emptyset \iff a+nx = b-my \text{ for some } x, y \in \mathbb{Z} \iff (m, n) \mid a-b,$$

where (m, n) denotes the greatest common divisor of m and n. Thus, if $a(n)$ and $b(m)$ are disjoint, then $(m, n) \nmid a-b$ and in particular $(m, n) > 1$.

A finite system

$$A = \{a_s(n_s)\}_{s=1}^k = \{a_1(n_1), \ldots, a_k(n_k)\}$$

of residue classes is said to be *disjoint* if the k members in A are pairwise disjoint, i.e. $a_i(n_i) \cap a_j(n_j) = \emptyset$ for all $1 \leqslant i < j \leqslant k$. Let $N_A = [n_1, \ldots, n_k]$ be the least common multiple of the moduli $n_1, \ldots, n_k$. If A is disjoint, then $\sum_{s=1}^k 1/n_s \leqslant 1$ because

$$\begin{aligned} N_A \geqslant & \left| \bigcup_{s=1}^k a_s(n_s) \cap \{0, 1, \ldots, N_A - 1\} \right| \\ = & \sum_{s=1}^k |a_s(n_s) \cap \{0, \ldots, N_A - 1\}| = \sum_{s=1}^k \frac{N_A}{n_s}. \end{aligned}$$

A finite sequence $\{n_s\}_{s=1}^k$ of positive integers is said to be *harmonic* if $n_1, \ldots, n_k$ are the moduli of a disjoint system, i.e., there exist integers $a_1, \cdots, a_k$ such that the residue classes $a_1(n_1), \cdots, a_k(n_k)$ are pairwise disjoint. Since $a_i(n_i) \cap a_j(n_j) \neq \emptyset$ if and only if $(n_i, n_j) \mid a_i - a_j$, whether $\{n_s\}_{s=1}^k$ $(k > 1)$ is harmonic or not, just depends on those greatest common divisors $d_{ij} = (n_i, n_j)$ $(1 \leqslant i < j \leqslant k)$. A necessary condition for $\{n_s\}_{s=1}^k$ to be harmonic, is that $d_{ij} > 1$ for all $1 \leqslant i < j \leqslant k$.

In 1982 A. P. Huhn and L. Megyesi [Discrete Math. 41(1982)] used a graph-theoretic argument to obtain the following result: *If all those $d_{ij} = (n_i, n_j)$ $(i < j)$ are distinct and greater than one, then $\{n_s\}_{s=1}^k$ is harmonic.*

In 1992 Z. W. Sun developed a new method to study harmonic sequences, a basic tool is the following lemma.

Lemma 1.1 [Sun, Discrete Math. 104(1992)]. *Let $n_1, \ldots, n_k, n_{k+1}$ be any positive integers. Suppose that $\{n_s\}_{s=1}^{k}$ is harmonic but $\{n_s\}_{s=1}^{k+1}$ is not. Then there are $a_1, \ldots, a_k \in \mathbb{Z}$ such that $B = \{a_i((n_i, n_k))\}_{i=1}^{k}$ forms a cover of $\mathbb{Z}$.*

Proof. Let $\{a_i(n_i)\}_{i=1}^{k}$ be disjoint. If $a_{k+1} \in \mathbb{Z}$ is not covered by B, then $a_{k+1}(n_{k+1}) \cap a_i(n_i) = \emptyset$ for all $i = 1, \ldots, k$, hence $\{a_s(n_s)\}_{s=1}^{k+1}$ is disjoint, which contradicts the condition that $\{n_s\}_{s=1}^{k+1}$ is not harmonic. □

With the help of this simple lemma, Sun studied harmonic sequences via covers of $\mathbb{Z}$ and deduced the following extension of the Huhn–Megyesi result.

Theorem 1.1 [Sun, Discrete Math. 104(1992)]. *Let $n_1, \cdots, n_k$ be positive integers. Then $\{n_s\}_{s=1}^{k}$ is harmonic if*

$$\#d = |\{\{i,j\} :\ 1 \leqslant i < j \leqslant k \ \&\ (n_i, n_j) = d\}| < \sqrt{\frac{d+7}{8}}$$

for those $d = 1, 2, \cdots, 2^{k-2}$ or those $d = \prod_{i=1}^{r} p_i^{\alpha_i}$ ($p_1, \cdots, p_r$ are distinct primes) with $\sum_{i=1}^{r} \alpha_i(p_i - 1) \leqslant k - 2$.

Sun's Conjecture 1.1 [Discrete Math., 104(1992)]. *$\sqrt{(d+7)/8}$ in Theorem 1.1 can be replaced by $2d - 1$.*

Though the conjecture is still open, following Sun's approach Y. G. Chen [Discrete Math. 162(1996)] proved the following result: *If $\#1 = 0$, $\#2 \leqslant 1$, $\#3 \leqslant 1$ and $\#d \leqslant d/4$ for all $d = 4, 5, \cdots, 2k-2$, then $\{n_s\}_{s=1}^{k}$ is harmonic.*

In their paper Huhn and Megyesi also posed two open problems on harmonic sequences. Both were solved by Sun [Chinese Ann. Math. 13A(1992)] negatively. The following challenging conjecture arose from Sun's solutions of the two problems of Huhn and Megyesi.

Conjecture 1.2. *Let* $\{n_s\}_{s=1}^k$ *be a harmonic sequence. Then* $(n_i, n_j) \geqslant k$ *for some* $1 \leqslant i < j \leqslant k$.

This conjecture is quite striking and far from transparent (of course the case $k = 2$ is obvious). Maybe it is extremely difficult. In 1988 Z. W. Sun verified it for $k \leqslant 5$, later one of his students checked the conjecture for $6 \leqslant k \leqslant 12$.

Here is an extension of Conjecture 1.2.

Sun's Conjecture 1.3 (`arXiv:math.GR/0501451`). *Let* $G_1, \ldots, G_k$ $(1 < k < \infty)$ *be subgroups of a group* G *with finite index. If* $a_1G_1, \ldots, a_kG_k$ *are pairwise disjoint for some* $a_1, \ldots, a_k \in G$, *then* $\gcd([G : G_i], [G : G_j]) \geqslant k$ *for some* $1 \leqslant i < j \leqslant k$.

Let $n_1, \ldots, n_k$ $(k > 1)$ be positive integers. Set $\tilde{n}_s = (n_s, [n_t]_{t \neq s}) = [(n_s, n_t)]_{t \neq s}$ for $s = 1, \ldots, k$, where we use $[m_i]_{i \in I}$ to denote the least common multiple of those m_i with $i \in I$. Clearly $(\tilde{n}_s, \tilde{n}_t) = (n_s, n_t)$ for $1 \leqslant s < t \leqslant k$, thus $\{n_s\}_{s=1}^k$ is harmonic if and only if $\{\tilde{n}_s\}_{s=1}^k$ is. If $n_s = \tilde{n}_s$ (i.e., $n_s \mid [n_t]_{t \neq s}$) for all $s = 1, \ldots, k$, then the sequence $\{n_s\}_{s=1}^k$ is said to be *normal* ([Sun, Acta Arith. 1997]).

If $\{n_s\}_{s=1}^k$ is harmonic, then for any $I \subseteq \{1, \ldots, k\}$ with $|I| \geqslant 2$, the sequence $\{\tilde{n}_s(I)\}_{s\in I}$ is harmonic where

$$\tilde{n}_s(I) = (n_s, [n_t]_{t\in I\setminus\{s\}}) = [(n_s, n_t)]_{t\in I\setminus\{s\}} \text{ for } s \in I,$$

therefore

$$\sum_{s\in I} \frac{1}{n_s} \leqslant \sum_{s\in I} \frac{1}{\tilde{n}_s(I)} \leqslant 1.$$

Example 1.1 ([Z. W. Sun, Chinese Ann. Math. Ser. A, 1992]). Let k, a, b, c, d be positive integers with $k, a \geqslant 5$ and $6, a, b, c, d$ pairwise coprime. Set

$$n_1 = 2a,\ n_2 = 3a,\ n_3 = 6b,\ n_4 = 6c,\ n_5 = 6d,\ n_6 = \cdots = n_k = 6kabcd.$$

Then, for any $I \subseteq \{1, \ldots, k\}$ with $|I| \geqslant 2$, we have

$$\sum_{s\in I} \frac{1}{\tilde{n}_s(I)} \leqslant 1,$$

and also $(n_s, n_t) \geqslant |I|$ for some $s, t \in I$ with $s \neq t$. But the sequence $\{n_s\}_{s=1}^k$ is not harmonic yet.

For a finite sequence $\{n_s\}_{s=1}^k$ of positive integers, if a subsequence $\{n_s\}_{s\in I}$ is not harmonic then the sequence $\{n_s\}_{s=1}^k$ is not harmonic either.

Theorem 1.2 [Z. W. Sun, Chinese Ann. Math. Ser. A, 1992]. *Let n_1, n_2, n_3, n_4 be positive integers. Then*

(i) $\{n_s\}_{s=1}^2$ *is not harmonic if and only if* $(n_1, n_2) = 1$.

(ii) $\{n_s\}_{s=1}^3$ *is not harmonic but any of its proper subsequences is, if and only if* $(n_1, n_2) = (n_1, n_3) = (n_2, n_3) = 2$.

(iii) $\{n_s\}_{s=1}^4$ *is not harmonic but any of its proper subsequences is, if and only if* $(n_i,n_j)=3$ *for all* $1\leqslant i<j\leqslant 4$*, or we can adjust the order of* n_1,n_2,n_3,n_4 *so that*

$$(n_1,n_2)=(n_1,n_3)=(n_1,n_4)=2 \quad \textit{and} \quad (n_2,n_3)=(n_2,n_4)=(n_3,n_4)=4.$$

Actually we are also able to characterize non-harmonic sequences with length 5 or 6.

The following result is interesting.

Theorem 1.3 [Z. W. Sun, Combinatorica 2003]**.** *If every integer lies in at most* m *members of* $A=\{a_s(n_s)\}_{s=1}^k$*, then there are positive integers* $m_1,\ldots,m_k$ *such that* $\sum_{s=1}^k m_s/n_s=m$*.*

Theorem 1.3 in the case $m=1$ was due to Chen and Porubský [Acta Arith. 1995].

2. On disjoint covers of the integers and groups

$A=\{a_s(n_s)\}_{s=1}^k$ is said to be a cover of $\mathbb{Z}$ if $\bigcup_{s=1}^k a_s(n_s)=\mathbb{Z}$. A disjoint system which is also a cover of $\mathbb{Z}$ is called a *disjoint cover* of $\mathbb{Z}$. If $A=\{a_s(n_s)\}_{s=1}^k$ is a disjoint cover of $\mathbb{Z}$, then we have $\sum_{s=1}^k 1/n_s=1$.

For any positive integer n, the system $\{r(n)\}_{r=0}^{n-1}$ is obviously a disjoint cover of $\mathbb{Z}$. Observe that

$$A_1=\{1(2),0(2)\},\ A_2=\{1(2),2(4),0(4)\},\ A_3=\{1(2),2(4),4(8),0(8)\},$$
$$\cdots\cdots,\ A_k=\{1(2),2(2^2),\ldots,2^{k-1}(2^k),0(2^k)\},\ \cdots\cdots$$

are also disjoint covers of $\mathbb{Z}$.

Soon after his invention of the concept of cover of $\mathbb{Z}$, Paul Erdős made the following conjecture: *If $A = \{a_s(n_s)\}_{s=1}^k$ ($k > 1$) is a disjoint system with the moduli $n_1, \ldots, n_k$ distinct, then it cannot be a cover of $\mathbb{Z}$.*

Theorem 2.1. *Let $A = \{a_s(n_s)\}_{s=1}^k$.*

(i) (H. Davenport, L. Mirsky, D. Newman and R. Radó) *If A is a disjoint cover of $\mathbb{Z}$ with $1 < n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$, then we must have $n_{k-1} = n_k$.*

(ii) [Z. W. Sun, Chinese Quart. J. Math. 1991] *Let n_0 be a positive period of the function $w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\}|$. For any positive integer d with $d \nmid n_0$ and $I(d) = \{1 \leqslant s \leqslant k : d \mid n_s\} \neq \emptyset$, we have*

$$|I(d)| \geqslant |\{a_s \bmod d : s \in I(d)\}| \geqslant_{\substack{0 \leqslant s \leqslant k \\ d \nmid n_s}} \frac{d}{(d, n_s)} \geqslant p(d),$$

where $p(d)$ is the least prime divisor of d.

Proof of part (i). Without loss of generality we let $0 \leqslant a_s < n_s$ ($1 \leqslant s \leqslant k$). For $|z| < 1$ we have

$$\sum_{s=1}^k \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{s=1}^k \sum_{q=0}^\infty z^{a_s + qn_s} = \sum_{n=0}^\infty z^n = \frac{1}{1-z}.$$

If $n_{k-1} < n_k$ then

$$\infty = \lim_{\substack{z \to e^{2\pi i/n_k} \\ |z|<1}} \frac{z^{a_k}}{1 - z^{n_k}} = \lim_{\substack{z \to e^{2\pi i/n_k} \\ |z|<1}} \left(\frac{1}{1-z} - \sum_{s=1}^{k-1} \frac{z^{a_s}}{1 - z^{n_s}} \right) < \infty,$$

a contradiction! $\square$

Part (ii) in the case $n_0 = 1$ and $d = n_k$ yields the Davenport-Mirsky-Newman-Radó result, a further extension of part (ii) was given by Z. W. Sun [Math. Res. Lett. 11(2004)].

The following theorem shows that disjoint covers of $\mathbb{Z}$ are related to unit fractions, actually further results were obtained by Z. W. Sun.

Theorem 2.2 [Z. W. Sun, Acta Arith. 1995; Trans. Amer. Math. Soc. 1996]**.** *Let $A = \{a_s(n_s)\}_{s=1}^k$ be a disjoint cover of $\mathbb{Z}$.*

(i) *If $\emptyset \neq J \subset \{1, \dots, k\}$, then there exists an $I \subseteq \{1, \dots, k\}$ with $I \neq J$ such that $\sum_{s\in I} 1/n_s = \sum_{s\in J} 1/n_s$.*

(ii) *For any $1 \leqslant t \leqslant k$ and $r \in \{0, 1, \dots, n_t - 1\}$, there is an $I \subseteq \{1, \dots, k\} \setminus \{t\}$ such that $\sum_{s\in I} 1/n_s = r/n_t$.*

Proof. Let $N = [n_1, \dots, n_k]$. Then

$$\prod_{s=1}^{k} \left(1 - z^{N/n_s} e^{2\pi i a_s/n_s}\right) = 1 - z^N$$

because each Nth root of unity is a single zero of the left hand side. Thus

$$\sum_{I\subseteq\{1,\dots,k\}} (-1)^{|I|} z^{\sum_{s\in I} N/n_s} e^{2\pi i \sum_{s\in I} a_s/n_s} = 1 - z^N.$$

Comparing the degrees of both sides we obtain the well-known equality $\sum_{s=1}^k 1/n_s = 1$. As $\emptyset \neq J \subset \{1, \dots, k\}$, $0 < \sum_{s\in J} N/n_s < N$ and hence

$$\sum_{\substack{I\subseteq\{1,\dots,k\} \\ \sum_{s\in I} 1/n_s = \sum_{s\in J} 1/n_s}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} a_s/n_s} = 0$$

which implies that $\sum_{s\in I} 1/n_s = \sum_{s\in J} 1/n_s$ for some $I \subseteq \{1, \dots, k\}$ with $I \neq J$.

Now fix $1 \leqslant t \leqslant k$. Observe that

$$\prod_{\substack{s=1\\ s\neq t}}^{k}\left(1-z^{N/n_s}e^{2\pi i a_s/n_s}\right)=\frac{1-z^N}{1-z^{N/n_t}e^{2\pi i a_t/n_t}}=\sum_{r=0}^{n_t-1} z^{Nr/n_t}e^{2\pi i a_t r/n_t}.$$

Thus, for any $r=0,1,\ldots,n_t-1$ we have

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\setminus\{t\}\\ \sum_{s\in I}1/n_s=r/n_t}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}a_s/n_s}=e^{2\pi i a_t r/n_t}\neq 0$$

and hence $\sum_{s\in I}1/n_s=r/n_t$ for some $I\subseteq\{1,\ldots,k\}\setminus\{t\}$. □

Note that $\mathbb{Z}$ is an additive cyclic group and the residue class $a(n)=a+n\mathbb{Z}$ is just a coset of the subgroup $n\mathbb{Z}$ with index n.

Let G be a group and $G_1,\cdots,G_k$ be subgroups of G. Let $a_1,\cdots,a_k\in G$. If the system $\mathcal{A}=\{a_iG_i\}_{i=1}^k$ of left cosets covers all the elements of G at least m times but none of its proper subsystems does, then all the indices $[G:G_i]$ are known to be finite and furthermore $[G:\bigcap_{i=1}^k G_i]\leqslant k!$.

The following conjecture extends the one of P. Erdős.

The Herzog-Schönheim Conjecture [1974, Canad. Math. Bull.]. *Let $\mathcal{A}=\{a_iG_i\}_{i=1}^k$ $(k>1)$ be a partition (i.e. disjoint cover) of a group G into left cosets of subgroups $G_1,\cdots,G_k$. Then the indices $n_1=[G:G_1],\cdots,n_k=[G:G_k]$ cannot be pairwise distinct.*

M. A. Berger, A. Felzenbaum and A. S. Fraenkel [1986, Canad. Math. Bull.; 1987, Fund. Math.] showed the conjecture for finite nilpotent groups and supersolvable groups. A quite recent progress was made by the speaker.

Theorem 2.3 [Z. W. Sun, J. Algebra 273(2004)]. *Let G be a group, and $\mathcal{A}=\{a_iG_i\}_{i=1}^k$ $(k>1)$ be a system of left cosets of subnormal subgroups. Suppose that $\mathcal{A}$ covers each $x\in G$ the same times, and*

$$n_1=[G:G_1]\leqslant\cdots\leqslant n_k=[G:G_k].$$

Then the indices $n_1,\cdots,n_k$ cannot be distinct. Moreover, if each index occurs in $n_1,\cdots,n_k$ at most M times, then

$$\log n_1\leqslant\frac{e^\gamma}{\log 2}M\log^2 M+O(M\log M\log\log M)$$

where $\gamma=0.577\cdots$ is the Euler constant and the O-constant is absolute.

The above theorem also answers a question analogous to a famous problem of Erdős negatively. Theorem 2.3 was established by a combined use of tools from group theory and number theory. One of the key lemmas is the following one which is the main reason why covers involving subnormal subgroups are better behaved than general covers.

Lemma 3.1 [Z. W. Sun, European J. Combin. 2001]. *Let G be a group, and let $P(n)$ denote the set of prime divisors of a positive integer n.*

(i) *If $G_1,\ldots,G_k$ are subnormal subgroups of G with finite index, then*

$$\left[G:\bigcap_{i=1}^k G_i\right]\ \Big|\ \prod_{i=1}^k[G:G_i]\ \text{and hence}\ P\left(\left[G:\bigcap_{i=1}^k G_i\right]\right)=\bigcup_{i=1}^k P([G:G_i]).$$

(ii) *Let H be a subnormal subgroup of G with finite index. Then*

$$P(|G/H_G|)=P([G:H]),$$

where $H_G = \bigcap_{g\in G} gHg^{-1}$ *is the largest normal subgroup of* G *contained in* H*.*

We mention that part (ii) is a consequence of the first part, and the word "*subnormal*" cannot be removed from part (i).

Here is another important lemma.

Lemma 3.2 [Z. W. Sun, J. Algebra 273(2004)]**.** *Let* G *be a group and* H *its subgroup with finite index* N*. Let* $a_1, \ldots, a_k \in G$*, and let* $G_1, \ldots, G_k$ *be subnormal subgroups of* G *containing* H*. Then* $\bigcup_{i=1}^k a_iG_i$ *contains at least* $|\bigcup_{i=1} 0(n_i) \cap \{0, 1, \ldots, N-1\}|$ *left cosets of* H*, where* $n_i = [G : G_i]$*.*