REVIEW OF C. REIHER'S SOLUTION TO THE KEMNITZ CONJECTURE

Zhi-Wei Sun

Department of Mathematics
Nanjing University
Nanjing 210093, P. R. China
E-mail: zwsun@nju.edu.cn
Homepage: http://pweb.nju.edu.cn/zwsun

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Abstract. We give a survey of research on the Kemnitz conjecture, and simplify C. Reiher's recent proof of the Kemnitz conjecture. We make the comments for those interested in zero-sums including C. Reiher, not for any publication purpose.

1. Previous work on the Kemnitz conjecture

Here are two famous theorems in the theory of zero-sums.

The Erdős-Ginzburg-Ziv Theorem [Bull. Research Council. Israel, 1961]. *For any* $c_1, \cdots, c_{2n-1} \in \mathbb{Z}$*, there is an* $I \subseteq [1, 2n-1] = \{1, \cdots, 2n-1\}$ *with* $|I| = n$ *such that* $\sum_{s\in I} c_s \equiv 0 \pmod{n}$*. In other words, given* $2n-1$ *(not necessarily distinct) elements of* $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$*, we can select* n *of them with the sum vanishing.*

Olson's Theorem [J. Number Theory, 1969]. *Let p be a prime and let G be an additive abelian p-group isomorphic to $\mathbb{Z}_{p_1^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p_r^{\alpha_r}}$. Set $k = 1 + \sum_{i=1}^{r}(p_i^{\alpha_i} - 1)$. Then for any $c, c_1, \cdots, c_k \in G$ we have*

$$\sum_{\substack{I \subseteq [1,k] \\ \sum_{s\in I} c_s = c}} (-1)^{|I|} \equiv 0 \pmod{p},$$

and in particular there exists a nonempty $I \subseteq [1,k]$ with $\sum_{s\in I} c_s = 0$.

What is the smallest integer $k = s(\mathbb{Z}_n^2)$ such that every sequence of k elements in $\mathbb{Z}_n^2 = \mathbb{Z}_n \oplus \mathbb{Z}_n$ contains a zero-sum subsequence of length n? In 1983 Kemnitz [Ars Combin.] conjectured that $s(\mathbb{Z}_n^2) = 4n - 3$, and the conjecture can be reduced to the case with n a prime. In 1993 Alon and Dubiner showed that $s(\mathbb{Z}_n^2) \leqslant 6n - 5$. In 2000 Rónyai [Combinatorica] was able to prove that $s(\mathbb{Z}_p^2) \leqslant 4p - 2$ for every prime p, in 2001 W. D. Gao [J. Combin. Theory Ser. A] used Olson's group ring approach to deduce that $s(\mathbb{Z}_q^2) \leqslant 4q - 2$ for any prime power q. All these results were obtained by various ingenious algebraic methods.

The following lemma plays an indispensable role in the study of the Kemnitz conjecture.

The Alon-Dubiner Lemma. *Let q be a prime power, and let $c_1, \cdots, c_{3q}$ be elements of $\mathbb{Z}_q^2$ with $c_1 + \cdots + c_{3q} = 0$. Then there is an $I \subseteq [1, 3q]$ with $|I| = q$ such that $\sum_{s\in I} c_s = 0$.*

Here is a useful formula due to Zhi-Wei Sun.

Theorem 1.1 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]. *Let R be a ring with identity, and let*

$f(x_1,\cdots,x_k)$ be a polynomial over R. If $J\subseteq[1,k]$ and $|J|\geqslant\deg f$, then we have the formula

$$\sum_{I\subseteq J}(-1)^{|J|-|I|}f([1\in I],\cdots,[k\in I])=\Big[\prod_{j\in J}x_j\Big]f(x_1,\cdots,x_k)$$

where we use $[x_1^{i_1}\cdots x_k^{i_k}]f(x_1,\cdots,x_k)$ to denote the coefficient of the monomial $x_1^{i_1}\cdots x_k^{i_k}$ in the polynomial $f(x_1,\cdots,x_k)$, and let $[i\in I]$ be 1 or 0 according to whether $i\in I$ or not.

The EGZ theorem, Olson's theorem and a special case of Alon's Combinatorial Nullstellenstaz [Combin. Probab. Comput. 1999] **are easy consequences of the above formula!**

By using the powerful formula, Sun [Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`] obtained the following result concerning the Kemnitz conjecture.

Theorem 1.2 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]. *Let p be a prime and let $h>0$ be an integer. Let $a_s,b_s\in\mathbb{Z}$ for $s=1,\cdots,4p^h-2$.*

(i) *Set $\mathcal{I}=\{I\subseteq[1,4p^h-2]\colon\sum_{s\in I}a_s\equiv\sum_{s\in I}b_s\equiv 0\pmod{p^h}\}$. Then*

$$|\{I\in\mathcal{I}\colon|I|=p^h\}|\equiv|\{I\in\mathcal{I}\colon|I|=3p^h\}|+2\pmod{p}.$$

(ii) *Suppose that*

$$\sum_{\substack{I,J\subseteq[1,4p^h-3]\\ |I|=|J|=p^h-1\\ I\cap J=\emptyset}}\Big(\prod_{i\in I}a_i\Big)\Big(\prod_{j\in J}b_j\Big)\not\equiv 2\pmod{p}.$$

Then there exists an $I\subseteq[1,4p^h-3]$ with $|I|=p^h$ such that $\sum_{s\in I}a_s\equiv\sum_{s\in I}b_s\equiv 0\pmod{p^h}$.

2. Reiher's recent work and related things

Recently I got C. Reiher's paper "*On Kemnitz's conjecture concerning lattice points in the plane*" from Prof. Alfred Geroldinger. In the paper Reiher completely proved the Kemnitz conjecture which had been open for 20 years! I'm much impressed by Reiher's cleverness, and I think his work represents one of the most important achievements in the theory of zero-sums.

Reiher's paper has 4 pages. Pages 1–3 are devoted to 5 corollaries to the Chevalley-Warning theorem which are needed later. Actually this can be significantly simplified by using Theorem 1.2(i) with $a_{4p-2} = b_{4p-2} = 0$.

A Consequence of Theorem 1.2(i). *Let p be a prime and let $h > 0$ be an integer. Let $a_s, b_s \in \mathbb{Z}$ for $s = 1, \cdots, 4p^h - 3$. Set $\mathcal{I} = \{I \subseteq [1, 4p^h - 3] \colon \sum_{s\in I} a_s \equiv \sum_{s\in I} b_s \equiv 0 \pmod{p^h}\}$. Then*

$$\begin{aligned}&|\{I \in \mathcal{I} \colon |I| = p^h\}| + |\{I \in \mathcal{I} \colon |I| = p^h - 1\}|\\ \equiv&|\{I \in \mathcal{I} \colon |I| = 3p^h\}| + |\{I \in \mathcal{I} \colon |I| = 3p^h - 1\}| + 2 \pmod{p}.\end{aligned}$$

On the last page of his paper, C. Reiher provided a key lemma which is obtained by a combinatorial method rather than an algebraic method.

Reiher's Lemma. *Let p be a prime and let $a_s, b_s \in \mathbb{Z}$ for $s = 1, \cdots, 4p-3$. Set*

$$\mathcal{I} = \left\{ I \subseteq [1, 4p-3] \colon \sum_{s\in I} a_s \equiv \sum_{s\in I} b_s \equiv 0 \pmod{p} \right\}.$$

Then, either $\{I \in \mathcal{I} \colon |I| = p\} \neq \emptyset$ or

$$|\{I \in \mathcal{I} \colon |I| = p - 1\}| \equiv |\{I \in \mathcal{I} \colon |I| = 3p - 1\}| \pmod{p}.$$

We remark that the prime power version of this lemma also holds.

Combining Reiher's Lemma, the Alon-Dubiner lemma and the above consequence of Theorem 1.2(i), we immediately obtain the following result of Reiher.

Reiher's Theorem. *The Kemnitz conjecture is true.*

What does Reiher's solution teach us? When we apply a powerful algebraic method in combinatorics, we should also realize its disadvantage and should not forget combinatorial methods. A combination of algebraic methods and combinatorial methods might be more powerful!

By the way, Z. W. Sun has extended the EGZ theorem and the Alon-Dubiner lemma in the following way.

Theorem 2.1 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]. *Let* $A = \{a_1 (\mathrm{mod}\, n_1), \cdots, a_k (\mathrm{mod}\, n_k)\}$ *be a finite system of residue classes, and let* q *be a prime power.*

(i) *If* A *covers some integers exactly* $2q - 1$ *times and other integers exactly* $2q$ *times, then for any* $c_1, \cdots, c_k \in \mathbb{Z}_q$ *there exists an* $I \subseteq [1, k]$ *such that* $\sum_{s \in I} 1/n_s = q$ *and* $\sum_{s \in I} c_s = 0$.

(ii) *If* A *covers every integer exactly* $3q$ *times, then for any* $c_1, \cdots, c_k \in \mathbb{Z}_q^2$ *with* $c_1 + \cdots + c_k = 0$, *there exists an* $I \subseteq [1, k]$ *such that* $\sum_{s \in I} 1/n_s = q$ *and* $\sum_{s \in I} c_s = 0$.

It seems that we cannot have a similar extension of Reiher's theorem.