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On Divisibility concerning Binomial Coefficients

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Abstract

Binomial coefficients arise naturally in combinatorics.

Recently the speaker initiated the study of certain divisibility properties of binomial coefficients, and products or sums of binomial coefficients.

In this talk we introduce the speaker's related results and various conjectures. The materials come from the author's two preprints:

1. Z. W. Sun, *Products and sums divisible by central binomial coefficients*, `arXiv:1004.4623`.

2. Z. W. Sun, *On divisibility concerning binomial coefficients*, `arXiv:1005.1054`.

Catalan numbers

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the nth Catalan number is given by

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}.$$

Recursion.

$$C_0 = 1 \quad \text{and} \quad C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k} \quad (n = 0, 1, 2, \ldots).$$

Generating Function.

$$\sum_{n=0}^{\infty} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}.$$

Combinatorial Interpretations. The Catalan numbers arise in many enumeration problems. For example, C_n is the number of binary parenthesizations of a string of $n+1$ letters, and it is also the number of ways to triangulate a convex $(n+2)$-gon into n triangles by $n-1$ diagonals that do not intersect in their interiors.

Generalized Catalan numbers

Let $h \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

The first-kind Catalan numbers of order h:

$$C_k^{(h)} = \frac{1}{hk+1}\binom{(h+1)k}{k} = \binom{(h+1)k}{k} - h\binom{(h+1)k}{k-1} \ (k \in \mathbb{N}).$$

(As usual, $\binom{x}{-n} = 0$ for $n = 1, 2, \ldots$.)

The second-kind Catalan numbers of order h:

$$\bar{C}_k^{(h)} = \frac{h}{k+1}\binom{(h+1)k}{k} = h\binom{(h+1)k}{k} - \binom{(h+1)k}{k+1} \ (k \in \mathbb{N}).$$

Remark. Those $C_k = C_k^{(1)} = \bar{C}_k^{(1)}$ are ordinary Catalan numbers.

When $\ell n+1 \mid \binom{kn+\ell n}{\ell n}$ for all $n=1,2,3,\ldots$?

Problem For what values of $k,\ell\in\mathbb{Z}^+$ we have

$$\ell n+1 \,\Big|\, \binom{kn+\ell n}{\ell n} \text{ for all } n=1,2,3,\ldots?$$

As $\binom{\ell n+n}{\ell n}=(\ell n+1)C_n^{(\ell)}$,

$$\ell n+1 \,\Big|\, \binom{n+\ell n}{\ell n} \text{ for all } n=1,2,3,\ldots.$$

Theorem. Let k and ℓ be positive integers. Then for any $n\in\mathbb{N}$ we have

$$\frac{\ell n+1}{(k,\ell n+1)} \,\Big|\, \binom{kn+\ell n}{kn}, \ \textit{i.e.},\ \ell n+1 \,\Big|\, k\binom{kn+\ell n}{\ell n}.$$

In particular, if all prime factors of k divides ℓ then $\ell n+1 \mid \binom{kn+\ell n}{\ell n}$ for every $n=0,1,2,\ldots$.

The p-adic order of $n!$

Let p be a prime. The p-adic order of an integer x is given by

$$\nu_p(x) = \sup\{a \in \mathbb{N} : \ p^a \mid x\}.$$

(In particular, $\nu_p(0) = +\infty$.)

A Known Fact. For any prime p and $n \in \mathbb{Z}^+$ we have

$$\nu_p(n!) = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.$$

Proof. Observe that

$$\begin{aligned}\nu_p(n!) &= \sum_{k=1}^{n} \nu_p(k) = \sum_{k=1}^{n} \sum_{\substack{i=1 \\ p^i \mid k}}^{k} 1 \\ &= \sum_{i=1}^{n} \sum_{\substack{k=1 \\ p^i \mid k}}^{n} 1 = \sum_{i=1}^{n} \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_{i=1}^{\infty} \left\lfloor \frac{n}{p^i} \right\rfloor.\end{aligned}$$

Legendre's Theorem

Legendre's Theorem. For any prime p and $n \in \mathbb{Z}^+$ we have

$$\nu_p(n!) = \frac{n - \rho_p(n)}{p-1},$$

where $\rho_p(n)$ is the sum of the digits of n in the expansion of n in base p.

Proof. Write $n = \sum_{i=0}^{k} a_i p^i$ with $a_i \in \{0, 1, \ldots, p-1\}$. For $i = 1, \ldots, k$ we have

$$n = p^i \sum_{j=i}^{k} a_j p^{j-i} + r_i \text{ with } r_i = \sum_{j=0}^{i-1} a_j p^j \leqslant \sum_{j=0}^{i-1} (p-1)p^j = p^i - 1.$$

Thus

$$\begin{aligned} \nu_p(n!) &= \sum_{i=1}^{k} \left\lfloor \frac{n}{p^i} \right\rfloor = \sum_{i=1}^{k} \sum_{j=i}^{k} a_j p^{j-i} \\ &= \sum_{j=1}^{k} a_j \sum_{i=1}^{j} p^{j-i} = \sum_{j=1}^{k} a_j \frac{p^j - 1}{p-1} = \frac{n - \rho_p(n)}{p-1}. \end{aligned}$$

A Lemma

Lemma. Let $m \in \mathbb{Z}^+$ and $k, l, n \in \mathbb{Z}$. Then

$$\left\lfloor \frac{kn+ln}{m} \right\rfloor - \left\lfloor \frac{kn}{m} \right\rfloor - \left\lfloor \frac{ln+1}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor - \left\lfloor \frac{k-1}{m} \right\rfloor \geqslant 0.$$

Proof. If $m \nmid kn$, then

$$\left\lfloor \frac{kn}{m} \right\rfloor + \left\lfloor \frac{ln+1}{m} \right\rfloor = \left\lfloor \frac{kn-1}{m} \right\rfloor + \left\lfloor \frac{ln+1}{m} \right\rfloor \leqslant \left\lfloor \frac{(kn-1)+(ln+1)}{m} \right\rfloor.$$

If $m \nmid (ln+1)$, then

$$\left\lfloor \frac{kn}{m} \right\rfloor + \left\lfloor \frac{ln+1}{m} \right\rfloor = \left\lfloor \frac{kn}{m} \right\rfloor + \left\lfloor \frac{ln}{m} \right\rfloor \leqslant \left\lfloor \frac{kn+ln}{m} \right\rfloor.$$

Now assume that $m \mid kn$ and $m \mid (ln+1)$. Clearly m is relatively prime to n. Thus $m \mid k$ and hence

$$\left\lfloor \frac{kn+ln}{m} \right\rfloor - \left\lfloor \frac{kn}{m} \right\rfloor - \left\lfloor \frac{ln+1}{m} \right\rfloor + \left\lfloor \frac{k}{m} \right\rfloor - \left\lfloor \frac{k-1}{m} \right\rfloor$$

$$= \frac{kn}{m} + \frac{ln+1}{m} - 1 - \frac{kn}{m} - \frac{ln+1}{m} + \frac{k}{m} - \left(\frac{k}{m} - 1 \right) = 0.$$

Proof of $\ell n+1 \mid k\binom{kn+\ell n}{\ell n}$

Let k,ℓ,n be any positive integers.

With the help of the above lemma, for any prime p we have

$$\nu_p\left(\frac{k\binom{kn+\ell n}{kn}}{\ell n+1}\right)=\nu_p\left(\frac{(kn+\ell n)!k!}{(kn)!(\ell n+1)!(k-1)!}\right)$$
$$=\sum_{j=1}^{\infty}\left(\left\lfloor\frac{kn+\ell n}{p^j}\right\rfloor-\left\lfloor\frac{kn}{p^j}\right\rfloor-\left\lfloor\frac{\ell n+1}{p^j}\right\rfloor+\left\lfloor\frac{k}{p^j}\right\rfloor-\left\lfloor\frac{k-1}{p^j}\right\rfloor\right)$$
$$\geqslant 0.$$

Therefore

$$\ell n+1 \,\Big|\, k\binom{kn+\ell n}{kn}.$$

A conjecture on divisibility of binomial coefficients

If all prime factors of $k \in \mathbb{Z}^+$ divides $\ell \in \mathbb{Z}^+$, then

$$\ell n + 1 \, \Big| \, \binom{kn + \ell n}{\ell n} \text{ for every } n = 0, 1, 2, \ldots$$

since $(\ell n + 1, k) = 1$.

Conjecture (Sun, 2010). Let k and ℓ be positive integers. If

$$\ell n + 1 \, \Big| \, \binom{kn + \ell n}{\ell n}$$

for all sufficiently large positive integers n, then each prime factor of k divides ℓ. In other words, if k has a prime factor not dividing ℓ then there are infinitely many positive integers n such that $\ell n + 1 \nmid \binom{kn+\ell n}{\ell n}$.

Some numerical examples

Let k and l be positive integers such that not all prime factors of k divides l. Define $f(k,l)$ as the smallest positive integer n such that $ln+1 \nmid \binom{kn+ln}{kn}$. Via Mathematica we obtained the following data:

$$f(7,36) = 279,\ f(10,192) = 362,\ f(11,100) = 1187,$$
$$f(13,144) = 2001,\ f(22,200) = 6462,\ f(31,171) = 1765;$$
$$f(43,26) = 640,\ f(53,32) = 790,\ f(67,56) = 2004,$$
$$f(73,61) = 2184,\ f(74,62) = 885,\ f(97,81) = 2904,$$
$$f(179,199) = 28989,\ f(223,93) = 13368,\ f(307,189) = 31915.$$

Bober's Recent Work

In 2009 J. W. Bober [J. London Math. Soc.] determined all those $a_1, \ldots, a_r, b_1, \ldots, b_{r+1} \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ with $a_1 + \cdots + a_r = b_1 + \cdots + b_{r+1}$ such that

$$\frac{(a_1 n)! \cdots (a_r n)!}{(b_1 n)! \cdots (b_{r+1} n)!} \in \mathbb{Z} \ \text{ for all } n \in \mathbb{Z}^+.$$

In particular, if k and ℓ are positive integers then

$$\frac{\binom{\ell n}{n}\binom{k\ell n}{\ell n}}{\binom{kn}{n}} = \frac{(k\ell n)!((k-1)n)!}{(kn)!((\ell-1)n)!((k-1)\ell n)!} \in \mathbb{Z} \ \text{ for all } n \in \mathbb{Z}^+,$$

$$\iff k = \ell, \text{ or } \{k, \ell\} \cap \{1, 2\} \neq \emptyset, \text{ or } \{k, \ell\} = \{3, 5\}.$$

More Results on Products of Two Binomial Coefficients

Theorem (Sun, 2010). Let $k, n \in \mathbb{Z}^+$.
(i) we have

$$2\binom{kn}{n} \,\Big|\, \binom{2n}{n} C_{2n}^{(k-1)}.$$

Moreover, $\binom{2n}{n} C_{2n}^{(k-1)}/(2\binom{kn}{n})$ is odd if and only if n is a power of two.
(ii) We have

$$\binom{kn}{n} \,\Big|\, (2k-1)C_n\binom{2kn}{2n},$$

and $(2k-1)C_n\binom{2kn}{2n}/\binom{kn}{n}$ is odd if and only if $n+1$ is a power of two.
(iii) Let $(k+1)'$ be the odd part of $k+1$. Then

$$\binom{2n}{n} \,\Big|\, (k+1)' C_n^{(k-1)}\binom{2kn}{kn},$$

and $(k+1)' C_n^{(k-1)}\binom{2kn}{kn}/\binom{2n}{n}$ is odd if and only if $(k-1)n+1$ is a power of two.

Some Lemmas

Lemma 1. Let $m \in \mathbb{Z}^+$ and $k, n \in \mathbb{Z}$. Then we have

$$\left\lfloor \frac{2kn}{m} \right\rfloor - \left\lfloor \frac{kn}{m} \right\rfloor + \left\lfloor \frac{(k-1)n}{m} \right\rfloor - \left\lfloor \frac{2(k-1)n}{m} \right\rfloor \geqslant \left\lfloor \frac{n+1}{m} \right\rfloor - \left\lfloor \frac{2k-1}{m} \right\rfloor + \left\lfloor \frac{2k-2}{m} \right\rfloor .$$

Lemma 2. Let $m > 2$ be an integer. For any $k, n \in \mathbb{Z}$ we have

$$\left\lfloor \frac{2kn}{m} \right\rfloor + \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{k+1}{m} \right\rfloor \geqslant \left\lfloor \frac{k}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor + \left\lfloor \frac{kn}{m} \right\rfloor + \left\lfloor \frac{(k-1)n+1}{m} \right\rfloor .$$

Another Theorem

Theorem (Sun, 2010). For any positive integers k and n, we have

$$2^{k-1}\binom{2n}{n} \,\Big|\, \binom{2(2^k-1)n}{(2^k-1)n} C_n^{(2^k-2)}.$$

In particular,

$$S_n = \frac{\binom{6n}{3n}\binom{3n}{n}}{2(2n+1)\binom{2n}{n}} \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots.$$

A Key Lemma. Let p be a prime and let $k \in \mathbb{N}$ and $n \in \mathbb{Z}^+$. Then

$$\frac{\rho_p((p^k-1)n)}{p-1} = \sum_{j=1}^{\infty} \left\{ \frac{(p^k-1)n}{p^j} \right\} \geqslant k$$

and hence the expansion of $(p^k-1)n$ in base p has at least k nonzero digits.

Proof of the Lemma

For any $m \in \mathbb{Z}^+$, by Legendre's theorem we have

$$\frac{\rho_p(m)}{p-1} = \frac{m}{p-1} - \nu_p(m!) = \sum_{j=1}^{\infty} \frac{m}{p^j} - \sum_{j=1}^{\infty} \left\lfloor \frac{m}{p^j} \right\rfloor = \sum_{j=1}^{\infty} \left\{ \frac{m}{p^j} \right\}.$$

Observe that

$$p^k \binom{p^k n - 1}{n-1} = \binom{p^k n}{n} = \frac{(p^k n)!}{n!((p^k-1)n)!}$$

and hence

$$\begin{aligned} k \leqslant & \nu_p((p^k n)!) - \nu_p(n!) - \nu_p(((p^k-1)n)!) \\ = & \sum_{j=1}^{\infty} \left\lfloor \frac{p^k n}{p^j} \right\rfloor - \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor - \sum_{j=1}^{\infty} \left\lfloor \frac{(p^k-1)n}{p^j} \right\rfloor \\ = & \sum_{j=1}^{k} p^{k-j} n - \sum_{j=1}^{\infty} \left\lfloor \frac{(p^k-1)n}{p^j} \right\rfloor = \sum_{j=1}^{\infty} \left\{ \frac{(p^k-1)n}{p^j} \right\}. \end{aligned}$$

A General Conjecture on Expansions in Base m

Conjecture. Let $m > 1$ be an integer and k and n be positive integers. Then the sum of all digits in the expansion of $(m^k - 1)n$ in base m is at least $k(m-1)$. Also, the expansion of $\frac{m^k-1}{m-1}n$ in base m has at least k nonzero digits.

Remark. Hao Pan has proved the conjecture fully.

Properties of S_n

It is interesting to investigate the integer sequence

$$S_n = \frac{\binom{6n}{3n}\binom{3n}{n}}{2(2n+1)\binom{2n}{n}} \quad (n = 1, 2, 3, \dots).$$

$$S_1 = 5,\ S_2 = 231,\ S_3 = 14568,\ S_4 = 1062347,\ S_5 = 84021990.$$

By Stirling's formula, $S_n \sim 108^n/(8n\sqrt{n\pi})$ as $n \to +\infty$.
Set $S_0 = 1/2$. Using `Mathematica` we find that

$$\sum_{k=0}^{\infty} S_k x^k = \frac{\sin(\frac{2}{3}\arcsin(6\sqrt{3x}))}{8\sqrt{3x}} \quad \left(0 < x < \frac{1}{108}\right)$$

and in particular

$$\sum_{k=0}^{\infty} \frac{S_k}{108^k} = \frac{3\sqrt{3}}{8}.$$

`Mathematica` also yields that

$$\sum_{k=0}^{\infty} \frac{S_k}{(2k+3)108^k} = \frac{27\sqrt{3}}{256}.$$

A Conjecture on the Sequence $\{S_n\}_{n\geqslant 1}$

One can easily show that

$$S_p \equiv 15 - 30p + 60p^2 \pmod{p^3}$$

for any odd prime p.

Conjecture (Sun, 2010) (i) Let $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then S_n is odd if and only if n is a power of two. Also, $2n + 3 \mid 3S_n$.

(ii) For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{S_k}{108^k} \equiv \begin{cases} 0 \pmod{p} & \text{if } p \equiv \pm 1 \pmod{12}, \\ -1 \pmod{p} & \text{if } p \equiv \pm 5 \pmod{12}. \end{cases}$$

On the Companion Sequence $\{T_n\}_{n\geqslant 1}$

Here is a conjecture on a companion sequence of $\{S_n\}_{n\geqslant 0}$.

Conjecture (Sun, 2010). There are positive integers $T_1, T_2, T_3, \ldots$ such that

$$\sum_{k=0}^{\infty} S_k x^{2k+1} + \frac{1}{24} - \sum_{k=1}^{\infty} T_k x^{2k} = \frac{\cos(\frac{2}{3}\arccos(6\sqrt{3}x))}{12}$$

for all real x with $|x| \leqslant 1/(6\sqrt{3})$. Also, $T_p \equiv -2 \pmod{p}$ for any prime p.

Values of $T_1, \ldots, T_8$:

1, 32, 1792, 122880, 9371648,
763363328, 65028489216, 5722507051008.

More Results

Theorem (Sun, 2010). For any $n \in \mathbb{Z}^+$ we have

$$(6n+1)\binom{5n}{n} \,\Big|\, \binom{3n-1}{n-1} C_{3n}^{(4)}, \quad \binom{3n}{n} \,\Big|\, \binom{5n-1}{n-1} C_{5n}^{(2)}.$$

A Lemma. (i) For any real number x we have

$$\{12x\} + \{5x\} + \{2x\} \geqslant \{4x\} + \{15x\}.$$

(ii) Let x be a real number with $\{5x\} \geqslant \{2x\} \geqslant 1/2$. Then $\{5x\} \geqslant 2/3$.

An Auxiliary Theorem. Let $m > 1$ and n be integers.
(i) If $3 \nmid m$, then

$$\left\lfloor \frac{15n-1}{m} \right\rfloor + \left\lfloor \frac{2}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor \geqslant \left\lfloor \frac{12n+2}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor + \left\lfloor \frac{5n-1}{m} \right\rfloor.$$

(ii) If $5 \nmid m$, then

$$\left\lfloor \frac{15n-1}{m} \right\rfloor + \left\lfloor \frac{2n}{m} \right\rfloor \geqslant \left\lfloor \frac{10n+1}{m} \right\rfloor + \left\lfloor \frac{4n}{m} \right\rfloor + \left\lfloor \frac{3n-1}{m} \right\rfloor.$$

More Conjectures

Define the new sequence $\{t_n\}_{n\geqslant 1}$ of integers by

$$t_n = \frac{\binom{5n-1}{n-1}C_{5n}^{(2)}}{\binom{3n}{n}} = \frac{\binom{5n-1}{n-1}\binom{15n}{5n}}{(10n+1)\binom{3n}{n}}.$$

Conjecture (Sun, 2010). Let n be any positive integer. We have

$$21t_n \equiv 0 \pmod{10n+3}.$$

If $3 \nmid n$, then $(10n+3) \mid 7t_n$. If $7 \nmid n+1$, then $(10n+3) \mid 3t_n$.

Conjecture (Sun, 2010). Let k and ℓ be integers greater than one.

(i) If

$$\binom{kn}{n} \mid \binom{\ell n}{n}\binom{k\ell n}{\ell n-1}$$

for all $n \in \mathbb{Z}^+$, then $k = \ell$, or $\ell = 2$, or $\{k,\ell\} = \{3,5\}$.

(ii) If

$$\binom{kn}{n} \mid \binom{\ell n}{n-1}\binom{k\ell n}{\ell n}$$

for all $n \in \mathbb{Z}^+$, then $k = 2$, and $\ell + 1$ is a power of two.

On Products of Three Binomial Coefficients

Theorem 1. For any nonnegative integers k and n we have

$$\binom{2k}{k} \Bigm| \binom{4n+2k+2}{2n+k+1}\binom{2n+k+1}{2k}\binom{2n-k+1}{n}$$

and

$$\binom{2k}{k} \Bigm| (2n+1)\binom{2n}{n} C_{n+k}\binom{n+k+1}{2k}.$$

Lemma 1. Let $m \in \mathbb{Z}^+$ and $k, n \in \mathbb{Z}$. Then we have

$$\left\lfloor \frac{4n+2k+2}{m} \right\rfloor - \left\lfloor \frac{2n+k+1}{m} \right\rfloor + 2\left\lfloor \frac{k}{m} \right\rfloor - 2\left\lfloor \frac{2k}{m} \right\rfloor$$
$$\geqslant \left\lfloor \frac{n}{m} \right\rfloor + \left\lfloor \frac{n-k+1}{m} \right\rfloor,$$

unless $2 \mid m$ and $k \equiv n+1 \equiv m/2 \pmod{m}$ in which case the right-hand side of the inequality equals the left-hand side plus one.

Another Lemma

Lemma 2. Let $m \in \mathbb{Z}^+$ and $k, n \in \mathbb{Z}$. Then we have

$$\left\lfloor \frac{2n+2k}{m} \right\rfloor - \left\lfloor \frac{n+k}{m} \right\rfloor + 2\left\lfloor \frac{k}{m} \right\rfloor - 2\left\lfloor \frac{2k}{m} \right\rfloor \geqslant 2\left\lfloor \frac{n}{m} \right\rfloor - \left\lfloor \frac{2n+1}{m} \right\rfloor + \left\lfloor \frac{n-k+1}{m} \right\rfloor,$$

unless $2 \mid m$ and $k \equiv n+1 \equiv m/2 \pmod{m}$ in which case the right-hand side of the inequality equals the left-hand side plus one.

A Theorem on Sums of Binomial Coefficients

In 1914 Ramanujan got that

$$\sum_{k=0}^{\infty} \frac{4k+1}{(-64)^k} \binom{2k}{k}^3 = \frac{2}{\pi}$$

and

$$\sum_{k=0}^{\infty} (20k+3) \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{10})^k} = \frac{8}{\pi}.$$

Via Theorem 1 and the WZ method, we obtain

Theorem 2. For any positive integer n we have

$$4(2n+1)\binom{2n}{n} \;\Big|\; \sum_{k=0}^{n} (4k+1)\binom{2k}{k}^3 (-64)^{n-k}$$

and

$$4(2n+1)\binom{2n}{n} \;\Big|\; \sum_{k=0}^{n} (20k+3)\binom{2k}{k}^2 \binom{4k}{2k} (-2^{10})^{n-k}.$$

A Further Conjecture

Conjecture (Sun, 2009) (i) For $n \in \mathbb{Z}^+$ set

$$a_n := \frac{1}{2n(2n+1)\binom{2n}{n}} \sum_{k=0}^{n-1} (20k+3) \binom{2k}{k}^2 \binom{4k}{2k} (-2^{10})^{n-1-k}.$$

Then $(-1)^{n-1} a_n \in \mathbb{Z}^+$ for all $n = 2, 3, 4, \ldots$.
(ii) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} \frac{20k+3}{(-2^{10})^k} \binom{2k}{k}^2 \binom{4k}{2k} \equiv 3p \left(\frac{-1}{p}\right) + 3p^3 E_{p-3} \pmod{p^4}$$

where $E_0, E_1, E_2, \ldots$ are Euler numbers, and

$$\sum_{k=0}^{(p-1)/2} \frac{20k+3}{(-2^{10})^k} \binom{2k}{k}^2 \binom{4k}{2k}$$
$$\equiv p \left(\frac{-1}{p}\right) \left(2^{p-1} + 2 - (2^{p-1} - 1)^2\right) \pmod{p^4}$$

provided $p > 3$.

A Further Conjecture (Continued)

(iii) For any prime $p \neq 2, 5$, we have We also have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{(-2^{10})^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \& \ p = x^2 + 5y^2, \\ 2(p - x^2) \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \ \& \ 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

Remark. Let $p \neq 2, 5$ be a prime. By the theory of binary quadratic forms, if $p \equiv 1, 9 \pmod{20}$ then $p = x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 3, 7 \pmod{20}$ then $2p = x^2 + 5y^2$ for some $x, y \in \mathbb{Z}$.

The speaker has made lots of other conjectures similar to the above one.

Thank you!