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Historical Remarks on my Conjectural Series

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Abstract

During 2010-2016, I posed 235 conjectural series for powers of π and other important constants (motivated by supercongruences). On my list there are 178 reasonable series for π^{-1}, four series for π^2, two series for π^{-2}, five series for π^4, two series for π^5, three series for π^6, seven series for $\zeta(3)$, one series for $\pi\zeta(3)$, two series for $\pi^2\zeta(3)$, one series for $\zeta(3)^2$, three series involving both $\zeta(3)^2$ and π^6, one series for $\zeta(5)$, three series involving $\zeta(7)$, and so on.

Almost all of the mentioned series $\sum_n a_n$ **converge fast** in the sense that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = r \in (0,1)$. I ever explained that these series came from a combination of *philosophy, intuition, inspiration, experience and computation*.

In this talk I'll give some historical remarks on my conjectural series, and reveal how I found some typical ones.

Main Reference

Zhi-Wei Sun, *List of conjectural series for powers of π and other constants*, preprint, arXiv:1102.5649.

My initial contact with π-series (1984-86)

When I was an undergraduate at Nanjing University, I learned from calculus the following classical results :

Leibniz:

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Euler:

$$\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

But I did not know any other π-series then.

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England
Modular equations and approximations to π,
Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for* $1/\pi$". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

Elliptic integrals

Complete elliptic integrals (with $0 < k < 1$):

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{(the first kind)},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \quad \text{(the seond kind)}.$$

Legendre's Relation:

$$E(k)K(\sqrt{1-k^2}) + E(\sqrt{1-k^2})K(k) - K(k)K(\sqrt{1-k^2}) = \frac{\pi}{2}.$$

A Central Result:

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \;\middle|\; k^2\right) = \frac{2}{\pi} K(k) = \varphi^2(q)$$

where $q = e^{-\pi K(\sqrt{1-k^2})/K(k)}$ and

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{(theta function)}.$$

Series for $1/\pi$ given by Ramanujan

Two of the 17 series for $1/\pi$ recorded by Ramanujan:

$$\sum_{k=0}^{\infty} \frac{6k+1}{4^k} \cdot \frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi},$$

(proved by S. Chowla in 1928)

$$\sum_{k=0}^{\infty} \frac{26390k+1103}{99^{4k}} \cdot \frac{(1/2)_k (1/4)_k (3/4)_k}{(1)_k^3}$$

$$= \sum_{k=0}^{\infty} \frac{26390k+1103}{396^{4k}} \binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

In 1985 Jr. R. W. Gosper used the last series of Ramanujan to calculate $17,526,100$ digits of π (a world record at that time).

In 1987 Jonathan Borwein and Peter Borwein succeeded in proving all the 17 Ramanujan series for $1/\pi$.

My first impression on Ramanujan-type series

In a year around 2003, I happened to see a paper on Ramanujan-type series. Here is one of Ramanujan series for $1/\pi$:

$$\sum_{k=0}^{\infty}(28k+3)\left(-\frac{27}{512}\right)^k\frac{(1/2)_k(1/6)_k(5/6)_k}{(1)_k^3}=\frac{32\sqrt{2}}{\pi}.$$

At that time I did not like this at all since it is too complicated! I only enjoy simple and beautiful results! Thus this paper gave me almost no impression and I could not remember what paper it is.

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are 36 known Ramanujan-type series for $1/\pi$ with $a,b,m\in\mathbb{Z}$. I prefer their forms in terms of binomial coefficients.

What is needed for proving $\sum_{n=0}^{\infty}(6n+1)\binom{2n}{n}^3/256^n = 4/\pi$

The proofs of Ramanujan series involve lots of things such as modulo forms, elliptic integrals, theta functions, hypergeometric series, modular equations and symbolic computation.

$$P(q) := 1 - 24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j} \quad \text{(Eisenstein series)},$$

$$\varphi(q) := \sum_{j=-\infty}^{\infty} q^{j^2} \quad \text{(theta function)},$$

$$X = X(q) = q\prod_{j=1}^{\infty}\frac{(1-q^j)^{24}(1-q^{4j})^{24}}{(1-q^{2j})^{48}}.$$

$$\varphi(q)^4 = \sum_{n=0}^{\infty}\binom{2n}{n}X^n,\ P(q^2) = \sqrt{1-64X}\sum_{n=0}^{\infty}(3n+1)\binom{2n}{n}^3 X^n.$$

$$X(e^{-\pi\sqrt{3}}) = \frac{1}{256} \text{ and } P(e^{-2\pi\sqrt{3}}) = \frac{\sqrt{3}}{\pi} + \frac{\sqrt{3}}{4}\varphi(e^{-\pi\sqrt{3}})^4.$$

On $a(p), b(p), c(p)$

For a power series $f(q)$ in q, we let $[q^n]f(q)$ denote the coefficient of q^n in $f(q)$.

For any prime $p > 3$, it is known that

$$a(p) := [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6 = \begin{cases} 4x^2 - 2p & \text{if } p = x^2 + y^2 \ (2 \nmid x), \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

$$\begin{aligned} b(p) :&= [q^p]q\prod_{n=1}^{\infty}(1-q^{6n})^3(1-q^{2n})^3 \\ &= \begin{cases} 4x^2 - 2p & \text{if } p = x^2 + 3y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 & \text{if } p \equiv 2 \pmod 3, \end{cases} \end{aligned}$$

$$\begin{aligned} c(p) :=&[q^p]q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2 \\ =& \begin{cases} 4x^2 - 2p & \text{if } (\frac{-2}{p}) = 1 \text{ and } p = x^2 + 2y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 & \text{if } (\frac{-2}{p}) = -1, \text{ i.e., } p \equiv 5, 7 \pmod 8. \end{cases} \end{aligned}$$

Conjectures of Rodriguez-Villegas

Let $p > 3$ be a prime. In 2003 Rodriguez-Villegas conjectured that

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}^3=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/3}{k}\binom{-2/3}{k}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/4}{k}\binom{-3/4}{k}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/6}{k}\binom{-5/6}{k}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

are congruent to $a(p), b(p), c(p)$ and $(\frac{p}{3})a(p)$ mod p^2 respectively. Actually the first one was proved by Ishikawa [Nagoya Math. J. 118(1990)]. E. Mortenson [Proc. AMS 133(2005)] provided partial solutions to the last three and the remaining thing were proved by Z.-W. Sun [156(2012)].

My joint work on congruences modulo prime powers

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod p\quad(d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod p\quad\text{for } p>3.$$

Sun & R. Tauraso [AAM 45(2010); IJNT 7(2011)].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\quad\text{for } p>3,$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers given by

$$B_0=1,\quad\sum_{k=0}^{n}\binom{n+1}{k}B_k=0\quad(n=1,2,3,\ldots).$$

What happened in November, 2009

I determined $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ modulo p^2 in 2009. After this I systematically investigate congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2/m^k$ and $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ modulo p^2. In particular, I formulated the following conjecture.

Conjecture (Z.-W. Sun, Nov. 2009). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3 \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1. \end{cases}$$

Prof. Ken Ono was very interested in this and he and one of his students worked on my conjecture. They claimed that they had a proof but in Jan. 2010 they replied me that they met real difficulties.

My above conjecture was finally confirmed by J. Kibelbek, L. Long, K. Moss, B. Sheller and H. Yuan [`arXiv:1210.4489`, JNT 164(2016)], as well as Z.-H. Sun [JNT 133(2013)].

What happened in Jan.-Feb. 2010

I visited India during Jan.-Feb. 2010. On Jan. 23 I suddenly realized that I should combine the congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ and $\sum_{k=0}^{p-1}k\binom{2k}{k}^3/m^k \bmod p^2$. This led me to conjecture that

$$\frac{1}{p}\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv 8+16p^3B_{p-3} \pmod{p^4} \qquad (*)$$

and that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3 \in \mathbb{Z}.$$

After reading my message to Number Theory List on Feb. 10, Kasper Andersen found on Feb. 11 that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3 = \sum_{k=0}^{n-1}\binom{n+k-1}{k}^2$$

via Sloane's OEIS (Online Encyclopedia of Integer Sequences). Inspired by this I finally proved $(*)$.

van Hamme's conjecture

After I found $\sum_{k=0}^{p-1}\binom{2k}{k}^3/4096^k$ mod p^2 and conjectured the congruence

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right)-p^3E_{p-3}\pmod{p^4}$$

(which was later confirmed by D.-W. Hu and G.-S. Mao [Ramanujan J. 42(2017)]), I got to know that van Hamme had the conjecture

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right)\pmod{p^3}$$

motivated by Ramanujan's identity

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi}.$$

Thus I became interested in Ramanujan-type series and wrote to several mathematicians to get Hamme's paper.

Rediscover Zeilberger's series $\sum_{k=1}^\infty \frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6}$

I proved that for any odd prime p we have

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv 8p + 16p^4 B_{p-3} \pmod{p^5}.$$

As the series $\sum_{k=0}^\infty (21k+8)\binom{2k}{k}^3$ diverges, it does not provide a Ramanujan-type series for $1/\pi$. However, I observe that

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 = 8 + \sum_{k=(p+1)/2}^{p-1}(21(p-k)+8)\binom{2(p-k)}{p-k}^3$$
$$\equiv 8 - \sum_{k=(p+1)/2}^{p-1}(21k-8)\left(\frac{2p}{k\binom{2k}{k}}\right)^3 \pmod{p}$$

and this led me to find that

$$\sum_{k=1}^\infty \frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6} \quad \text{(D. Zeilberger, 1993)}.$$

Conjecture: $\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2$

Conjecture (Z.-W. Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{11})=-1,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{11k+3}{64^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 3p+\frac{7}{2}p^4B_{p-3} \pmod{p^5},$$

$$p\sum_{k=1}^{(p-1)/2}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}\equiv 32\frac{2^{p-1}-1}{p}-\frac{64}{3}p^2B_{p-3} \pmod{p^3}.$$

Also,

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^2\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2 \quad \text{(confirmed by J. Guillera in 2013).}$$

Conjecture: $\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K$

Conjecture (Z.-W. Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1;\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 4p\left(\frac{p}{3}\right)+\frac{4}{3}p^3B_{p-2}\left(\frac{1}{3}\right) \pmod{p^4}.$$

Also,

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K:=\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2}$$ (confirmed by Kh. Hessami Pilehrood and T. Hessami Pilehrood in 2012).

More such conjectural series

Conjecture (Z.-W. Sun, 2010; Sci. China Math. 54(2011))

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},$$

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K.$$

The three conjectural identities were finally confirmed by J. Guillera and M. Rogers [J. Austral. Math. Soc. 97(2014)].

A curious identity with \$480 prize for the solution

Conjecture (Z.-W. Sun) (i) (2009-11-29) For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv 0 \pmod{p^2}.$$

(ii) (2014-07-07) For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv \frac{5}{12} p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3},$$

$$p^2 \sum_{k=1}^{p-1} \frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}} \equiv 4\left(\frac{p}{3}\right) + 4p \pmod{p^2}.$$

(iii) (2014-08-12, **\$480 prize for the solution**) We have

$$\sum_{k=1}^{\infty} \frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}} = \frac{15}{2} \sum_{k=1}^{\infty} \frac{\left(\frac{k}{3}\right)}{k^2}.$$

Three more conjectural series

Motivated by corresponding congruences, I made the following conjecture in 2010-2011.

Conjecture (Z.-W. Sun) (i) [Sci. China Math. 54(2011)] We have

$$\sum_{n=0}^{\infty} \frac{18n^2+7n+1}{(-128)^n} \binom{2n}{n}^2 \sum_{k=0}^{n} \binom{-1/4}{k}^2 \binom{-3/4}{n-k}^2 = \frac{4\sqrt{2}}{\pi^2}$$

$$\sum_{n=0}^{\infty} \frac{40n^2+26n+5}{(-256)^n} \binom{2n}{n}^2 \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} = \frac{24}{\pi^2}.$$

(In 2004 H.H. Chan, S.H. Chan and Z. Liu [Adv. Math.] proved that $\sum_{n=0}^{\infty} \frac{5n+1}{64^n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} = \frac{8}{\sqrt{3}\pi}$.)

(ii) [Electron. J. Combin. 20(2013)] We have

$$\sum_{k=1}^{\infty} \frac{(28k^2-18k+3)(-64)^k}{k^5 \binom{2k}{k}^4 \binom{3k}{k}} = -14\zeta(3).$$

My Philosophy about Series for $1/\pi$

Part I of the Philosophy (2010). Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k}\equiv 0 \pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1,2,2^2,2^3,\ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d>1$ and for any prime $p>3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k}\equiv\begin{cases}(\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p=x^2+dy^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p})=-1,\end{cases}$$

or $d=1$, $\gcd(15,m)>1$, and for any prime $p\equiv 3 \pmod{4}$ with $p\nmid 3m$ we have $\sum_{k=0}^{p-1}a_k/m^k\equiv 0 \pmod{p^2}$.

Philosophy about Series for $1/\pi$ (continued)

Part II of the Philosophy (2011). Let $b, c, m, a_0, a_1, \ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k} \equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right) \pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

Remark. Almost all identities of the stated form are *regular*.

An Example Illustrating the Philosophy

Ramanujan Series:

$$\sum_{k=0}^{\infty}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}=\frac{16}{\sqrt{3}\,\pi}.$$

Conjecture (Sun [Sci. China Math. 54(2011)]). For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{12}3)^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 12\mid p-1,\ p=x^2+y^2,\ 3\nmid x \text{ and } 3\mid y,\\ -(\frac{xy}{3})4xy\pmod{p^2} & \text{if } 12\mid p-5 \text{ and } p=x^2+y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3p\left(\frac{p}{3}\right)+\frac{5}{24}p^3B_{p-2}\left(\frac13\right)\pmod{p^4}.$$

Another Example Illustrating the Philosophy

I would like to offer \$90 for the first proof of the identity in the following conjecture and \$105 for the first proof of congruences in the conjecture.

Conjecture (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

Another Example Illustrating the Philosophy (continued)

And

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } 2p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 2p-6x^2 \pmod{p^2} & \text{if } 2p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 10x^2-2p \pmod{p^2} & \text{if } 2p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 28x^2-2p \pmod{p^2} & \text{if } p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 14x^2-2p \pmod{p^2} & \text{if } 2p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p})=-1.\end{cases}$$

Remark. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

One more Example Illustrating the Philosophy

Conjecture (Z.-W. Sun, Jan. 2012) (i) For any prime $p > 3$ we have

$$\sum_{n=0}^{p-1}\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^{n}5^k\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(3+2\left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

(ii) We have the identity

$$\sum_{n=0}^{\infty}\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^{n}5^k\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}=\frac{9}{\pi}(2+\sqrt{2}).$$

Conjecture (Sun). For any prime $p > 5$, we have

$$\left(\frac{-1}{p}\right)\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{576^n}\sum_{k=0}^{n}\frac{5^k\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1,\ p=x^2+30y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1,\ p=2x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1,\ p=3x^2+10y^2,\\ 20x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1,\ p=5x^2+6y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

Generalized central trinomial coefficients

For real numbers b and c, we define

$$T_n(b,c) := [x^n](x^2+bx+c)^n$$

$$\text{(the coefficient of } x^n \text{ in } (x^2+bx+c)^n)$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k.$$

Recursion: $T_0(b,c)=1$, $T_1(b,c)=b$, and

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c) \ (n>0),$$

where $d=b^2-4c$. It is known that if $d \neq 0$ then

$$T_n(b,c) = \sqrt{d}^n P_n\left(\frac{b}{\sqrt{d}}\right)$$

where

$$P_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k$$

is the Legendre polynomial of degree n.

Replace $\binom{2k}{k}$ by $T_k(b,c)$

As $T_k(2,1)=\binom{2k}{k}$, in 2010 I viewed $T_k(b,c)$ as a natural extension of the central binomial coefficients. In contrast with my conjectures on $\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k}$ and $\sum_{k=0}^{p-1}(a+dk)\frac{\binom{2k}{k}^3}{m^k}$ modulo p^2 (with p an odd prime not dividing m), in December 2010 I formulated many conjectures with some $\binom{2k}{k}$ replaced by $T_k(b,c)$. For example, I made the following conjecture.

Conjecture (Sun, 2010-12-25). Let p be any odd prime. Then

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1 \text{ and } p=x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6\pmod 7.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}\equiv 7p\left(\frac{-1}{p}\right)\pmod{p^2}.$$

Asymptotic Behavior of $T_n(b,c)$

By the Laplace-Heine formula, for $x \notin [-1,1]$ we have

$$P_n(x) \sim \frac{(x+\sqrt{x^2-1})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{x^2-1}} \quad \text{as } n \to +\infty.$$

It follows that if $b > 0$ and $c > 0$ then

$$T_n(b,c) \sim f_n(b,c) := \frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}.$$

as $n \to +\infty$. Note that $T_n(-b,c) = (-1)^n T_n(b,c)$.

Conjecture (Sun, 2011; proved by S. Wagner): For $b, c > 0$,

$$T_n(b,c) = f_n(b,c)\left(1 + \frac{b-4\sqrt{c}}{16n\sqrt{c}} + O\left(\frac{1}{n^2}\right)\right)$$

as $n \to +\infty$. If $c > 0$ and $b = 4\sqrt{c}$, then

$$\frac{T_n(b,c)}{\sqrt{c}^n} = \frac{3 \times 6^n}{\sqrt{6n\pi}}\left(1 + \frac{1}{8n^2} + \frac{15}{64n^3} + \frac{21}{32n^4} + O\left(\frac{1}{n^5}\right)\right).$$

If $c < 0$ and $b \in \mathbb{R}$ then

$$\lim_{n\to\infty} \sqrt[n]{|T_n(b,c)|} = \sqrt{b^2-4c}.$$

Story happened around Jan 1, 2011

$$T_n(1,16) \sim \frac{(1+2\sqrt{16})^{n+1/2}}{2\sqrt[4]{16}\sqrt{n\pi}} = \frac{9^{n+1/2}}{4\sqrt{n\pi}} = \frac{9^n}{12\sqrt{n\pi}}.$$

This is very similar to the fact that $\binom{2n}{n} \sim \frac{4^n}{\sqrt{n\pi}}$. On Dec. 25, 2010, I conjectured that for any odd prime p we have

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2 T_k(1,16)}{(-256)^k} \equiv 7p\left(\frac{-1}{p}\right) \pmod{p^2},$$

which is very similar to Ramanujan-type congruences.

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which is very similar to Ramanujan-type congruences.

Conjecture (Z. W. Sun, Jan. 2, 2011). We have

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2 T_k(1,16) = \frac{24}{\pi},$$

```
T[n_]:=If[n>0,Coefficient[(x^2+x+16)^n,x^n],1]
S[n_]:=Sum[(30k+7)Binomial[2k,k]^2*T[k]/(-256)^k,{k,0,n}]
Print[N[S[200]Pi,20]]
```

Output: 24.000000000000000000

New series for $1/\pi$ involving $T_k(b,c)$

For $b, c \in \mathbb{Z}$ let $T_k(b,c)$ be the coefficient of x^k in $(x^2+bx+c)^k$. In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a, b, c, d, m integers and $mbcd(b^2-4c)$ nonzero. In August I added 8 new series for $1/\pi$ of type III.

`Type I.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

`Type II.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

`Type III.` $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

`Type IV.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

`Type V.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

New series for $1/\pi$ involving $T_k(b,c)$

For $b,c \in \mathbb{Z}$ let $T_k(b,c)$ be the coefficient of x^k in $(x^2+bx+c)^k$. In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a,b,c,d,m integers and $mbcd(b^2-4c)$ nonzero. In August I added 8 new series for $1/\pi$ of type III.

Type I. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

Type II. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

Type III. $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

Type IV. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

Type V. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

In October 2011, I found 10 conjectural series for $1/\pi$ of two new types:

Type VI. $\sum_{k=0}^{\infty}(a+dk)T_k^3(b,c)/m^k$.

Type VII. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k} T_k^2(b,c)/m^k$.

This stimulated several papers by H.-H. Chan, J. Wan, W. Zudilin.

My conjectural series of type VI

$$\sum_{k=0}^{\infty} \frac{66k+17}{(2^{11}3^3)^k} T_k^3(10, 11^2) = \frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty} \frac{126k+31}{(-80)^{3k}} T_k^3(22, 21^2) = \frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty} \frac{3990k+1147}{(-288)^{3k}} T_k^3(62, 95^2) = \frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

I would like to offer \$300 as the prize for the person who can provide first rigorous proofs of all the above three identities. The last one was inspired by my following conjecture for primes $p > 3$.

$$\sum_{k=0}^{p-1} \frac{3990k+1147}{(-288)^{3k}} T_k^3(62, 95^2)$$
$$\equiv \frac{p}{19}\left(17563\left(\frac{-14}{p}\right) + 4230\left(\frac{-2}{p}\right)\right) \pmod{p^2}.$$

My unsolved conjectural series of type VII

Conjecture (Sun, 2011). (i) For any $n \in \mathbb{Z}^+$, the number

$$\frac{1}{n\binom{2n-1}{n-1}} \sum_{k=0}^{n-1} (2800512k + 435257)434^{2(n-1-k)} \binom{2k}{k} T_k(73, 576)^2$$

is an odd integer, and

$$n\binom{2n-1}{n-1} \Bigm| \sum_{k=0}^{n-1} (24k + 5)28^{2(n-1-k)} \binom{2k}{k} T_k(4, 9)^2.$$

(ii) We have

$$\sum_{k=0}^{p-1} \frac{2800512k + 435257}{434^{2k}} \binom{2k}{k} T_k(73, 576)^2 = \frac{10406669}{2\sqrt{6}\,\pi},$$

$$\sum_{k=0}^{\infty} \frac{24k + 5}{28^{2k}} \binom{2k}{k} T_k(4, 9)^2 = \frac{49}{9\pi}(\sqrt{3} + \sqrt{6}).$$

Conjecture (Sun). (i) If $p > 3$ is a prime with $p \neq 7, 11, 17, 31$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k(73,576)^2}{434^{2k}}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{17}) = 1,\ p = x^2 + 102y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{17}) = 1, (\frac{2}{p}) = (\frac{p}{3}) = -1,\ p = 2x^2 + 51y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1, (\frac{2}{p}) = (\frac{p}{17}) = -1,\ p = 3x^2 + 34y^2, \\ 24x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, (\frac{p}{3}) = (\frac{p}{17}) = -1,\ p = 6x^2 + 17y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-102}{p}) = -1, \end{cases}$$

where x and y are integers.

(ii) For any odd prime $p \neq 7, 31$, we have

$$\sum_{k=0}^{p-1} \frac{2800512k + 435257}{434^{2k}} \binom{2k}{k} T_k(73,576)^2$$

$$\equiv p\left(466752\left(\frac{-6}{p}\right) - 31495\right) \pmod{p^2}.$$

Conjecture (Sun). (i) For any prime $p > 7$, we have

$$\sum_{k=0}^{p-1} \binom{2k}{k} \frac{T_k(4,9)^2}{28^{2k}}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{5}) = 1,\ p = x^2 + 30y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1, (\frac{2}{p}) = (\frac{p}{5}) = -1,\ p = 3x^2 + 10y^2, \\ 2p - 8x^2 \pmod{p^2} & \text{if } (\frac{2}{p}) = 1, (\frac{p}{3}) = (\frac{p}{5}) = -1,\ p = 2x^2 + 15y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{5}) = 1, (\frac{2}{p}) = (\frac{p}{3}) = -1,\ p = 5x^2 + 6y^2, \\ p\delta_{p,7} \pmod{p^2} & \text{if } (\frac{-30}{p}) = -1. \end{cases}$$

where x and y are integers.

(ii) For any odd prime $p \neq 7$, we have

$$\sum_{k=0}^{p-1} \frac{24k+5}{28^{2k}} \binom{2k}{k} T_k(4,9)^2 \equiv p \left(\frac{-6}{p}\right) \left(4 + \left(\frac{2}{p}\right)\right) \pmod{p^2}.$$

520-Series

In 1895 J. Franel introduced the Franel numbers
$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n \in \mathbb{N})$. In view of Strehl's identity
$f_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}$, in 2011 I introduced the Franel polynomials

$$f_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} x^k = \sum_{k=0}^{n} \binom{n}{k} \binom{k}{n-k} \binom{2k}{k} x^k \ (n \geqslant 0).$$

520-Series:

$$\sum_{k=0}^{\infty} \frac{1054k + 233}{3840^k} \binom{2k}{k} f_k(-64) = \frac{520}{\pi}.$$

As May 20 is the day for Nanjing University, I offered \$520 as the prize for proving this 520-series.

In 2013, M. Rogers and A. Straub [Int. J. Number Theory 9(2013)] won the prize via their following paper.

M. Rogers and A. Straub, *A solution of Sun's \$520 challenge concerning* $520/\pi$, Int. J. Number Theory 9(2013), 1273–1288.

Series for $1/\pi$ involving a kind of polynomials

In March 2011, I introduced the polynomials

$$p_n(x)=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}x^{n-k}\quad (n=0,1,2,\ldots)$$

and proved that $\sum_{k=0}^{\infty}k\binom{2k}{k}p_k(4)/128^k=\sqrt{2}/\pi$,

$$\sum_{k=0}^{\infty}\frac{8k+1}{576^k}\binom{2k}{k}p_k(4)=\frac{9}{2\pi},\ \sum_{k=0}^{\infty}\frac{8k+1}{(-4032)^k}\binom{2k}{k}p_k(4)=\frac{9\sqrt{7}}{8\pi}.$$

via Ramanujan-type series for $1/\pi$. I noted that

$$\binom{2n}{n}p_n(4)=\sum_{k=0}^{n}\binom{2k}{k}^2\binom{4k}{2k}\binom{k}{n-k}(-64)^{n-k}.$$

Conjecture (Sun, 2011).

$$\sum_{k=0}^{\infty}\frac{4k+1}{(-192)^k}\binom{2k}{k}p_k(4)=\frac{\sqrt{3}}{\pi}.$$

Series for $1/\pi$ involving a kind of polynomials

Conjecture (Sun, 2011) We have

$$\sum_{k=0}^{\infty}\frac{17k-224}{(-225)^k}\binom{2k}{k}p_k(-14)=\frac{1800}{\pi},\ \sum_{k=0}^{\infty}\frac{15k-256}{17^{2k}}\binom{2k}{k}p_k(18)=\frac{2312}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{20k-11}{(-576)^k}\binom{2k}{k}p_k(-32)=\frac{90}{\pi},\ \sum_{k=0}^{\infty}\frac{3k-2}{640^k}\binom{2k}{k}p_k(36)=\frac{5\sqrt{10}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{20k-67}{(-3136)^k}\binom{2k}{k}p_k(-192)=\frac{490}{\pi},\ \sum_{k=0}^{\infty}\frac{7k-24}{3200^k}\binom{2k}{k}p_k(196)=\frac{125\sqrt{2}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{5k-32}{(-6336)^k}\binom{2k}{k}p_k(-392)=\frac{495}{2\pi},\sum_{k=0}^{\infty}\frac{66k-427}{6400^k}\binom{2k}{k}p_k(396)=\frac{1000\sqrt{11}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{34k-7}{(-18432)^k}\binom{2k}{k}p_k(-896)=\frac{54\sqrt{2}}{\pi},\ \sum_{k=0}^{\infty}\frac{24k-5}{136^{2k}}\binom{2k}{k}p_k(900)=\frac{867}{16\pi}.$$

A New One Found in 2019:

$$\sum_{k=0}^{\infty}\frac{12k+1}{100^k}\binom{2k}{k}p_k\left(\frac{9}{4}\right)=\frac{75}{4\pi}.$$

Comments from Shaun Cooper

In 2017, Prof. Shaun Cooper published the following book:
`S. Cooper, Ramanujan Theta Functions, Springer, Cham, 2017.`

In his Notes for Chapter 14 (Ramanujan's series for $1/\pi$), he wrote the following comments:

"The theory of Ramanujan's series for $1/\pi$ was extended significantly by the announcement of a large number of conjectures by Z.-W. Sun that are summarized in [279]. Sun's conjectures have stimulated and inspired works by W. Zudilin and coauthors, including work with H.H. Chan and J. Wan [93], the paper [125], works with J. Guillera [172], with J. Wan [294] and the paper [308]. See also the work of M. Rogers and A. Straub [258] and the works of J. Wan [292], [293]."

Main references:

1. Z.-W. Sun, *List of conjectural series for powers of* π *and other constants*, preprint, `arXiv:1102.5649`, 2011-2014.

2. Z.-W. Sun, *Conjectures and results on* x^2 *mod* p^2 *with* $4p = x^2 + dy^2$, in: Number Theory and Related Area (eds., Y. Ouyang, C. Xing, F. Xu and P. Zhang), Adv. Lect. Math. 27, Higher Education Press and Internat. Press, Beijing-Boston, 2013, pp. 149–197.

Thank you!