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Historical Remarks on my Conjectural Series

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Abstract

During 2010-2016, I have formulated 235 conjectural series for powers of π and other important constants. On my list there are 178 reasonable series for π^{-1}, four series for π^2, two series for π^{-2}, five series for π^4, two series for π^5, three series for π^6, seven series for $\zeta(3)$, one series for $\pi\zeta(3)$, two series for $\pi^2\zeta(3)$, one series for $\zeta(3)^2$, three series involving both $\zeta(3)^2$ and π^6, one series for $\zeta(5)$, three series involving both $\zeta(5)$ and $\zeta(2)\zeta(3)$, two series involving both $\pi\zeta(5)$ and $\pi^3\zeta(3)$, three series involving $\zeta(7)$, three series for $K = L(2, (\frac{\cdot}{3}))$, one series for the Catalan constant G, two series for πG, one series involving both $\pi^3 G$ and $\pi^2\zeta(3)$, two series for πK, two series involving $L = L(4, (\frac{\cdot}{3}))$, three series involving $\beta(4) = L(4, (\frac{-4}{\cdot}))$, and four series for $\pi^2 \log a$ ($a = 2, 3, \frac{\sqrt{5}+1}{2}$).

In this talk I'll give some historical remarks on my conjectural series, and reveal how I found some typical ones.

Main Reference

Zhi-Wei Sun, *List of conjectural series for powers of π and other constants*, preprint, arXiv:1102.5649.

My initial contact with π-series (1984-86)

When I was an undergraduate at Nanjing University, I learned from calculus the following classical results :

Leibniz:

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Euler:

$$\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

But I did not know any other π-series then.

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

My first impression on Ramanujan-type series

In a year around 2003, I happened to see a paper on Ramanujan-type series. Here is one of Ramanujan series for $1/\pi$:

$$\sum_{k=0}^{\infty}(28k+3)\frac{(-27)^k}{512^k}\cdot\frac{(1/2)_k(1/6)_k(5/6)_k}{(1)_k^3}=\frac{32\sqrt{2}}{\pi}.$$

At that time I did not like this at all since it is too complicated! I only enjoy simple and beautiful results! Thus this paper gave me almost no impression and I could not remember what paper it is.

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are 36 known Ramanujan-type series for $1/\pi$ with $a,b,m\in\mathbb{Z}$. I prefer their forms in terms of binomial coefficients.

Conjectures of Rodriguez-Villegas

It is easy to see that for any $k \in \mathbb{N} = \{0, 1, 2, \dots\}$ we have

$$\binom{-1/2}{k} = \prod_{j=1}^{k} \frac{1/2 - j}{j} = (-1)^k \frac{(2k-1)!!}{k!2^k} = \frac{(-1)^k (2k)!}{(k!2^k)^2} = \frac{\binom{2k}{k}}{(-4)^k}.$$

In 2003 Rodriguez-Villegas conjectured the following congruences for primes $p > 3$ (which were soon confirmed by E. Mortenson):

$$\sum_{k=0}^{p-1} \binom{-1/2}{k}^2 = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \binom{-1/3}{k}\binom{-2/3}{k} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \binom{-1/4}{k}\binom{-3/4}{k} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \binom{-1/6}{k}\binom{-5/6}{k} = \sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

On $a(p), b(p), c(p)$

For a power series $f(q)$ in q, we let $[q^n]f(q)$ denote the coefficient of q^n in $f(q)$.

For any prime $p > 3$, it is known that

$$a(p) := [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6 = \begin{cases} 4x^2-2p & \text{if } p = x^2+y^2 \ (2 \nmid x), \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

$$\begin{aligned} b(p) :&= [q^p]q\prod_{n=1}^{\infty}(1-q^{6n})^3(1-q^{2n})^3 \\ &= \begin{cases} 4x^2-2p & \text{if } p = x^2+3y^2 \text{ with } x,y \in \mathbb{Z}, \\ 0 & \text{if } p \equiv 2 \pmod 3, \end{cases} \end{aligned}$$

$$\begin{aligned} c(p) :=& [q^p]q\prod_{n=1}^{\infty}(1-q^{n})^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2 \\ =& \begin{cases} 4x^2-2p & \text{if } (\frac{-2}{p}) = 1 \text{ and } p = x^2+2y^2 \text{ with } x,y \in \mathbb{Z}, \\ 0 & \text{if } (\frac{-2}{p}) = -1, \text{ i.e., } p \equiv 5,7 \pmod 8). \end{cases} \end{aligned}$$

Conjectures of Rodriguez-Villegas

Let $p > 3$ be a prime. In 2003 Rodriguez-Villegas conjectured that

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}^3 = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/3}{k}\binom{-2/3}{k} = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/4}{k}\binom{-3/4}{k} = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/6}{k}\binom{-5/6}{k} = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

are congruent to $a(p), b(p), c(p)$ and $(\frac{p}{3})a(p)$ mod p^2 respectively. Actually the first one was proved by Ishikawa [Nagoya Math. J. 118(1990)]. E. Mortenson [Proc. AMS 133(2005)] provided partial solutions to the last three and the remaining thing were proved by Z.-W. Sun [156(2012)].

My joint work on congruences modulo prime powers

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod p\ \ (d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod p\ \ \text{for } p>3.$$

Sun & R. Tauraso [AAM 45(2010); IJNT 7(2011)].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\ \ \text{for } p>3,$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers given by

$$B_0=1,\ \ \sum_{k=0}^{n}\binom{n+1}{k}B_k=0\ \ (n=1,2,3,\ldots).$$

My result on $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod p^2

Sun [Sci. China Math. 53(2010)]: Let p be an odd prime and let $m \in \mathbb{Z}$ with $p \nmid m$. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m^2-4m}{p}\right) + u_{p-(\frac{m^2-4m}{p})} \pmod{p^2},$$

where $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0 = 0,\ u_1 = 1,\ \text{and}\ u_{n+1} = (m-2)u_n - u_{n-1}\ (n = 1, 2, 3, \ldots).$$

In particular,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} \quad (\text{mod } p^2). \qquad (*)$$

Remark. Later I found that $\sum_{k=0}^{p-1}\binom{2k}{k}/2^k$ mod p^3 is related to Euler numbers. Recently V.J.W. Guo proved a q-analogue of $(*)$.

What happened in November, 2009

After I determined $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ modulo p^2 in 2009, I systematically investigate congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^2/m^k$ and $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ modulo p^2. In particular, I formulated the following conjecture.

Conjecture (Z.-W. Sun, Nov. 2009). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3 \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1. \end{cases}$$

Prof. Ken Ono was very interested in this and he and one of his students worked on my conjecture. They claimed that they had a proof but in Jan. 2010 they replied me that they met real difficulties.

My above conjecture was finally confirmed by Z.-H. Sun [JNT 133(2013)], as well as J. Kibelbek, L. Long, K. Moss, B. Sheller and H. Yuan [`arXiv:1210.4489`, JNT 164(2016)].

What happened in Jan.-Feb. 2010

I visited India during Jan.-Feb. 2010. On Jan. 23 I suddenly realized that I should combine the congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ and $\sum_{k=0}^{p-1}k\binom{2k}{k}^3/m^k$ mod p^2. This led me to conjecture that

$$\frac{1}{p}\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3\equiv 8+16p^3B_{p-3}\pmod{p^4}\qquad(*)$$

and that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3\in\mathbb{Z}.$$

After reading my message to Number Theory List on Feb. 10, Kasper Andersen found on Feb. 11 that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3=\sum_{k=0}^{n-1}\binom{n+k-1}{k}^2$$

via Sloane's OEIS (Online Encyclopedia of Integer Sequences). Inspired by this I finally proved $(*)$.

van Hamme's conjecture

After I found $\sum_{k=0}^{p-1}\binom{2k}{k}^3/4096^k \bmod p^2$ and conjectured the congruence

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right)-p^3E_{p-3}\pmod{p^4},$$

I got to know that van Hamme had the conjecture

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right)\pmod{p^3}$$

motivated by Ramanujan's identity

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi}.$$

Thus I became interested in Ramanujan-type series and wrote to several mathematicians to get Hamme's paper.

Rediscover Zeilberger's series $\sum_{k=1}^\infty \frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6}$

I proved that for any odd prime p we have

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv 8p+16p^4B_{p-3} \pmod{p^5}.$$

As the series $\sum_{k=0}^\infty(21k+8)\binom{2k}{k}^3$ diverges, it does not provide a Ramanujan-type series for $1/\pi$. However, I observe that

$$\begin{aligned}\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 =&8+\sum_{k=(p+1)/2}^{p-1}(21(p-k)+8)\binom{2(p-k)}{p-k}^3\\ \equiv&8-\sum_{k=(p+1)/2}^{p-1}(21k-8)\left(\frac{2p}{k\binom{2k}{k}}\right)^3 \pmod{p}\end{aligned}$$

and this led me to find that

$$\sum_{k=1}^\infty \frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6} \quad \text{(D. Zeilberger, 1993)}.$$

Conjecture: $\sum_{k=1}^\infty \frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} = 8\pi^2$

Conjecture (Z.-W. Sun, 2010) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k} \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{11}) = 1 \ \&\ 4p = x^2 + 11y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1, \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{11k+3}{64^k}\binom{2k}{k}^2\binom{3k}{k} \equiv 3p + \frac{7}{2}p^4 B_{p-3} \pmod{p^5},$$

$$p\sum_{k=1}^{(p-1)/2} \frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} \equiv 32\frac{2^{p-1}-1}{p} - \frac{64}{3}p^2 B_{p-3} \pmod{p^3}.$$

Also,

$$\sum_{k=1}^\infty \frac{(11k-3)64^k}{k^2\binom{2k}{k}^2\binom{3k}{k}} = 8\pi^2 \quad \text{(confirmed by J. Guillera in 2013).}$$

Conjecture: $\sum_{k=1}^\infty \frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}} = K$

Conjecture (Z.-W. Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1;\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 4p\left(\frac{p}{3}\right)+\frac{4}{3}p^3B_{p-2}\left(\frac{1}{3}\right) \pmod{p^4}.$$

Also,

$$\sum_{k=1}^\infty\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}} = K := \sum_{k=1}^\infty\frac{(\frac{k}{3})}{k^2}$$ (confirmed by

Kh. Hessami Pilehrood and T. Hessami Pilehrood in 2012).

More such conjectural series

Conjecture (Z.-W. Sun, 2010; Sci. China Math. 54(2011))

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},$$

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K.$$

The three conjectural identities were finally confirmed by J. Guillera and M. Rogers [J. Austral. Math. Soc. 97(2014)].

A curious identity with \$480 prize for the solution

Conjecture (Z.-W. Sun) (i) (2009) For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{48^k}\equiv\begin{cases}2x-p/(2x)\pmod{p^2} & \text{if } p=x^2+3y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

(ii) (2012) For any prime $p>3$, we have

$$\sum_{k=1}^{p-1}\frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k}\equiv\frac{5}{12}p^2B_{p-2}\left(\frac{1}{3}\right)\pmod{p^3},$$

$$p^2\sum_{k=1}^{p-1}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}\equiv 4\left(\frac{p}{3}\right)+4p\pmod{p^2}.$$

(iii) (2012) We have

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}=\frac{15}{2}K\quad\text{(with \$480 prize for the solution).}$$

Three more conjectural series

Motivated by corresponding congruences, I made the following conjecture in 2010-2011.

Conjecture (Z.-W. Sun) (i) [Sci. China Math. 54(2011)] We have

$$\sum_{n=0}^{\infty}\frac{18n^2+7n+1}{(-128)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2=\frac{4\sqrt{2}}{\pi^2}$$

$$\sum_{n=0}^{\infty}\frac{40n^2+26n+5}{(-256)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{24}{\pi^2}.$$

(In 2004 H.H. Chan, S.H. Chan and Z. Liu [Adv. Math.] proved that $\sum_{n=0}^{\infty}\frac{5n+1}{64^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{8}{\sqrt{3}\pi}$.)

(ii) [Electron. J. Combin. 20(2013)] We have

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

My Philosophy about Series for $1/\pi$

Part I of the Philosophy. Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we must have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k} \equiv 0 \pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, 2^3, \ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k} \equiv \begin{cases} (\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+dy^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1, \end{cases}$$

or $d = 1$, $\gcd(15, m) > 1$, and for any prime $p \equiv 3 \pmod 4$ with $p \nmid 3m$ we have $\sum_{k=0}^{p-1} a_k/m^k \equiv 0 \pmod{p^2}$.

Philosophy about Series for $1/\pi$ (continued)

Part II of the Philosophy. Let $b, c, m, a_0, a_1, \ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k} \equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right) \pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

An Example Illustrating the Philosophy

I would like to offer \$90 for the first proof of the identity in the following conjecture and \$105 for the first proof of congruences in the conjecture.

Conjecture (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

An Example Illustrating the Philosophy (continued)

And

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } 2p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 2p-6x^2 \pmod{p^2} & \text{if } 2p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 10x^2-2p \pmod{p^2} & \text{if } 2p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 28x^2-2p \pmod{p^2} & \text{if } p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 14x^2-2p \pmod{p^2} & \text{if } 2p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p})=-1.\end{cases}$$

Remark. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

Another Example Illustrating the Philosophy

Conjecture (Z.-W. Sun, Jan. 2012) (i) For any prime $p>3$ we have

$$\sum_{n=0}^{p-1}\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^{n}5^k\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}} \equiv p\left(\frac{-1}{p}\right)\left(3+2\left(\frac{2}{p}\right)\right)\pmod{p^2}.$$

(ii) We have the identity

$$\sum_{n=0}^{\infty}\frac{28n+5}{576^n}\binom{2n}{n}\sum_{k=0}^{n}5^k\frac{\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2}{\binom{n}{k}}=\frac{9}{\pi}(2+\sqrt{2}).$$

New series for $1/\pi$ involving $T_k(b,c)$

For $b,c\in\mathbb{Z}$ let $T_k(b,c)$ be the coefficient of x^k in $(x^2+bx+c)^k$. As $T_k(2,1)=\binom{2k}{k}$, I view $T_k(b,c)$ as a natural extension of the central binomial coefficients.

In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a,b,c,d,m integers and $mbcd(b^2-4c)$ nonzero. In August I added 8 new series for $1/\pi$ of type III.

`Type I.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

`Type II.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

`Type III.` $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

`Type IV.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

`Type V.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

In October I found 10 conjectural series for $1/\pi$ of two new types:

`Type VI.` $\sum_{k=0}^{\infty}(a+dk)T_k^3(b,c)/m^k$.

`Type VII.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}T_k^2(b,c)/m^k$.

This stimulated several papers by H.-H. Chan, J. Wan, et al.

My conjectural series of type VI

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

I would like to offer \$300 as the prize for the person who can provide first rigorous proofs of all the above three identities. The last one was inspired by my following conjecture for primes $p>3$.

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)$$
$$\equiv\frac{p}{19}\left(17563\left(\frac{-14}{p}\right)+4230\left(\frac{-2}{p}\right)\right)\pmod{p^2}.$$

An idea to find more Ramanujan-type series

Idea. If there are Ramanujan-type series

$$\sum_{n=0}^{\infty}(bn+c)\sum_{k=0}^{n}a_k=\frac{C}{\pi}$$

or

$$\sum_{n=0}^{\infty}(bn+c)\binom{2n}{n}\sum_{k=0}^{n}a_k=\frac{C}{\pi},$$

then we should also seek for identities of the form

$$\sum_{n=0}^{\infty}(bn+c)\sum_{k=0}^{n}a_kx^k=\frac{C}{\pi}$$

or

$$\sum_{n=0}^{\infty}(bn+c)\binom{2n}{n}\sum_{k=0}^{n}a_kx^k=\frac{C}{\pi}.$$

An example

Example. Let

$$g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \quad \text{and} \quad g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k.$$

Since there are Ramanujan-type series

$$\sum_{n=0}^{\infty} (bk+c)\binom{2k}{k} g_k = \frac{C}{\pi},$$

I found in 2011 some Ramanujan-type identities of the form

$$\sum_{k=0}^{\infty} (bk+c) g_k(x) = \frac{C}{\pi}$$

such as

$$\sum_{k=0}^{\infty} \frac{16k+5}{324^k} \binom{2k}{k} g_k(-20) = \frac{189}{25\pi} \quad (\textbf{open}).$$

Franel polynomials

In 1895 J. Franel introduced the numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots).$$

In view of Strehl's identity $f_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n}$, I introduced the Franel polynomials

$$f_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} x^k = \sum_{k=0}^{n} \binom{n}{k} \binom{k}{n-k} \binom{2k}{k} x^k \ (n \geqslant 0)$$

and proved that $\sum_{k=0}^{n} \binom{n}{k} f_k(x) = g_n(x)$.

Since there are Ramanujan-type series

$$\sum_{n=0}^{\infty} (bk + c) \binom{2k}{k} f_k = \frac{C}{\pi},$$

I found in 2011 some Ramanujan-type identities of the form

$$\sum_{k=0}^{\infty} (bk + c) f_k(x) = \frac{C}{\pi}.$$

520-series

For example, I conjectured that

$$\sum_{k=0}^{\infty} \frac{1054k + 233}{3840^k} \binom{2k}{k} f_k(-64) = \frac{520}{\pi}.$$

As May 20 is the day for Nanjing University, I offered \$520 as the prize for proving this 520-series.

In 2013, M. Rogers and A. Straub [Int. J. Number Theory 9(2013)] won the prize via their following paper.

M. Rogers and A. Straub, *A solution of Sun's \$520 challenge concerning* $520/\pi$, Int. J. Number Theory 9(2013), 1273–1288.

Some known series involving harmonic numbers

Harmonic numbers are those $H_n = \sum_{0<k\leqslant n} 1/k$ $(n = 0, 1, 2, \ldots)$. For $m = 2, 3, \ldots$ the harmonic numbers of order m are defined by $H_n^{(m)} = \sum_{0<k\leqslant n} 1/k^m$ $(n = 0, 1, 2, \ldots)$.

$$\arcsin^2 \frac{x}{2} = \frac{1}{2}\sum_{k=0}^{\infty} \frac{x^{2k}}{k^2\binom{2k}{k}},\ \arcsin^4 \frac{x}{2} = \frac{3}{2}\sum_{k=0}^{\infty} \frac{H_{k-1}^{(2)} x^{2k}}{k^2\binom{2k}{k}}\ (|x| \leqslant 2).$$

$$\zeta(3) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\ \text{(Markov, 1890)},\ \zeta(4) = \frac{36}{17}\sum_{k=1}^{\infty} \frac{1}{k^4\binom{2k}{k}},$$

$$\zeta(5) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\left(\frac{2}{k^2} - \frac{5}{2}H_{k-1}^{(2)}\right)\ \text{(Koecher and Leshchiner, 1980)}$$

$$\zeta(7) = \frac{5}{2}\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\left(5H_{k-1}^{(4)} + \frac{1}{k^4}\right)$$

(conjectured by J. M. Borwein and D. M. Bradlay in 1996, and proved by G. Almkvist and A. Granville in 1999).

Few typical conjectural series involving harmonic numbers

Conjecture (Z.-W. Sun, 2014) We have

$$\sum_{k=1}^{\infty}\frac{H_{2k}-H_k+2/k}{k^4\binom{2k}{k}}=\frac{11}{9}\zeta(5)$$

(confirmed by J. Ablinger [Experiment. Math. 26(2017)]). Also,

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{10H_k-3/k}{k^3\binom{2k}{k}}=\frac{\pi^4}{30},$$

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_k^{(3)}+1/(5k^3)}{k^3\binom{2k}{k}}=\frac{2}{5}\zeta(3)^2,$$

$$\sum_{n=1}^{\infty}\frac{3H_{k-1}^2+4H_{k-1}/k}{k^2\binom{2k}{k}}=\frac{\pi^4}{360}\quad\text{(found in 2016)},$$

$$\sum_{k=1}^{\infty}\frac{L_{2k}}{k^2\binom{2k}{k}}\left(\frac{1}{k}+\frac{1}{k+1}+\cdots+\frac{1}{2k}\right)=\frac{41\zeta(3)+4\pi^2\log\phi}{25},$$

where $\phi=(\sqrt{5}+1)/2$, and $L_0,L_1,L_2,\ldots$ are the Lucas numbers.

Thank you!