Two Local-Global Theorems related to Covers of the Integers

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/~zwsun

June 30, 2020
Abstract

A system $A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k$ of $k$ residue classes is called a cover of $\mathbb{Z}$ if any integer belongs to one of the $k$ residue classes. This concept was introduced by P. Erdős in the 1930s. Erdős ever conjectured that $A$ is a cover of $\mathbb{Z}$ whenever it covers $1, \ldots, 2^k$.

In this talk we introduce the speaker’s two local-global theorems arising from his study of covers of $\mathbb{Z}$. One of them states that if $\psi_1, \ldots, \psi_k$ are maps from $\mathbb{Z}$ to an additive abelian group $G$ with positive periods $n_1, \ldots, n_k$ respectively then the sum function $\psi = \psi_1 + \ldots + \psi_k$ is a constant function whenever $\psi(x) = \psi(x + 1) = \ldots = \psi(x + |S| - 1)$ for some $x \in \mathbb{Z}$, where

$$S = \bigcup_{s=1}^k \left\{ \frac{r}{n_s} : r = 0, \ldots, n_s - 1 \right\}.$$
Part I. The First Local-Global Theorem
Covering systems of residue classes

For \( a \in \mathbb{Z} \) and \( n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\} \), let \( a(\text{mod } n) = a + n\mathbb{Z} \) and

\[
[x \equiv a \pmod{n}] = \begin{cases} 
1 & \text{if } x \equiv a \pmod{n}, \\
0 & \text{otherwise}.
\end{cases}
\]

For a finite system \( A = \{a_s(\text{mod } n_s)\}_{s=1}^k \) of residue classes, if \( \bigcup_{s=1}^k a_s(\text{mod } n_s) = \mathbb{Z} \) then we call \( A \) a covering system or a cover of \( \mathbb{Z} \); if \( A \) covers each integer exactly once then \( A \) is called an exact cover of \( \mathbb{Z} \).

The concept of covering system was introduced by Paul Erdős who gave the following example:

\[ \{0(\text{mod } 2), 0(\text{mod } 3), 1(\text{mod } 4), 5(\text{mod } 6), 7(\text{mod } 12)\}. \]

**Another Example.**

\[ A = \{1(\text{mod } 2), 2(\text{mod } 2^2), \ldots, 2^{k-1}(\text{mod } 2^k), 0(\text{mod } 2^k)\} \]

is an exact cover of \( \mathbb{Z} \). Note that \( B = \{2^{s-1}(\text{mod } 2^s)\}_{s=1}^k \) covers \( 1, \ldots, 2^k - 1 \) but it does not cover 0.
An application of covers with distinct moduli

P. Erdős: Some residue class \( a \pmod{d} \) with \( d \) even and \( a \) odd contains no numbers of the form \( p + 2^n \) with \( p \) prime and \( n \in \mathbb{N} \).

Proof. Let \( A = \{ a_1 \pmod{n_1}, \ldots, a_6 \pmod{n_6} \} \) be
\[{0 \pmod{2}, 0 \pmod{3}, 1 \pmod{4}, 3 \pmod{8}, 7 \pmod{12}, 23 \pmod{24}}\].
This is a cover of \( \mathbb{Z} \) with all the moduli distinct. Let
\( p_1 = 3, \ p_2 = 7, \ p_3 = 5, \ p_4 = 17, \ p_5 = 13, \ p_6 = 241 \). Then
\( p_s \mid 2^{n_s} - 1 \) but \( p_s \nmid 2^n - 1 \) for \( 0 < n < n_s \). As \( 2^5 \equiv 1 \pmod{31} \),
we have \( |\{p_s + 2^n \pmod{31} : 1 \leq s \leq 6, \ n \in \mathbb{N}\}| \leq 6 \times 5 < 31 \). In fact, \( p_s + 2^n \not\equiv 3 \pmod{31} \). Let \( a \pmod{d} \) denote the residue class
\( 1 \pmod{2} \cap 3 \pmod{31} \cap 2^{a_1} \pmod{p_1} \cap \ldots \cap 2^{a_6} \pmod{p_6} \).
(This intersection is nonempty by the Chinese Remainder Theorem.) If \( x \equiv a \pmod{d} \) and \( x = p + 2^n \) with \( p \) prime and
\( n \in \mathbb{N} \). For some \( 1 \leq s \leq 6 \), we have \( n \equiv a_s \pmod{n_s} \) and hence
\( 2^n \equiv 2^{a_s} \equiv x \pmod{p_s} \). Thus \( p_s \mid p \) and \( p_s = p \). But \( x \not\equiv p_s + 2^n \pmod{31} \), so we get a contradiction.
Covering function

For \( A = \{a_s(\text{mod } n_s)\}_{s=1}^{k} \), its covering function \( w_A : \mathbb{Z} \to \mathbb{Z} \) is defined by

\[
w_A(x) = |\{ 1 \leq s \leq k : x \equiv a_s \pmod{n_s} \}|.
\]

Clearly \( W_A \) is periodic modulo \( N_A = [n_1, \ldots, n_k] \). Observe that

\[
\frac{1}{N_A} \sum_{x=0}^{N_A-1} w_A(x) = \frac{1}{N_A} \sum_{x=0}^{N_A-1} \sum_{s=1}^{k} [x \equiv a_s \pmod{n_s}]
\]

\[
= \sum_{s=1}^{k} \frac{1}{N_A} |\{ 0 \leq x < N_A : x \equiv a_s \pmod{n_s} \}|
\]

\[
= \sum_{s=1}^{k} \frac{1}{N_A} \cdot \frac{N_A}{n_s} = \sum_{s=1}^{k} \frac{1}{n_s}.
\]

If \( A \) covers each integer at least \( m \) times, then we call \( A \) an \( m \)-cover (of \( \mathbb{Z} \)) and note that \( \sum_{s=1}^{k} \frac{1}{n_s} \geq m \). If \( A \) covers each integer exactly \( m \) times, then we call \( A \) an exact \( m \)-cover (of \( \mathbb{Z} \)) and note that \( \sum_{s=1}^{k} \frac{1}{n_s} = m \) in this case.
Davenport-Mirsky-Newman-Rado Result

In the 1960s Paul Erdős made the following conjecture: If $A = \{a_s(\text{mod } n_s)\}_{s=1}^{k} (k > 1)$ is a disjoint system with the moduli $n_1, \ldots, n_k$ distinct, then it cannot be a cover of $\mathbb{Z}$.

H. Davenport, L. Mirsky, D. Newman and R. Radó (1960s): If $A = \{a_s(\text{mod } n_s)\}_{s=1}^{k} (k > 1)$ is a disjoint cover of $\mathbb{Z}$ with $1 < n_1 \leq n_2 \leq \cdots \leq n_{k-1} \leq n_k$, then we must have $n_{k-1} = n_k$.

**Proof.** Without loss of generality we let $0 \leq a_s < n_s (1 \leq s \leq k)$. For $|z| < 1$ we have

$$\sum_{s=1}^{k} \frac{z^{a_s}}{1 - z^{n_s}} = \sum_{s=1}^{k} \sum_{q=0}^{\infty} z^{a_s + qn_s} = \sum_{n=0}^{\infty} z^n = \frac{1}{1 - z}.$$ 

If $n_{k-1} < n_k$ then

$$\infty = \lim_{z \to e^{2\pi i/n_k}} \frac{z^{a_k}}{1 - z^{n_k}} = \lim_{z \to e^{2\pi i/n_k}} \left( \frac{1}{1 - z} - \sum_{s=1}^{k-1} \frac{z^{a_s}}{1 - z^{n_s}} \right) < \infty,$$

a contradiction!
Herzog-Schöheim Conjecture

Herzog-Schönheim Conjecture [Canad. Math. Bull. 17(1974)]. Let \( \{a_i G_i\}_{i=1}^k \) \((k > 1)\) be a partition of a group \( G \) into left cosets of subgroups \( G_1, \ldots, G_k \). Then the (finite) indices \( n_1 = [G : G_1], \ldots, n_k = [G : G_k] \) cannot be distinct.


Z.-W. Sun [J. Algebra 273(2004)]. Let \( A = \{a_i G_i\}_{i=1}^k \) be a finite system of left cosets in a group \( G \) with not all the \( G_i \) equal to \( G \). Suppose that \( A \) covers all the elements of \( G \) the same number of times, and that among the (finite) indices

\[
n_1 = [G : G_1] \leq \ldots \leq n_k = [G : G_k],
\]

each occurs at most \( M \in \mathbb{Z}^+ \) times. If all the \( G_i \) are subnormal in \( G \), then \( M > 1 \) and

\[
\log n_1 \leq \frac{e^\gamma}{\log 2} M \log^2 M + O(M \log M \log \log M).
\]
A Conjecture of S.K. Stein [Math. Ann. 134(1958)]: If \( A = \{a_s(\mod n_s)\}_{s=1}^k \) \((1 < n_1 < \ldots < n_k)\) is a disjoint system, then one of 1, \ldots, 2^k is not covered by \( A \).

Note that if \( A = \{a_s(\mod n_s)\}_{s=1}^k \) \((1 < n_1 < \ldots < n_k)\) is a disjoint system then it is not a cover of \( \mathbb{Z} \) by the Davenport-Mirsky-Newman-Rado result.

P. Erdős [Mat. Lapok 13(1962)] proved a weaker version of Stein’s Conjecture with 2^k replaced by \( k2^k \).
In 1965, P. Erdős offered $25 prize for a proof of his following conjecture which is a refinement of Stein’s conjecture.

**Erdős’ Conjecture** (1962). $A = \{a_s (\mod n_s)\}_{s=1}^k$ is a covering system if it covers all those integers from 1 to $2^k$.

*Remark.* The $2^k$ in Erdős’ conjecture is best possible because $\{2^{s-1}(\mod 2^s)\}_{s=1}^k$ covers $1, \ldots, 2^k - 1$ but does not cover any multiple of $2^k$.

A local-global theorem

As usual, the fractional part of a real number $x$ is denoted by $\{x\}$. For real numbers $\alpha$ and $\beta > 0$, we define

$$\alpha + \beta \mathbb{Z} := \{\alpha + \beta x : x \in \mathbb{Z}\}.$$ 

The First Local-Global Theorem (Z.-W. Sun [Acta Arith. 72(1995)]). Let $\alpha_1, \ldots, \alpha_k$ be real numbers and $\beta_1, \ldots, \beta_k$ be positive real numbers. Then $A = \{\alpha_s + \beta_s \mathbb{Z}\}_{s=1}^k$ covers all the integers at least $m$ times if it covers $|S|$ consecutive integers at least $m$ times, where

$$S = \left\{\left\{\sum_{s \in I} \frac{1}{\beta_s}\right\} : I \subseteq \{1, \ldots, k\}\right\}.$$ 

Remark. For $1 \leq m \leq k$, clearly an integer $x$ is covered by $A = \{\alpha_s + \beta_s \mathbb{Z}\}_{s=1}^k$ at least $m$ times if and only if it is covered by $\{\alpha_s + \beta_s \mathbb{Z}\}_{s=1}^k$ for all $J \subseteq \{1, \ldots, k\}$ with $|J| = m - 1$. So the theorem is reduced to the case $m = 1$. 


Proof of the Local-Global Theorem with $m = 1$

For any integer $x$, clearly

\[ x \text{ is covered by } A \iff e^{2\pi i(\alpha_s - x)/\beta_s} = 1 \text{ for some } s = 1, \ldots, k \]

\[ \iff \prod_{s=1}^{k} \left(1 - e^{2\pi i(\alpha_s - x)/\beta_s} \right) = 0 \]

\[ \iff \sum_{I\subseteq\{1,\ldots,k\}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} \alpha_s / \beta_s} \cdot e^{-2\pi i x \sum_{s\in I} 1 / \beta_s} = 0 \]

\[ \iff \sum_{\theta \in \mathcal{S}} e^{-2\pi i x \theta} z_\theta = 0, \]

where

\[ z_\theta = \sum_{\substack{I\subseteq\{1,\ldots,k\} \\{\sum_{s\in I} 1 / \beta_s\} = \theta}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} \alpha_s / \beta_s}. \]
Proof of the Local-Global Theorem with \( m = 1 \)

Suppose that \( A \) covers \(|S|\) consecutive integers

\[
a, a + 1, \ldots, a + |S| - 1
\]

where \( a \in \mathbb{Z} \). By the above,

\[
\sum_{\theta \in S} (e^{-2\pi i \theta})^r (e^{-2\pi i a \theta} z_\theta) = 0
\]

for \( r = 0, 1, \ldots, |S| - 1 \). As the determinant

\[
\begin{vmatrix}
(e^{-2\pi i \theta})^r \\
0 \leq r < |S|, \theta \in S
\end{vmatrix}
\]

is of Vandermonde’s type and hence nonzero, by Cramer’s rule we have \( z_\theta = 0 \) for all \( \theta \in S \). Therefore

\[
\sum_{\theta \in S} e^{-2\pi i x \theta} z_\theta = 0
\]

for all \( x \in \mathbb{Z} \), i.e., any \( x \in \mathbb{Z} \) is covered by \( A \).
A corollary

**Corollary.** Let $A = \{a_s (\mod n_s)\}_{s=1}^k$ and $M = \max_{n \in \mathbb{Z}^+} |\{1 \leq s \leq k : n_s = n\}|$. If $A$ covers $2^{k-M}(M + 1)$ consecutive integers at least $m$ times then $A$ is an $m$-cover.

**Proof.** Choose $n \in \mathbb{Z}^+$ with $J = \{1 \leq s \leq k : n_s = n\}$ of cardinality $M$. Then

$$\left|\left\{\left\{\sum_{s \in I} \frac{1}{n_s}\right\} : I \subseteq \{1, \ldots, k\}\right\}\right|$$

$$\leq \left|\left\{\sum_{s \in I \cap J} \frac{1}{n_s} + \sum_{s \in I \setminus J} \frac{1}{n_s} : I \subseteq \{1, \ldots, k\}\right\}\right|$$

$$\leq \left|\left\{\sum_{s \in I} \frac{1}{n_s} : I \subseteq J\right\}\right| \times \left|\left\{\sum_{s \in I} \frac{1}{n_s} : I \subseteq \{1, \ldots, k\} \setminus J\right\}\right|$$

$$\leq \left|\left\{|I| n : I \subseteq J\right\}\right| \times |\{I : I \subseteq \{1, \ldots, k\} \setminus J\}|$$

$$= (|J| + 1)2^{k-|J|} = (M + 1)2^{k-M}.$$
Crittenden-Vanden Eynden Conjecture

**Example:** Let $1 \leq l \leq k$. Then the residue classes

$$2^{i-1} \pmod{2^i} \quad (i = 1, \ldots, k - l + 1)$$

are disjoint and their union is $\mathbb{Z} \setminus 0 \pmod{2^{k-l+1}}$. Thus the $k$ residue classes

$$1 \pmod{l}, \ldots, l - 1 \pmod{l}, l \pmod{2l}, \ldots, 2^{k-l} l \pmod{2^{k-l+1} l}$$

are disjoint and their union is $\mathbb{Z} \setminus 0 \pmod{2^{k-l+1} l}$. So, the system $A$ of these $k$ residue classes covers $1, \ldots, 2^{k-l+1} l - 1$ but it is not a cover of $\mathbb{Z}$. Note that each modulus in $A$ occurs at most $l - 1$ times and also every modulus of $A$ is at least $l$.

**Crittenden-Vanden Eynden Conjecture** [Amer. Math. Monthly 79(1972)]. Let $A = \{a_s \pmod{n_s}\}_{s=1}^k$ with each modulus at least $l$, where $1 \leq l \leq k$. $A$ is a cover of $\mathbb{Z}$ if it covers $1, \ldots, 2^{k-l+1} l$.

**Remark.** When $l = 1, 2$ this reduces to Erdős’ conjecture. The above conjecture in the case $l = 3$ was proved by R.J. Simpson [J. Austral Math. Soc. 63(1972)].
An application of the First Local-Global Theorem

**Theorem.** Let $m_1, \ldots, m_{n-1} \ (n > 1)$ be integers. If there is a permutation $\sigma \in S_{n-1}$ such that $n \nmid sm_{\sigma(s)}$ for all $s = 1, \ldots, n - 1$, then the set

$$\left\{ \sum_{i \in I} m_i : I \subseteq \{1, \ldots, n - 1\} \right\}$$

contains a complete system of residues modulo $n$.

**Proof.** $A = \left\{ s + \left( n/m_{\sigma(s)} \right) \mathbb{Z} \right\}_{s=1}^{n-1}$ covers $1, \ldots, n - 1$ but it does not cover 0. By the Local-Global Theorem, the fractional parts

$$\left\{ \sum_{s \in I} \frac{1}{n/m_{\sigma(s)}} \right\} \quad (I \subseteq \{1, \ldots, n - 1\})$$

must have more than $n - 1$ distinct values. Thus, the set

$$\left\{ \sum_{i \in I} m_i : I \subseteq \{1, \ldots, n - 1\} \right\} = \left\{ \sum_{s \in I} m_{\sigma(s)} : I \subseteq \{1, \ldots, n - 1\} \right\}$$

contains a complete system of residues modulo $n$. 
A natural question

Let \( m_1, \ldots, m_{n-1} \) (\( n > 1 \)) be integers. Now it is natural to ask when there is a permutation \( \sigma \in S_{n-1} \) such that \( n \nmid s m_{\sigma(s)} \) for all \( s = 1, \ldots, n - 1 \). Clearly, this happens if \( (m_s, n) = 1 \) for all \( s = 1, \ldots, n - 1 \). So we have

**Corollary** (Z.-W. Sun [Eletron. Res. Announc. Amer. Math. Soc., 9(2003)]). Let \( m_1, \ldots, m_{n-1} \) (\( n > 1 \)) be integers all relatively prime to \( n \). Then the subset sums \( \sum_{i \in I} m_i \) \((I \subseteq \{1, \ldots, n-1\})\) contain a complete system of residues modulo \( n \).

If there is such a permutation \( \sigma \in S_{n-1} \) such that \( n \nmid s m_{\sigma(s)} \) for all \( s = 1, \ldots, n - 1 \), then for each positive divisor \( d \) of \( n \) we have

\[
\left| \left\{ 1 \leq c < d : d \nmid m_{\sigma(cn/d)} \right\} \right| \geq \left| \left\{ 1 \leq c < d : n \nmid \frac{cn}{d} m_{\sigma(cn/d)} \right\} \right| = d - 1,
\]

and hence the sequence \( \{m_s\}_{s=1}^{n-1} \) has the following property:

\[
\left| \left\{ 1 \leq s < n : d \nmid m_s \right\} \right| \geq d - 1 \quad \text{for any } d \in D(n),
\]

where \( D(n) \) denotes the set of all positive divisors of \( n \).
A conjecture of Sun

**Conjecture** (Z.-W. Sun, May 1, 2004). Let $m_1, \ldots, m_{n-1}$ ($n > 1$) be integers satisfying the condition

$$\left| \left\{ 1 \leq s < n : d \nmid m_s \right\} \right| \geq d - 1 \text{ for any } d \in D(n)$$

(where $D(n)$ denotes the set of all positive divisors of $n$). Then there is a permutation $\sigma \in S_{n-1}$ such that $n \nmid sm_{\sigma(s)}$ for all $s = 1, \ldots, n - 1$.

Note that

$$n \nmid sm_t \iff \bar{s} \bar{m}_t \neq \bar{0},$$

where $\bar{a} = a + n\mathbb{Z}$ belongs to the additive cyclic group $\mathbb{Z}/n\mathbb{Z}$. 
An extended version for finite abelian groups

For a finite multiplicative group $G$, its exponent $\exp(G)$ is defined to be the least positive integer such that $x^n = e$ for all $x \in G$, where $e$ is the identity of $G$. For a finite abelian group $G$, $\exp(G)$ is known to be $\max\{o(x) : x \in G\}$, where $o(x)$ denotes the order of $x$. If $G$ is an additive group, then for $k \in \mathbb{Z}^+$ and $a \in G$ we write $ka$ for the sum $a_1 + \ldots + a_k$ with $a_1 = \cdots = a_k = a$.

**Theorem** (F. Ge and Z.-W. Sun [Electron. J. Combin. 24(2017)]). Let $G$ be a finite additive abelian group with exponent $n > 1$. For any $a_1, \ldots, a_{n-1} \in G$, there is a permutation $\sigma \in S_{n-1}$ such that all the elements $sa_{\sigma(s)}$ $(s = 1, \ldots, n-1)$ are nonzero if and only if

$$\left|\left\{1 \leq s < n : \frac{n}{d} a_s \neq 0\right\}\right| \geq d - 1 \text{ for all } d \in D(n).$$

**Remark.** Applying this theorem to the cyclic group $\mathbb{Z}/n\mathbb{Z}$, we immediately confirm the conjecture of Sun.
About the Proof

Proof of the Necessariness. If there is a permutation $\sigma \in S_{n-1}$ such that $sa_{\sigma(s)} \neq 0$ for all $s = 1, \ldots, n-1$, then for any $d \in D(n)$ we have

$$\left| \left\{ 1 \leq s < n : \frac{n}{d} a_s \neq 0 \right\} \right| \geq \left| \left\{ 1 \leq c < d : \frac{cn}{d} a_{\sigma(cn/d)} \neq 0 \right\} \right| = d - 1.$$

The sufficiency is difficult to prove. We omit the details of the proof.
Part II. The Second Local-Global Theorem
Periodic arithmetical maps

The characteristic function of a residue class is a periodic arithmetical map. Dirichlet characters are also periodic functions.

If an element $a$ in an additive abelian group $G$ has order $n$, then the map $\psi : \mathbb{Z} \to G$ given by $\psi(x) = xa$ is periodic mod $n$. 
The Second Local-Global Theorem (Z.-W. Sun [J. Algebra, 293(2005)]). Let $G$ be any abelian group written additively, and let $\psi_1, \ldots, \psi_k$ be maps from $\mathbb{Z}$ to $G$ with periods $n_1, \ldots, n_k \in \mathbb{Z}^+$ respectively. Set $\psi = \psi_1 + \cdots + \psi_k$ and

$$S(n_1, \ldots, n_k) = \bigcup_{s=1}^{k} \left\{ \frac{r}{n_s} : r = 0, \ldots, n_s - 1 \right\}.$$ 

(i) There are periodic maps $f_0, \ldots, f_{|S(n_1,\ldots,n_k)|-1} : \mathbb{Z} \to \mathbb{Z}$ only depending on $S(n_1, \ldots, n_k)$ such that

$$\psi(x) = \sum_{0 \leq r < |S(n_1,\ldots,n_k)|} f_r(x)\psi(r) \text{ for all } x \in \mathbb{Z}.$$ 

In particular, values of $\psi$ are completely determined by the set $S(n_1, \ldots, n_k)$ and the initial values $\psi(0), \ldots, \psi(|S(n_1,\ldots,n_k)| - 1)$.

(ii) $\psi$ is constant if $\psi(x)$ equals a constant for $|S(n_1, \ldots, n_k)| (\leq n_1 + \cdots + n_k - k + 1)$ consecutive integers $x$. 
Remarks on $|S(n_1, \ldots, n_k)|$

Let $D = \{d \in \mathbb{Z}^+ : d \mid n_s \text{ for some } s = 1, \ldots, k\}$. Then

$$|S(n_1, \ldots, n_k)| = \left| \bigcup_{d \in D} \left\{ \frac{c}{d} : 0 \leq c < d \& (c, d) = 1 \right\} \right| = \sum_{d \in D} \varphi(d),$$

where $\varphi$ is the well-known Euler function.

As $|\bigcap_{s \in l}\{r/n_s : r = 0, \ldots, n_s - 1\}| = \gcd(n_s : s \in l)$ for all $\emptyset \neq l \subseteq \{1, \ldots, k\}$, by the inclusion-exclusion principle, we have

$$|S(n_1, \ldots, n_k)| = \sum_{\emptyset \neq l \subseteq \{1, \ldots, k\}} (-1)^{|l|-1} \gcd(n_s : s \in l).$$
Two corollaries

As $|S(m, n)| = m + n - \gcd(m, n)$, we have the following consequence.

**Corollary 1.** Let $g$ and $h$ be maps from $\mathbb{Z}$ to an additive abelian group $G$ with positive periods $m$ and $n$ respectively. Then $\{g(x) - h(x) : x \in \mathbb{Z}\}$ is contained in the subgroup of $G$ generated by those $g(r) - h(r)$ with $0 \leq r < m + n - \gcd(m, n)$; in particular, $g$ and $h$ are identical if $g(r) = h(r)$ for all $r = 0, \ldots, m + n - \gcd(m, n) - 1$.

**Fine-Wilf Theorem** (N.J. Fine and H.S. Wilf [Proc. Amer. Math. Soc. 16(1965)]). Let $g$ and $h$ be maps from $\mathbb{Z}$ to the real field $\mathbb{R}$ with positive periods $m$ and $n$ respectively. If $g(r) = h(r)$ for all $r = 0, \ldots, m + n - \gcd(m, n) - 1$, then we have $g = h$.

**Corollary 2.** $A = \{a_s (\text{mod } n_s)\}_{s=1}^{k}$ is an exact $m$-cover of $\mathbb{Z}$ if it covers $|\bigcup_{s=1}^{k}{r/n_s : r = 0, \ldots, n_s - 1}| \leq \sum_{s=1}^{k}{n_s - k + 1}$ consecutive integers exactly $m$ times.
One more corollary

**Corollary 3.** Let $G$ be an additive abelian group. Let $c_1, \ldots, c_k$ be elements of $G$ with orders $n_1, \ldots, n_k \in \mathbb{Z}^+$ respectively. For any $P_1(x), \ldots, P_k(x) \in \mathbb{Z}[x]$, the sum

$$P_1(x)c_1 + \ldots + P_k(x)c_k$$

vanishes for all $x \in \mathbb{Z}$ if it vanishes for $|S(n_1, \ldots, n_k)|$ consecutive integers $x$.

**Proof.** For each $s = 1, \ldots, k$, the arithmetical function

$$\psi_s(x) = P_s(x)c_s$$

is periodic modulo $n_s$. By the Second Local-Global Theorem, $\psi = \psi_1 + \ldots + \psi_k$ is the zero function if $\psi(x) = 0$ for $|S(n_1, \ldots, n_k)|$ consecutive integers $x$. 


A lemma

Let $\Omega$ denote the ring of all algebraic integers. Clearly all roots of unity belong to $\Omega$.

**Lemma 1.** Let $\psi(x) = \sum_{s=1}^{k} c_s \omega_s^x$ for $x \in \mathbb{Z}$, where $c_1, \ldots, c_k \in \mathbb{C}$, and $\omega_1, \ldots, \omega_k$ are roots of unity. Suppose that $\prod_{\zeta \in \{\omega_1, \ldots, \omega_k\}} (x - \zeta) \in R[x]$ where $R$ is a subring of $\Omega$ containing $\mathbb{Z}$. Then we have $\psi = \psi(0)f_0 + \cdots + \psi(l-1)f_{l-1}$, where $l = |\{\omega_1, \ldots, \omega_k\}|$, and $f_0, \ldots, f_{l-1}$ are suitable periodic maps from $\mathbb{Z}$ to $R$ only depending on the set $\{\omega_1, \ldots, \omega_k\}$.

**Proof.** Let $\zeta_1, \ldots, \zeta_l$ be all the distinct roots of unity among $\omega_1, \ldots, \omega_k$, and write

$$P(z) = \prod_{t=1}^{l} (z - \zeta_t) = z^l - a_1z^{l-1} - \cdots - a_{l-1}z - a_l,$$  

where

$$a_j = (-1)^{j-1} \sum_{1 \leq i_1 < \cdots < i_j \leq l} \zeta_{i_1} \cdots \zeta_{i_j} \in R \quad \text{for } j = 1, \ldots, l.$$
Proof of Lemma 1

Set $u_n = \sum_{t=1}^{l} c_t' \zeta_t^n$ for all $n \in \mathbb{Z}$, where $c_t' = \sum_{1 \leq s \leq k, \omega_s = \zeta_t} c_s$. Clearly $u_n = \sum_{s=1}^{k} c_s \omega_s^n = \psi(n)$. Also, $\{u_n\}_{n \in \mathbb{Z}}$ is a linear recurrence because

$$
\sum_{j=1}^{l} a_j u_{n-j} = \sum_{j=1}^{l} a_j \sum_{t=1}^{l} c_t' \zeta_t^{n-j} = \sum_{t=1}^{l} c_t' \zeta_t^{n-l} \sum_{j=1}^{l} a_j \zeta_t^{l-j} \\
= \sum_{t=1}^{l} c_t' \zeta_t^{n-l} \left( \zeta_t^l - P(\zeta_t) \right) = u_n.
$$

Let $N$ be the least positive integer with $\zeta_t^N = 1$ for all $t = 1, \ldots, l$. Then $N \geq l$ since $\zeta_1, \ldots, \zeta_l$ are distinct. Construct $f_r : \mathbb{Z} \to R \ (0 \leq r \leq l - 1)$ with period $N$ as follows:

$$
f_r(n) = \begin{cases} 
\delta_{n,r} & \text{if } n \in \{0, \ldots, l - 1\}, \\
\sum_{j=1}^{l} a_j f_r(n-j) & \text{if } n \in \{l, \ldots, N - 1\}.
\end{cases}
$$
Proof of Lemma 1

For $0 \leq n < l$, we have

$$\sum_{r=0}^{l-1} \psi(r)f_r(n) = \sum_{r=0}^{l-1} \psi(r)\delta_{n,r} = \psi(n).$$

If $l \leq n < N$ and $\sum_{r=0}^{l-1} \psi(r)f_r(m) = \psi(m)$ for all $m = 0, \ldots, n - 1$, then

$$\psi(n) = u_n = \sum_{j=1}^{l} a_j u_{n-j} = \sum_{j=1}^{l} a_j \psi(n-j)$$

$$= \sum_{j=1}^{l} a_j \sum_{r=0}^{l-1} \psi(r)f_r(n-j) = \sum_{r=0}^{l-1} \psi(r) \sum_{j=1}^{l} a_j f_r(n-j)$$

$$= \sum_{r=0}^{l-1} \psi(r)f_r(n).$$

So $\sum_{r=0}^{l-1} \psi(r)f_r(n) = \psi(n)$ for all $0 \leq n < N$, and hence

$$\psi = \sum_{r=0}^{l-1} \psi(r)f_r.$$
Lemma 2. Let $\psi = \psi_1 + \cdots + \psi_k$ where each $\psi_s$ $(1 \leq s \leq k)$ is a complex-valued function on $\mathbb{Z}$ with period $n_s \in \mathbb{Z}^+$. Then $\psi$ can be written in the form $\sum_{0 \leq r < |S(n_1, \ldots, n_k)|} \psi(r)f_r$, where $f_0, \ldots, f_{|S(n_1, \ldots, n_k)|-1}$ are suitable periodic maps from $\mathbb{Z}$ to $\mathbb{Z}$ only depending on $S(n_1, \ldots, n_k)$.

Proof. If $x \in \mathbb{Z}$ then

$$\psi(x) = \sum_{s=1}^{k} \sum_{0 \leq a < n_s \atop n_s \mid x-a} \psi_s(a) = \sum_{s=1}^{k} \frac{1}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) \sum_{r=0}^{n_s-1} e^{2\pi i \frac{r}{n_s} (x-a)}$$

$$= \sum_{s=1}^{k} \sum_{r=0}^{n_s-1} \left( \frac{1}{n_s} \sum_{a=0}^{n_s-1} \psi_s(a) e^{-2\pi i a \frac{r}{n_s}} \right) \left( e^{2\pi i \frac{r}{n_s}} \right)^x.$$

Observe that

$$\prod_{\theta \in S(n_1, \ldots, n_k)} (x - e^{2\pi i \theta}) = \prod_{d \mid n_s} \Phi_d(x) \in \mathbb{Z}[x],$$

where $\Phi_d(x) = \prod_{0 \leq c < d, \gcd(c,d)=1} (x - e^{2\pi i c/d}) \in \mathbb{Z}[x]$ is the $d$th cyclotomic polynomial. Now it suffices to apply Lemma 1.
Remarks

Let $m, n_1, \ldots, n_k \in \mathbb{Z}^+$, and let $f_0, \ldots, f_{|S(n_1,\ldots,n_k)|-1}$ be as in Lemma 2. For each $s = 1, \ldots, k$ let $\psi_s : \mathbb{Z} \to \mathbb{Z}$ be a map which has period $n_s$ modulo $m$ (i.e., $\psi_s(a) \equiv \psi_s(b) \pmod{m}$ whenever $a \equiv b \pmod{n_s}$). Let $x$ be any integer. By Lemma 2 we have

$$\sum_{s=1}^{k} \psi'_s(x) = \sum_{0 \leq r < |S(n_1,\ldots,n_k)|} f_r(x) \sum_{s=1}^{k} \psi'_s(r),$$

where $\psi'_s(x) = \sum_{0 \leq a < n_s, n_s | x-a} \psi_s(a)$. As $\psi'_s(x) \equiv \psi_s(x) \pmod{m}$ for each $s = 1, \ldots, k$, this yields that

$$\psi(x) \equiv \sum_{0 \leq r < |S(n_1,\ldots,n_k)|} f_r(x) \psi(r) \pmod{m},$$

where $\psi = \psi_1 + \cdots + \psi_k$. For $a \in \mathbb{Z}$ let $\bar{a}$ denote the residue class $a \pmod{m}$ in the ring $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Then we have

$$\sum_{s=1}^{k} \overline{\psi_s(x)} = \overline{\psi(x)} = \sum_{0 \leq r < |S(n_1,\ldots,n_k)|} f_r(x) \overline{\psi(r)}.$$

So part (i) of the Second Local-Global Theorem holds for $G = \mathbb{Z}_m$. 
Proof of the Second Local-Global Theorem

Without any loss of generality, we simply let $G$ coincide with its subgroup generated by the finite set

$$\{\psi_s(x) : x = 0, \ldots, n_s - 1; \ s = 1, \ldots, k\}.$$ 

Since $G$ is finitely generated, there are $m_1, \ldots, m_l \in \mathbb{Z}^+$ and $n \in \{0, 1, \ldots\}$ such that $G$ is isomorphic to the direct sum $\mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_l} \oplus \mathbb{Z}^n$. Let us identify $G$ with $G_1 \oplus \cdots \oplus G_{l+n}$, where $G_t = \mathbb{Z}_{m_t}$ for $t = 1, \ldots, l$, and $G_{l+1} = \cdots = G_{l+n} = \mathbb{Z}$.

Let $f_0, \ldots, f_{|S(n_1, \ldots, n_k)|-1}$ be as in Lemma 2, and let $x$ be any integer. For $s = 1, \ldots, k$ we write $\psi_s(x)$ in the vector form

$$\langle \psi_s, 1(x), \ldots, \psi_s, l+n(x) \rangle,$$

where $\psi_{s, t}(x) \in G_t$ for $t = 1, \ldots, l + n$. 
Proof of the Second Local-Global Theorem

Set $\psi(t) = \sum_{s=1}^{k} \psi_{s,t}$ for $t = 1, \ldots, l + n$. Since $\psi_{s,t} : \mathbb{Z} \to G_t$ also has period $n_s$, we have

$$\psi(t)(x) = \sum_{0 \leq r < |S(n_1,\ldots,n_k)|} f_r(x) \psi(t)(r)$$

Therefore,

$$\psi(x) = \langle \psi^{(1)}(x), \ldots, \psi^{(l+n)}(x) \rangle$$

$$= \sum_{0 \leq r < |S(n_1,\ldots,n_k)|} f_r(x) \langle \psi^{(1)}(r), \ldots, \psi^{(l+n)}(r) \rangle$$

$$= \sum_{0 \leq r < |S(n_1,\ldots,n_k)|} f_r(x) \psi(r).$$

This proves the first part of the Second Local-Global Theorem.
Proof of the Second Local-Global Theorem

Now we prove the second part. Suppose that $\psi(a + r) = c$ for all $r = 0, \ldots, |S(n_1, \ldots, n_k)| - 1$, where $a \in \mathbb{Z}$ and $c \in G$.

For $x \in \mathbb{Z}$, let $\psi^*(x) = \psi_s(a + x)$ for $1 \leq s < k$, and $\psi_k^*(x) = \psi_k(a + x) - c$.

Set

$$\psi^*(x) = \psi_1^*(x) + \cdots + \psi_k^*(x) = \psi(a + x) - c.$$

By the first part of the Second Local-Global Theorem, the range of $\psi^*$ is contained in the subgroup of $G$ generated by

$$\{\psi^*(r) : 0 \leq r < |S(n_1, \ldots, n_k)|\} = \{0\}.$$

Thus $\psi(a + x) - c = \psi^*(x) = 0$ for all $x \in \mathbb{Z}$. This concludes our proof.
Part III. Connections between Covers of $\mathbb{Z}$ and Unit Fractions
Zhang’s result and its extensions

**Ming-Zhi Zhang** (1989): If \( A = \{a_s (\mod n_s)\}_{s=1}^{k} \) is a cover of \( \mathbb{Z} \) then \( \sum_{s \in I} \frac{1}{n_s} \in \mathbb{Z}^+ \) for some \( \emptyset \neq I \subseteq \{1, \ldots, k\} \).

**Z.-W. Sun** [Israel J. Math. 77(1992)]: If \( A = \{a_s (\mod n_s)\}_{s=1}^{k} \) is an exact \( m \)-cover of \( \mathbb{Z} \), then for any \( n = 0, \ldots, m \) we have

\[
\left| \left\{ I \subseteq \{1, \ldots, k\} : \sum_{s \in I} \frac{1}{n_s} = n \right\} \right| \geq \binom{m}{n}.
\]

**Z.-W. Sun** [Trans. Amer. Math. Soc. 348(1996)]: If \( A = \{a_s (\mod n_s)\}_{s=1}^{k} \) is an \( m \)-cover of \( \mathbb{Z} \), then for any \( m_1, \ldots, m_k \in \mathbb{Z}^+ \) there are at least \( m \) positive integers in the form \( \sum_{s \in I} m_s/n_s \) with \( I \subseteq \{1, \ldots, k\} \).
Other results

**Z.-W. Sun** [Acta Arith. 72(1995)]: If $A = \{a_s(\text{mod } n_s)\}_{s=1}^k$ is an exact $m$-cover of $\mathbb{Z}$, then for any $\emptyset \neq J \subset \{1, \ldots, k\}$ there is an $I \subseteq \{1, \ldots, k\}$ with $I \neq J$ such that $\sum_{s \in I} \frac{1}{n_s} = \sum_{s \in J} \frac{1}{n_s}$.

**Z.-W. Sun** [Combinatorica 23(2003)]: If the covering function of $A = \{a_s(\text{mod } n_s)\}_{s=1}^k$ is periodic modulo $n_0 \in \mathbb{Z}^+$, then

$$\left\{ \sum_{s \in J} \frac{1}{n_s} : J \subseteq \{1, \ldots, k-1\} \right\} \supseteq \left\{ \frac{r}{n_k} : 0 \leq r < \frac{n_k}{(n_k, n_0)} \right\}.$$  

**Z.-W. Sun** [Adv. Appl. Math. 38(2007)]: If the covering function of $A = \{a_s(\text{mod } n_s)\}_{s=1}^k$ is periodic modulo $n_k$, then for any $r = 0, \ldots, n_k - 1$ we have

$$\left| \left\{ \sum_{s \in I} \frac{1}{n_s} : I \subseteq \{1, \ldots, k\} \& \left\{ \sum_{s \in I} \frac{1}{n_s} \right\} = \frac{r}{n_k} \right\} \right| \geq m.$$
Main References


Thank you!