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Two Local-Global Theorems related to Covers of the Integers

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Abstract

A system $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ of k residue classes is called a *cover of* $\mathbb{Z}$ if any integer belongs to one of the k residue classes. This concept was introduced by P. Erdős in the 1930s. Erdős ever conjectured that A is a cover of $\mathbb{Z}$ whenever it covers $1, \ldots, 2^k$.

In this talk we introduce the speaker's two local-global theorems arising from his study of covers of $\mathbb{Z}$. One of them states that if $\psi_1, \ldots, \psi_k$ are maps from $\mathbb{Z}$ to an additive abelian group G with positive periods $n_1, \ldots, n_k$ respectively then the sum function $\psi = \psi_1 + \ldots + \psi_k$ is a constant function whenever $\psi(x) = \psi(x+1) = \ldots = \psi(x + |S| - 1)$ for some $x \in \mathbb{Z}$, where

$$S = \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : \ r = 0, \ldots, n_s - 1 \right\}.$$

Part I. The First Local-Global Theorem

Covering systems of residue classes

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, let $a(\mathrm{mod}\ n) = a + n\mathbb{Z}$ and

$$[\![x \equiv a \pmod{n}]\!] = \begin{cases} 1 & \text{if } x \equiv a \pmod{n}, \\ 0 & \text{otherwise.} \end{cases}$$

For a finite system $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ of residue classes, if $\bigcup_{s=1}^k a_s(\mathrm{mod}\ n_s) = \mathbb{Z}$ then we call A a *covering system* or a *cover of* $\mathbb{Z}$; if A covers each integer exactly once then A is called an *exact cover* of $\mathbb{Z}$.

The concept of covering system was introduced by Paul Erdős who gave the following example:

$$\{0(\mathrm{mod}\ 2),\ 0(\mathrm{mod}\ 3),\ 1(\mathrm{mod}\ 4),\ 5(\mathrm{mod}\ 6),\ 7(\mathrm{mod}\ 12)\}.$$

Another Example.

$$A = \{1(\mathrm{mod}\ 2), 2(\mathrm{mod}\ 2^2), \ldots, 2^{k-1}(\mathrm{mod}\ 2^k), 0(\mathrm{mod}\ 2^k)\}$$

is an exact cover of $\mathbb{Z}$. Note that $B = \{2^{s-1}(\mathrm{mod}\ 2^s)\}_{s=1}^k$ covers $1, \ldots, 2^k - 1$ but it does not cover 0.

An application of covers with distinct moduli

P. Erdős: Some residue class $a \pmod d$ with d even and a odd contains no numbers of the form $p + 2^n$ with p prime and $n \in \mathbb{N}$.

Proof. Let $A = \{a_1 \pmod{n_1}, \ldots, a_6 \pmod{n_6}\}$ be

$\{0 \pmod 2, 0 \pmod 3, 1 \pmod 4, 3 \pmod 8, 7 \pmod{12}, 23 \pmod{24}\}$.

This is a cover of $\mathbb{Z}$ with all the moduli distinct. Let $p_1 = 3,\ p_2 = 7,\ p_3 = 5,\ p_4 = 17,\ p_5 = 13,\ p_6 = 241$. Then $p_s \mid 2^{n_s} - 1$ but $p_s \nmid 2^n - 1$ for $0 < n < n_s$. As $2^5 \equiv 1 \pmod{31}$, we have $|\{p_s + 2^n \bmod 31 :\ 1 \leqslant s \leqslant 6,\ n \in \mathbb{N}\}| \leqslant 6 \times 5 < 31$. In fact, $p_s + 2^n \not\equiv 3 \pmod{31}$. Let $a \pmod d$ denote the residue class

$$1 \pmod 2 \cap 3 \pmod{31} \cap 2^{a_1} \pmod{p_1} \cap \ldots \cap 2^{a_6} \pmod{p_6}.$$

(This intersection is nonempty by the Chinese Remainder Theorem.) If $x \equiv a \pmod d$ and $x = p + 2^n$ with p prime and $n \in \mathbb{N}$. For some $1 \leqslant s \leqslant 6$, we have $n \equiv a_s \pmod{n_s}$ and hence $2^n \equiv 2^{a_s} \equiv x \pmod{p_s}$. Thus $p_s \mid p$ and $p_s = p$. But $x \not\equiv p_s + 2^n \pmod{31}$, so we get a contradiction.

Covering function

For $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$, its *covering function* $w_A : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$w_A(x) = |\{1 \leqslant s \leqslant k : \ x \equiv a_s \pmod{n_s}\}|.$$

Clearly W_A is periodic modulo $N_A = [n_1, \ldots, n_k]$. Observe that

$$\begin{aligned}\frac{1}{N_A}\sum_{x=0}^{N_A-1} w_A(x) =& \frac{1}{N_A}\sum_{x=0}^{N_A-1}\sum_{s=1}^{k} [\![x \equiv a_s \pmod{n_s}]\!] \\ =& \sum_{s=1}^{k} \frac{1}{N_A} |\{0 \leqslant x < N_A : \ x \equiv a_s \pmod{n_s}\}| \\ =& \sum_{s=1}^{k} \frac{1}{N_A} \cdot \frac{N_A}{n_s} = \sum_{s=1}^{k} \frac{1}{n_s}.\end{aligned}$$

If A covers each integer at least m times, then we call A an *m-cover* (of $\mathbb{Z}$) and note that $\sum_{s=1}^k \frac{1}{n_s} \geqslant m$. If A covers each integer exactly m times, then we call A an *exact m-cover* (of $\mathbb{Z}$) and note that $\sum_{s=1}^k \frac{1}{n_s} = m$ in this case.

Davenport-Mirsky-Newman-Rado Result

In the 1960s Paul Erdős made the following conjecture: *If* $A = \{a_s (\mathrm{mod}\ n_s)\}_{s=1}^k$ $(k > 1)$ *is a disjoint system with the moduli* $n_1, \ldots, n_k$ *distinct, then it cannot be a cover of* $\mathbb{Z}$.

H. Davenport, L. Mirsky, D. Newman and R. Radó (1960s): If $A = \{a_s (\mathrm{mod}\ n_s)\}_{s=1}^k$ $(k > 1)$ is a disjoint cover of $\mathbb{Z}$ with $1 < n_1 \leqslant n_2 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$, then we must have $n_{k-1} = n_k$.

Proof. Without loss of generality we let $0 \leqslant a_s < n_s$ $(1 \leqslant s \leqslant k)$. For $|z| < 1$ we have

$$\sum_{s=1}^{k} \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{s=1}^{k} \sum_{q=0}^{\infty} z^{a_s + qn_s} = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.$$

If $n_{k-1} < n_k$ then

$$\infty = \lim_{\substack{z \to e^{2\pi i/n_k} \\ |z|<1}} \frac{z^{a_k}}{1 - z^{n_k}} = \lim_{\substack{z \to e^{2\pi i/n_k} \\ |z|<1}} \left(\frac{1}{1 - z} - \sum_{s=1}^{k-1} \frac{z^{a_s}}{1 - z^{n_s}} \right) < \infty,$$

a contradiction!

Herzog-Schöheim Conjecture

Herzog-Schönheim Conjecture [Canad. Math. Bull. 17(1974)]. Let $\{a_iG_i\}_{i=1}^k$ ($k > 1$) be a partition of a group G into left cosets of subgroups $G_1, \ldots, G_k$. Then the (finite) indices $n_1 = [G : G_1], \ldots, n_k = [G : G_k]$ cannot be distinct.

Berger, Felzenbaum and Fraenkel [Canad. Bull. Math. 1986] The H-S Conjecture holds for finite nilpotent groups.

Z.-W. Sun [J. Algebra 273(2004)]. Let $\mathcal{A} = \{a_iG_i\}_{i=1}^k$ be a finite system of left cosets in a group G with not all the G_i equal to G. Suppose that $\mathcal{A}$ covers all the elements of G the same number of times, and that among the (finite) indices

$$n_1 = [G : G_1] \leqslant \ldots \leqslant n_k = [G : G_k].$$

each occurs at most $M \in \mathbb{Z}^+$ times. If all the G_i are subnormal in G, then $M > 1$ and

$$\log n_1 \leqslant \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log\log M).$$

A conjecture of Stein

A Conjecture of S.K. Stein [Math. Ann. 134(1958)]: If $A = \{a_s (\mathrm{mod}\ n_s)\}_{s=1}^{k}$ $(1 < n_1 < \ldots < n_k)$ is a disjoint system, then one of $1, \ldots, 2^k$ is not covered by A.

Note that if $A = \{a_s (\mathrm{mod}\ n_s)\}_{s=1}^{k}$ $(1 < n_1 < \ldots < n_k)$ is a disjoint system then it is not a cover of $\mathbb{Z}$ by the Davenport-Mirsky-Newman-Rado result.

P. Erdős [Mat. Lapok 13(1962)] proved a weaker version of Stein's Conjecture with 2^k replaced by $k2^k$.

Erdős' Conjecture

In 1965, P. Erdős offered \$25 prize for a proof of his following conjecture which is a refinement of Stein's conjecture.

Erdős' Conjecture (1962). $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a covering system if it covers all those integers from 1 to 2^k.

Remark. The 2^k in Erdős' conjecture is best possible because $\{2^{s-1}(\mathrm{mod}\ 2^s)\}_{s=1}^k$ covers $1, \ldots, 2^k - 1$ but does not cover any multiple of 2^k.

In 1969–1970 R. B. Crittenden and C. L. Vanden Eynden [Bull. Amer. Math. Soc. 1969; Proc. Amer. Math. Soc. 1970] supplied a long and awkward proof of the Erdős conjecture for $k \geqslant 20$, which involves some deep results concerning the distribution of primes.

A local-global theorem

As usual, the fractional part of a real number x is denoted by $\{x\}$.

For real numbers α and $\beta > 0$, we define

$$\alpha + \beta\mathbb{Z} := \{\alpha + \beta x : \ x \in \mathbb{Z}\}.$$

The First Local-Global Theorem (Z.-W. Sun [Acta Arith. 72(1995)]). Let $\alpha_1, \ldots, \alpha_k$ be real numbers and $\beta_1, \ldots, \beta_k$ be positive real numbers. Then $A = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k$ covers all the integers at least m times if it covers $|S|$ consecutive integers at least m times, where

$$S = \left\{ \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} : \ I \subseteq \{1, \ldots, k\} \right\}.$$

Remark. For $1 \leqslant m \leqslant k$, clearly an integer x is covered by $A = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k$ at least m times if and only if it is covered by $\{\alpha_s + \beta_s\mathbb{Z}\}_{\substack{s=1 \\ s \notin J}}^k$ for all $J \subseteq \{1, \ldots, k\}$ with $|J| = m - 1$. So the theorem is reduced to the case $m = 1$.

Proof of the Local-Global Theorem with $m = 1$

For any integer x, clearly

$$\begin{aligned}
& x \text{ is covered by } A \\
\iff & e^{2\pi i(\alpha_s - x)/\beta_s} = 1 \text{ for some } s = 1, \ldots, k \\
\iff & \prod_{s=1}^{k} \left(1 - e^{2\pi i(\alpha_s - x)/\beta_s}\right) = 0 \\
\iff & \sum_{I \subseteq \{1,\ldots,k\}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} \alpha_s/\beta_s} \cdot e^{-2\pi i x \sum_{s\in I} 1/\beta_s} = 0 \\
\iff & \sum_{\theta \in S} e^{-2\pi i x\theta} z_\theta = 0,
\end{aligned}$$

where

$$z_\theta = \sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\sum_{s\in I} 1/\beta_s\} = \theta}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} \alpha_s/\beta_s}.$$

Proof of the Local-Global Theorem with $m = 1$

Suppose that A covers $|S|$ consecutive integers

$$a, a+1, \ldots, a+|S|-1$$

where $a \in \mathbb{Z}$. By the above,

$$\sum_{\theta \in S} (e^{-2\pi i\theta})^r (e^{-2\pi i a\theta} z_\theta) = 0$$

for $r = 0, 1, \ldots, |S|-1$. As the determinant

$$\left| (e^{-2\pi i\theta})^r \right|_{0 \leqslant r < |S|,\, \theta \in S}$$

is of Vandermonde's type and hence nonzero, by Cramer's rule we have $z_\theta = 0$ for all $\theta \in S$. Therefore

$$\sum_{\theta \in S} e^{-2\pi i x\theta} z_\theta = 0$$

for all $x \in \mathbb{Z}$, i.e., any $x \in \mathbb{Z}$ is covered by A.

A corollary

Corollary. Let $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ and $M=\max_{n\in\mathbb{Z}^+}|\{1\leqslant s\leqslant k:\ n_s=n\}|$. If A covers $2^{k-M}(M+1)$ consecutive integers at least m times then A is an m-cover.

Proof. Choose $n\in\mathbb{Z}^+$ with $J=\{1\leqslant s\leqslant k:\ n_s=n\}$ of cardinality M. Then

$$\begin{aligned}
&\left|\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}:\ I\subseteq\{1,\ldots,k\}\right\}\right| \\
\leqslant&\left|\left\{\sum_{s\in I\cap J}\frac{1}{n_s}+\sum_{s\in I\setminus J}\frac{1}{n_s}:\ I\subseteq\{1,\ldots,k\}\right\}\right| \\
\leqslant&\left|\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq J\right\}\right|\times\left|\left\{\sum_{s\in I}\frac{1}{n_s}:\ I\subseteq\{1,\ldots,k\}\setminus J\right\}\right| \\
\leqslant&\left|\left\{\frac{|I|}{n}:\ I\subseteq J\right\}\right|\times|\{I:\ I\subseteq\{1,\ldots,k\}\setminus J\}| \\
=&(|J|+1)2^{k-|J|}=(M+1)2^{k-M}.
\end{aligned}$$

Crittenden-Vanden Eynden Conjecture

Example: Let $1 \leqslant l \leqslant k$. Then the residue classes

$$2^{i-1}(\mathrm{mod}\ 2^i) \quad (i = 1, \ldots, k - l + 1)$$

are disjoint and their union is $\mathbb{Z} \setminus 0(\mathrm{mod}\ 2^{k-l+1})$. Thus the k residue classes

$$1(\mathrm{mod}\ l), \ldots, l-1(\mathrm{mod}\ l), l(\mathrm{mod}\ 2l), \ldots, 2^{k-l}l(\mathrm{mod}\ 2^{k-l+1}l)$$

are disjoint and their union is $\mathbb{Z} \setminus 0(\mathrm{mod}\ 2^{k-l+1}l)$. So, the system A of these k residue classes covers $1, \ldots, 2^{k-l+1}l - 1$ but it is not a cover of $\mathbb{Z}$. Note that each modulus in A occurs at most $l - 1$ times and also every modulus of A is at least l.

Crittenden-Vanden Eynden Conjecture [Amer. Math. Monthly 79(1972)]. Let $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ with each modulus at least l, where $1 \leqslant l \leqslant k$. A is a cover of $\mathbb{Z}$ if it covers $1, \ldots, 2^{k-l+1}l$.

Remark. When $l = 1, 2$ this reduces to Erdős' conjecture. The above conjecture in the case $l = 3$ was proved by R.J. Simpson [J. Austral Math. Soc. 63(1972)].

An application of the First Local-Global Theorem

Theorem. Let $m_1, \dots, m_{n-1}$ $(n > 1)$ be integers. If there is a permutation $\sigma \in S_{n-1}$ such that $n \nmid sm_{\sigma(s)}$ for all $s = 1, \dots, n-1$, then the set

$$\left\{ \sum_{i \in I} m_i : \ I \subseteq \{1, \dots, n-1\} \right\}$$

contains a complete system of residues modulo n.

Proof. $A = \{s + (n/m_{\sigma(s)})\mathbb{Z}\}_{s=1}^{n-1}$ covers $1, \dots, n-1$ but it does not cover 0. By the Local-Global Theorem, the fractional parts

$$\left\{ \sum_{s \in I} \frac{1}{n/m_{\sigma(s)}} \right\} \quad (I \subseteq \{1, \dots, n-1\})$$

must have more than $n-1$ distinct values. Thus, the set

$$\left\{ \sum_{i \in I} m_i : \ I \subseteq \{1, \dots, n-1\} \right\} = \left\{ \sum_{s \in I} m_{\sigma(s)} : \ I \subseteq \{1, \dots, n-1\} \right\}$$

contains a complete system of residues modulo n.

A natural question

Let $m_1, \ldots, m_{n-1}$ $(n > 1)$ be integers. Now it is natural to ask when there is a permutation $\sigma \in S_{n-1}$ such that $n \nmid sm_{\sigma(s)}$ for all $s = 1, \ldots, n-1$. Clearly, this happens if $(m_s, n) = 1$ for all $s = 1, \ldots, n-1$. So we have

Corollary (Z.-W. Sun [Eletron. Res. Announc. Amer. Math. Soc., 9(2003)]). Let $m_1, \ldots, m_{n-1}$ $(n > 1)$ be integers all relatively prime to n. Then the subset sums $\sum_{i \in I} m_i \ \ (I \subseteq \{1, \ldots, n-1\})$ contain a complete system of residues modulo n.

If there is such a permutation $\sigma \in S_{n-1}$ such that $n \nmid sm_{\sigma(s)}$ for all $s = 1, \ldots, n-1$, then for each positive divisor d of n we have

$$|\{1 \leqslant c < d : \ d \nmid m_{\sigma(cn/d)}\}| \geqslant \left|\left\{1 \leqslant c < d : \ n \nmid \frac{cn}{d} m_{\sigma(cn/d)}\right\}\right| = d-1,$$

and hence the sequence $\{m_s\}_{s=1}^{n-1}$ has the following property:

$$\left|\{1 \leqslant s < n : \ d \nmid m_s\}\right| \geqslant d-1 \ \text{ for any } d \in D(n),$$

where $D(n)$ denotes the set of all positive divisors of n.

A conjecture of Sun

Conjecture (Z.-W. Sun, May 1, 2004). Let $m_1, \ldots, m_{n-1}$ ($n > 1$) be integers satisfying the condition

$$\left|\{1 \leqslant s < n : \ d \nmid m_s\}\right| \geqslant d - 1 \ \text{ for any } d \in D(n)$$

(where $D(n)$ denotes the set of all positive divisors of n). Then there is a permutation $\sigma \in S_{n-1}$ such that $n \nmid s m_{\sigma(s)}$ for all $s = 1, \ldots, n-1$.

Note that

$$n \nmid s m_t \iff s \overline{m}_t \neq \bar{0},$$

where $\bar{a} = a + n\mathbb{Z}$ belongs to the additive cyclic group $\mathbb{Z}/n\mathbb{Z}$.

An extended version for finite abelian groups

For a finite multiplicative group G, its exponent $\exp(G)$ is defined to be the least positive integer such that $x^n = e$ for all $x \in G$, where e is the identity of G. For a finite abelian group G, $\exp(G)$ is known to be $\max\{o(x) : x \in G\}$, where $o(x)$ denotes the order of x. If G is an additive group, then for $k \in \mathbb{Z}^+$ and $a \in G$ we write ka for the sum $a_1 + \ldots + a_k$ with $a_1 = \cdots = a_k = a$.

Theorem (F. Ge and Z.-W. Sun [Electron. J. Combin. 24(2017)]). Let G be a finite additive abelian group with exponent $n > 1$. For any $a_1, \ldots, a_{n-1} \in G$, there is a permutation $\sigma \in S_{n-1}$ such that all the elements $sa_{\sigma(s)}$ $(s = 1, \ldots, n-1)$ are nonzero if and only if

$$\left|\left\{1 \leqslant s < n : \frac{n}{d} a_s \neq 0\right\}\right| \geqslant d - 1 \text{ for all } d \in D(n).$$

Remark. Applying this theorem to the cyclic group $\mathbb{Z}/n\mathbb{Z}$, we immediately confirm the conjecture of Sun.

About the Proof

Proof of the Necessariness. If there is a permutation $\sigma \in S_{n-1}$ such that $sa_{\sigma(s)} \neq 0$ for all $s = 1, \ldots, n-1$, then for any $d \in D(n)$ we have

$$\begin{aligned} &\left|\left\{1 \leqslant s < n : \ \frac{n}{d} a_s \neq 0\right\}\right| \\ \geqslant &\left|\left\{1 \leqslant c < d : \ \frac{cn}{d} a_{\sigma(cn/d)} \neq 0\right\}\right| = d - 1. \end{aligned}$$

The sufficiency is difficult to prove. We omit the details of the proof.

Part II. The Second Local-Global Theorem

Periodic arithmetical maps

The characteristic function of a residue class is a periodic arithmatical map. Dirichlet characters are also periodic functions.

If an element a in an additive abelian group G has order n, then the map $\psi : \mathbb{Z} \to G$ given by $\psi(x) = xa$ is periodic mod n.

The Second Local-Global Theorem

The Second Local-Global Theorem (Z.-W. Sun [J. Algebra, 293(2005)]). Let G be any abelian group written additively, and let $\psi_1, \ldots, \psi_k$ be maps from $\mathbb{Z}$ to G with periods $n_1, \ldots, n_k \in \mathbb{Z}^+$ respectively. Set $\psi = \psi_1 + \cdots + \psi_k$ and

$$S(n_1, \ldots, n_k) = \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : \ r = 0, \ldots, n_s - 1 \right\}.$$

(i) There are periodic maps $f_0, \ldots, f_{|S(n_1,\ldots,n_k)|-1} : \mathbb{Z} \to \mathbb{Z}$ only depending on $S(n_1, \ldots, n_k)$ such that $\psi(x) = \sum_{0 \leqslant r < |S(n_1,\ldots,n_k)|} f_r(x)\psi(r)$ for all $x \in \mathbb{Z}$. In particular, values of ψ are completely determined by the set $S(n_1, \ldots, n_k)$ and the initial values $\psi(0), \ldots, \psi(|S(n_1, \ldots, n_k)| - 1)$.

(ii) ψ is constant if $\psi(x)$ equals a constant for $|S(n_1, \ldots, n_k)|$ ($\leqslant n_1 + \cdots + n_k - k + 1$) consecutive integers x.

Remarks on $|S(n_1,\ldots,n_k)|$

Let $D=\{d\in\mathbb{Z}^+ :\ d\mid n_s \text{ for some } s=1,\ldots,k\}$. Then

$$|S(n_1,\ldots,n_k)|=\left|\bigcup_{d\in D}\left\{\frac{c}{d}:\ 0\leqslant c<d\ \&\ (c,d)=1\right\}\right|=\sum_{d\in D}\varphi(d),$$

where φ is the well-known Euler function.

As $|\bigcap_{s\in I}\{r/n_s:\ r=0,\ldots,n_s-1\}|=\gcd(n_s:\ s\in I)$ for all $\emptyset\neq I\subseteq\{1,\ldots,k\}$, by the inclusion-exclusion principle, we have

$$|S(n_1,\ldots,n_k)|=\sum_{\emptyset\neq I\subseteq\{1,\ldots,k\}}(-1)^{|I|-1}\gcd(n_s:\ s\in I).$$

Two corollaries

As $|S(m,n)| = m + n - \gcd(m,n)$, we have the following consequence.

Corollary 1. Let g and h be maps from $\mathbb{Z}$ to an additive abelian group G with positive periods m and n respectively. Then $\{g(x) - h(x) : x \in \mathbb{Z}\}$ is contained in the subgroup of G generated by those $g(r) - h(r)$ with $0 \leqslant r < m + n - \gcd(m,n)$; in particular, g and h are identical if $g(r) = h(r)$ for all $r = 0, \ldots, m + n - \gcd(m,n) - 1$.

Fine-Wilf Theorem (N.J. Fine and H.S. Wilf [Proc. Amer. Math. Soc. 16(1965)]). Let g and h be maps from $\mathbb{Z}$ to the real field $\mathbb{R}$ with positive periods m and n respectively. If $g(r) = h(r)$ for all $r = 0, \ldots, m + n - \gcd(m,n) - 1$, then we have $g = h$.

Corollary 2. $A = \{a_s(\mathrm{mod}\ n_s\}_{s=1}^k$ is an exact m-cover of $\mathbb{Z}$ if it covers $|\bigcup_{s=1}^k \{r/n_s :\ r = 0, \ldots, n_s - 1\}|$ ($\leqslant \sum_{s=1}^k n_s - k + 1$) consecutive integers exactly m times.

One more corollary

Corollary 3. Let G be an additive abelian group. Let $c_1, \ldots, c_k$ be elements of G with orders $n_1, \ldots, n_k \in \mathbb{Z}^+$ respectively. For any $P_1(x), \ldots, P_k(x) \in \mathbb{Z}[x]$, the sum

$$P_1(x)c_1 + \ldots + P_k(x)c_k$$

vanishes for all $x \in \mathbb{Z}$ if it vanishes for $|S(n_1, \ldots, n_k)|$ consecutive integers x.

Proof. For each $s = 1, \ldots, k$, the arithmetical function $\psi_s(x) = P_s(x)c_s$ is periodic modulo n_s. By the Second Local-Global Theorem, $\psi = \psi_1 + \ldots + \psi_k$ is the zero function if $\psi(x) = 0$ for $|S(n_1, \ldots, n_k)|$ consecutive integers x.

A lemma

Let Ω denote the ring of all algebraic integers. Clearly all roots of unity belong to Ω.

Lemma 1. Let $\psi(x) = \sum_{s=1}^{k} c_s \omega_s^x$ for $x \in \mathbb{Z}$, where $c_1, \ldots, c_k \in \mathbb{C}$, and $\omega_1, \ldots, \omega_k$ are roots of unity. Suppose that $\prod_{\zeta \in \{\omega_1, \ldots, \omega_k\}} (x - \zeta) \in R[x]$ where R is a subring of Ω containing $\mathbb{Z}$. Then we have $\psi = \psi(0) f_0 + \cdots + \psi(l-1) f_{l-1}$, where $l = |\{\omega_1, \ldots, \omega_k\}|$, and $f_0, \ldots, f_{l-1}$ are suitable periodic maps from $\mathbb{Z}$ to R only depending on the set $\{\omega_1, \ldots, \omega_k\}$.

Proof. Let $\zeta_1, \ldots, \zeta_l$ be all the distinct roots of unity among $\omega_1, \ldots, \omega_k$, and write

$$P(z) = \prod_{t=1}^{l} (z - \zeta_t) = z^l - a_1 z^{l-1} - \cdots - a_{l-1} z - a_l,$$

where

$$a_j = (-1)^{j-1} \sum_{1 \leqslant i_1 < \cdots < i_j \leqslant l} \zeta_{i_1} \cdots \zeta_{i_j} \in R \quad \text{for } j = 1, \ldots, l.$$

Proof of Lemma 1

Set $u_n = \sum_{t=1}^{l} c'_t \zeta_t^n$ for all $n \in \mathbb{Z}$, where $c'_t = \sum_{1 \leqslant s \leqslant k,\, \omega_s = \zeta_t} c_s$. Clearly $u_n = \sum_{s=1}^{k} c_s \omega_s^n = \psi(n)$. Also, $\{u_n\}_{n \in \mathbb{Z}}$ is a linear recurrence because

$$\sum_{j=1}^{l} a_j u_{n-j} = \sum_{j=1}^{l} a_j \sum_{t=1}^{l} c'_t \zeta_t^{n-j} = \sum_{t=1}^{l} c'_t \zeta_t^{n-l} \sum_{j=1}^{l} a_j \zeta_t^{l-j}$$
$$= \sum_{t=1}^{l} c'_t \zeta_t^{n-l} \left(\zeta_t^l - P(\zeta_t) \right) = u_n.$$

Let N be the least positive integer with $\zeta_t^N = 1$ for all $t = 1, \ldots, l$. Then $N \geqslant l$ since $\zeta_1, \ldots, \zeta_l$ are distinct. Construct $f_r : \mathbb{Z} \to R$ $(0 \leqslant r \leqslant l-1)$ with period N as follows:

$$f_r(n) = \begin{cases} \delta_{n,r} & \text{if } n \in \{0, \ldots, l-1\}, \\ \sum_{j=1}^{l} a_j f_r(n-j) & \text{if } n \in \{l, \ldots, N-1\}. \end{cases}$$

Proof of Lemma 1

For $0 \leqslant n < l$, we have

$$\sum_{r=0}^{l-1} \psi(r) f_r(n) = \sum_{r=0}^{l-1} \psi(r) \delta_{n,r} = \psi(n).$$

If $l \leqslant n < N$ and $\sum_{r=0}^{l-1} \psi(r) f_r(m) = \psi(m)$ for all $m = 0, \ldots, n-1$, then

$$\begin{aligned} \psi(n) =& u_n = \sum_{j=1}^{l} a_j u_{n-j} = \sum_{j=1}^{l} a_j \psi(n-j) \\ &= \sum_{j=1}^{l} a_j \sum_{r=0}^{l-1} \psi(r) f_r(n-j) = \sum_{r=0}^{l-1} \psi(r) \sum_{j=1}^{l} a_j f_r(n-j) \\ &= \sum_{r=0}^{l-1} \psi(r) f_r(n). \end{aligned}$$

So $\sum_{r=0}^{l-1} \psi(r) f_r(n) = \psi(n)$ for all $0 \leqslant n < N$, and hence $\psi = \sum_{r=0}^{l-1} \psi(r) f_r$.

Lemma 2

Lemma 2. Let $\psi = \psi_1 + \cdots + \psi_k$ where each ψ_s $(1 \leqslant s \leqslant k)$ is a complex-valued function on $\mathbb{Z}$ with period $n_s \in \mathbb{Z}^+$. Then ψ can be written in the form $\sum_{0 \leqslant r < |S(n_1,\ldots,n_k)|} \psi(r) f_r$, where $f_0, \ldots, f_{|S(n_1,\ldots,n_k)|-1}$ are suitable periodic maps from $\mathbb{Z}$ to $\mathbb{Z}$ only depending on $S(n_1, \ldots, n_k)$.

Proof. If $x \in \mathbb{Z}$ then

$$\psi(x) = \sum_{s=1}^{k} \sum_{\substack{0 \leqslant a < n_s \\ n_s \mid x-a}} \psi_s(a) = \sum_{s=1}^{k} \frac{1}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) \sum_{r=0}^{n_s-1} e^{2\pi i \frac{r}{n_s}(x-a)}$$

$$= \sum_{s=1}^{k} \sum_{r=0}^{n_s-1} \left(\frac{1}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) e^{-2\pi i a \frac{r}{n_s}} \right) \left(e^{2\pi i \frac{r}{n_s}} \right)^x .$$

Observe that

$$\prod_{\theta \in S(n_1,\ldots,n_k)} \left(x - e^{2\pi i \theta} \right) = \prod_{d \mid n_s \text{ for some } s=1,\ldots,k} \Phi_d(x) \in \mathbb{Z}[x],$$

where $\Phi_d(x) = \prod_{0 \leqslant c < d,\, \gcd(c,d)=1} (x - e^{2\pi i c/d}) \in \mathbb{Z}[x]$ is the dth cyclotomic polynomial. Now it suffices to apply Lemma 1.

Remarks

Let $m, n_1, \ldots, n_k \in \mathbb{Z}^+$, and let $f_0, \ldots, f_{|S(n_1,\ldots,n_k)|-1}$ be as in Lemma 2. For each $s = 1, \ldots, k$ let $\psi_s : \mathbb{Z} \to \mathbb{Z}$ be a map which has period n_s modulo m (i.e., $\psi_s(a) \equiv \psi_s(b) \pmod{m}$ whenever $a \equiv b \pmod{n_s}$). Let x be any integer. By Lemma 2 we have

$$\sum_{s=1}^{k} \psi_s'(x) = \sum\nolimits_{0 \leqslant r < |S(n_1,\ldots,n_k)|} f_r(x) \sum_{s=1}^{k} \psi_s'(r),$$

where $\psi_s'(x) = \sum_{0 \leqslant a < n_s,\, n_s \mid x-a} \psi_s(a)$. As $\psi_s'(x) \equiv \psi_s(x) \pmod{m}$ for each $s = 1, \ldots, k$, this yields that

$$\psi(x) \equiv \sum\nolimits_{0 \leqslant r < |S(n_1,\ldots,n_k)|} f_r(x)\psi(r) \pmod{m},$$

where $\psi = \psi_1 + \cdots + \psi_k$. For $a \in \mathbb{Z}$ let $\bar{a}$ denote the residue class $a \pmod{m}$ in the ring $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Then we have

$$\sum_{s=1}^{k} \overline{\psi_s(x)} = \overline{\psi(x)} = \sum\nolimits_{0 \leqslant r < |S(n_1,\ldots,n_k)|} f_r(x)\overline{\psi(r)}.$$

So part (i) of the Second Local-Global Theorem holds for $G = \mathbb{Z}_m$.

Proof of the Second Local-Global Theorem

Without any loss of generality, we simply let G coincide with its subgroup generated by the finite set

$$\{\psi_s(x) :\ x = 0, \dots, n_s - 1;\ s = 1, \dots, k\}.$$

Since G is finitely generated, there are $m_1, \dots, m_l \in \mathbb{Z}^+$ and $n \in \{0, 1, \dots\}$ such that G is isomorphic to the direct sum $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_l} \oplus \mathbb{Z}^n$. Let us identify G with $G_1 \oplus \cdots \oplus G_{l+n}$, where $G_t = \mathbb{Z}_{m_t}$ for $t = 1, \dots, l$, and $G_{l+1} = \cdots = G_{l+n} = \mathbb{Z}$.

Let $f_0, \dots, f_{|S(n_1,\dots,n_k)|-1}$ be as in Lemma 2, and let x be any integer. For $s = 1, \dots, k$ we write $\psi_s(x)$ in the vector form

$$\langle \psi_{s,1}(x), \dots, \psi_{s,\,l+n}(x) \rangle,$$

where $\psi_{s,\,t}(x) \in G_t$ for $t = 1, \dots, l + n$.

Proof of the Second Local-Global Theorem

Set $\psi^{(t)} = \sum_{s=1}^{k} \psi_{s,\,t}$ for $t = 1, \ldots, l + n$. Since $\psi_{s,\,t} : \mathbb{Z} \to G_t$ also has period n_s, we have

$$\psi^{(t)}(x) = \sum\nolimits_{0 \leqslant r < |S(n_1,\ldots,n_k)|} f_r(x)\psi^{(t)}(r)$$

Therefore,

$$\begin{aligned}\psi(x) =& \langle \psi^{(1)}(x), \ldots, \psi^{(l+n)}(x)\rangle \\ =& \sum\nolimits_{0 \leqslant r < |S(n_1,\ldots,n_k)|} f_r(x)\langle \psi^{(1)}(r), \ldots, \psi^{(l+n)}(r)\rangle \\ =& \sum\nolimits_{0 \leqslant r < |S(n_1,\ldots,n_k)|} f_r(x)\psi(r).\end{aligned}$$

This proves the first part of the Second Local-Global Theorem.

Proof of the Second Local-Global Theorem

Now we prove the second part. Suppose that $\psi(a + r) = c$ for all $r = 0, \ldots, |S(n_1, \ldots, n_k)| - 1$, where $a \in \mathbb{Z}$ and $c \in G$.

For $x \in \mathbb{Z}$, let $\psi^*(x) = \psi_s(a + x)$ for $1 \leqslant s < k$, and $\psi_k^*(x) = \psi_k(a + x) - c$.

Set

$$\psi^*(x) = \psi_1^*(x) + \cdots + \psi_k^*(x) = \psi(a + x) - c.$$

By the first part of the Second Local-Global Theorem, the range of ψ^* is contained in the subgroup of G generated by

$$\{\psi^*(r) :\ 0 \leqslant r < |S(n_1, \ldots, n_k)|\} = \{0\}.$$

Thus $\psi(a + x) - c = \psi^*(x) = 0$ for all $x \in \mathbb{Z}$. This concludes our proof.

Part III. Connections between Covers of $\mathbb{Z}$ and Unit Fractions

Zhang's result and its extensions

Ming-Zhi Zhang (1989): If $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ then $\sum_{s\in I}\frac{1}{n_s} \in \mathbb{Z}^+$ for some $\emptyset \neq I \subseteq \{1,\ldots,k\}$.

Z.-W. Sun [Israel J. Math. 77(1992)]: If $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is an exact m-cover of $\mathbb{Z}$, then for any $n = 0,\ldots,m$ we have

$$\left|\left\{I \subseteq \{1,\ldots,k\} :\ \sum_{s\in I}\frac{1}{n_s} = n\right\}\right| \geqslant \binom{m}{n}.$$

Z.-W. Sun [Trans. Amer. Math. Soc. 348(1996)]: If $A = \{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is an m-cover of $\mathbb{Z}$, then for any $m_1,\ldots,m_k \in \mathbb{Z}^+$ there are at least m positive integers in the form $\sum_{s\in I} m_s/n_s$ with $I \subseteq \{1,\ldots,k\}$.

Other results

Z.-W. Sun [Acta Arith. 72(1995)]: If $A = \{a_s(\bmod\ n_s)\}_{s=1}^k$ is an exact m-cover of $\mathbb{Z}$, then for any $\emptyset \neq J \subset \{1, \ldots, k\}$ there is an $I \subseteq \{1, \ldots, k\}$ with $I \neq J$ such that $\sum_{s\in I} \frac{1}{n_s} = \sum_{s\in J} \frac{1}{n_s}$.

Z.-W. Sun [Combinatorica 23(2003)]: If the covering function of $A = \{a_s(\bmod\ n_s)\}_{s=1}^k$ is periodic modulo $n_0 \in \mathbb{Z}^+$, then

$$\left\{ \sum_{s\in J} \frac{1}{n_s} :\ J \subseteq \{1, \ldots, k-1\} \right\} \supseteq \left\{ \frac{r}{n_k} :\ 0 \leqslant r < \frac{n_k}{(n_k, n_0)} \right\}.$$

Z.-W. Sun [Adv. Appl. Math. 38(2007)]: If the covering function of $A = \{a_s(\bmod\ n_s)\}_{s=1}^k$ is periodic modulo n_k, then for any $r = 0, \ldots, n_k - 1$ we have

$$\left| \left\{ \sum_{s\in I} \frac{1}{n_s} :\ I \subseteq \{1, \ldots, k\} \ \& \ \left\{ \sum_{s\in I} \frac{1}{n_s} \right\} = \frac{r}{n_k} \right\} \right| \geqslant m.$$

References

Main References

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Thank you!