A talk given at the Wuhan Conference on q-Series and Special Functions

New Series for Powers of π and Their q-Analogues

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Abstract

Motivated by classical Ramanujan-type series for $1/\pi$, in 2019 the speaker obtained some new kinds of series for powers of π involving rational functions, for example

$$\sum_{k=1}^{\infty}\frac{(3k+1)16^k}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{\pi^2-8}{2}. \qquad (*)$$

In this talk we introduce how to deduce those series, and provide q-analogue of four of them (including $(*)$) due to Qing-Hu Hou and the speaker.

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0,\ldots,\alpha_r;\beta_1,\ldots,\beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n\cdots(\alpha_r)_n}{(\beta_1)_n\cdots(\beta_r)_n}\cdot\frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

$$y = {}_{r+1}F_r\left(\alpha_0,\ldots,\alpha_r;\beta_1,\ldots,\beta_r \mid x\right)$$

satisfies the differential equation:

$$\left(\theta\prod_{t=1}^{r}(\theta+\beta_t-1) - x\prod_{s=0}^{r}(\theta+\alpha_s)\right)y = 0$$

where $\theta = x\frac{d}{dx}$.

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England *Modular equations and approximations to* π, Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for* $1/\pi$". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

Series for $1/\pi$ given by Ramanujan

S. Ramanujan: "An equation for me has no meaning, unless it represents a thought of God."

Two of the 17 series for $1/\pi$ recorded by Ramanujan:

$$\sum_{k=0}^{\infty} \frac{6k+1}{4^k} \cdot \frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi},$$

(proved by S. Chowla in 1928)

$$\sum_{k=0}^{\infty} \frac{26390k+1103}{99^{4k}} \cdot \frac{(1/2)_k (1/4)_k (3/4)_k}{(1)_k^3}$$

$$= \sum_{k=0}^{\infty} \frac{26390k+1103}{396^{4k}} \binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

In 1985 Jr. R. W. Gosper used the last series of Ramanujan to calculate $17,526,100$ digits of π (a world record at that time).

In 1987 J. Borwein and P. Borwein succeeded in proving all the 17 Ramanujan series for $1/\pi$.

My first impression on Ramanujan-type series

In a year around 2003, I happened to see a paper on Ramanujan-type series. Here is one of Ramanujan series for $1/\pi$:

$$\sum_{k=0}^{\infty}(28k+3)\left(-\frac{27}{512}\right)^k\frac{(1/2)_k(1/6)_k(5/6)_k}{(1)_k^3}=\frac{32\sqrt{2}}{\pi}.$$

At that time I did not like this at all since it is too complicated! I only enjoy simple and beautiful results! Thus this paper gave me almost no impression and I could not remember what paper it is.

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are 36 known Ramanujan-type series for $1/\pi$ with $a,b,m\in\mathbb{Z}$. I prefer their forms in terms of binomial coefficients.

van Hamme's Philosophy

In 1997, van Hamme conjectured some p-analogues of certain series involving π. As $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, he viewed π as $\Gamma(\frac{1}{2})^2$ and considered its p-analogue

$$\Gamma_p\left(\frac{1}{2}\right)^2 = -\left(\frac{-1}{p}\right)$$

with $(\frac{\cdot}{p})$ the Legendre symbol and $\Gamma_p(x)$ the p-adic Γ-function.

For example, motivated by Ramanujan's identity

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} = \frac{16}{\pi},$$

van Hamme conjectured that

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv 5p\left(\frac{-1}{p}\right) \pmod{p^3}$$

which was proved by R. Osburn and W. Zudilin [J. Math. Anal. Appl. 433 (2016)].

My Philosophy involving Bernoulli or Euler numbers

My Philosophy (Sun, 2010): If $(a_k)_{k\geqslant 0}$ is a sequence of rational numbers with $\sum_{k=0}^{p-1} a_k$ related to the zeta function or powers of π, then for large primes p, the partial sum $\sum_{k=0}^{p-1} a_k$ modulo powers of p is related to Bernoulli numbers or Bernoulli polynomials.

Z.-W. Sun [Sci. China Math. 54(2011)] conjectured that

$$\sum_{k=0}^{p-1} \frac{42k+5}{4096^k}\binom{2k}{k}^3 \equiv 5p\left(\frac{-1}{p}\right) - p^3 E_{p-3} \pmod{p^4}$$

(with $p>3$ prime) which was confirmed by D.-W. Hu and G.-S. Mao [Ramanujan J. 42(2017)].

Another Example: In contrast with the Apéry series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3\binom{2k}{k}} = -\frac{2}{5}\zeta(3)$, Z.-W. Sun [J. Number Theory 134(2014)] proved that for any prime $p>5$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3\binom{2k}{k}} \equiv -2B_{p-3} \pmod{p}.$$

Zeilberger-type series

In 1993, D. Zeilberger used the Wilf-Zeilberger method to obtain the new identity

$$\sum_{k=1}^{\infty} \frac{21k-8}{k^3\binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}.$$

J. Guillera [Ramanujan J. 15(2008)] used the WZ method to give three new Zeilberger-type series:

$$\sum_{k=1}^{\infty} \frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3} = -16G,$$

$$\sum_{k=1}^{\infty} \frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3} = -2G,$$

$$\sum_{k=1}^{\infty} \frac{(3k-1)16^k}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{2},$$

where G denotes the Catalan constant $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$.

More such series

Conjecture [Z.-W. Sun, 2010; Sci. China Math. 54(2011)]:

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^2\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K:=\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2}$$

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},\quad \sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K.$$

The first and the second were confirmed by J. Guillera in 2013, and Kh. Hessami Pilehrood and T. Hessami Pilehrood in 2012 respectively. The last three were finally confirmed by J. Guillera and M. Rogers [J. Austral. Math. Soc. 97(2014)].

Series involving $\binom{2k}{k}/(2k-1)$

Recall that the Catalan numbers are given by

$$C_n := \frac{\binom{2n}{n}}{n+1} = \binom{2n}{n} - \binom{2n}{n+1} \ \ (n \in \mathbb{N}).$$

For $k \in \mathbb{Z}^+$ it is easy to see that $\binom{2k}{k}/(2k-1) = 2C_{k-1}$.

J.W.L. Glaisher (1905):

$$\sum_{k=0}^{\infty} \frac{(4k-1)\binom{2k}{k}^4}{256^k(2k-1)^4} = -\frac{8}{\pi^2}.$$

Z.-W. Sun [J. Number Theory 134(2014)] observed that

$$\sum_{k=0}^{n} \frac{(4k-1)\binom{2k}{k}^4}{256^k(2k-1)^4} = -(8n^2+4n+1)\frac{\binom{2n}{n}^4}{256^n}.$$

Z.-W. Sun [Colloq. Math. 154(2018)] studied congruences for

$$R_n := \sum_{k=0}^{n} \binom{n+k}{2k} \frac{\binom{2k}{k}}{2k-1} \ (n = 0, 1, 2, \ldots).$$

Congruences involving $\binom{2k}{k}/(2k-1)$

V.J.W. Guo and J.-C. Liu [Integral Transforms Spec. Funct. 31(2020)]: For any prime $p>3$ we have

$$\sum_{k=0}^{(p+1)/2}(-1)^{k-1}\frac{(4k-1)(-1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{(p+1)/2}\frac{(4k-1)\binom{2k}{k}^3}{(2k-1)^3(-64)^k}$$
$$\equiv p\left(\frac{-1}{p}\right)+p^3(E_{p-3}-2)\pmod{p^4}.$$

C. Wang [J. Math. Anal. Appl. 488(2020)] For any prime $p>3$ we have

$$\sum_{k=0}^{(p+1)/2}\frac{(3k-1)\binom{2k}{k}^3}{(2k-1)^2 16^k}\equiv p+2p^3\left(\frac{-1}{p}\right)(E_{p-3}-3)\pmod{p^4}.$$

(This extends a conjecture of Guo and M.J. Schlosser [Constr. Approx., to appear]).

New series for $1/\pi$ involving rational functions

For **each** rational Ramanujan series for $1/\pi$, Z.-W. Sun [Electron. Res. Arch., to appear] obtained a corresponding series for $1/\pi$ with the rational part $bk+c$ replaced by a proper rational function

$$(a_0k^2+a_1k+a_2)/(c_0k^3+c_1k^2+c_2k+c_3).$$

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Theorem (Sun, 2019). We have

$$\sum_{k=0}^{\infty}\frac{P(k)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}=\frac{18\times 557403^3\sqrt{10005}}{5\pi},$$

$$\begin{aligned}P(k):=&\,637379600041024803108k^2+657229991696087780968k\\&+19850391655004126179.\end{aligned}$$

Proof of the Theorem

As

$$\sum_{k=0}^{n}\frac{(10939058860032072k^3-36k^2-2k+1)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-640320)^{3k}}$$
$$=\frac{6n+1}{(-640320)^{3n}}\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n},$$

we have

$$\sum_{k=0}^{\infty}\frac{(10939058860032072k^3-36k^2-2k+1)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-640320)^{3k}}=0.$$

Note that

$$\begin{aligned}&10939058860032072k^3-36k^2-2k+1\\=&1672209(2k-1)(6k-1)(545140134k+13591409)\\&+426880(16444841148k^2-1709536232k-53241371)\end{aligned}$$

Continue the proof

and hence

$$\sum_{k=0}^{\infty} \frac{(16444841148k^2 - 1709536232k - 53241371)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(2k-1)(6k-1)(-640320)^{3k}}$$

$$= -\frac{1672209}{426880} \times \frac{3 \times 53360^2}{2\pi\sqrt{10005}} = -1672209\frac{\sqrt{10005}}{\pi}.$$

Observe that

$$\sum_{k=0}^{n} \frac{(10939058860032072k^3 + 10939058860031964k^2 - 2k + 1)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}$$

$$= \frac{6n+1}{(n+1)(-640320)^{3n}}\binom{2n}{n}\binom{3n}{n}\binom{6n}{3n}$$

and hence

$$\sum_{k=0}^{\infty} \frac{(10939058860032072k^3 + 10939058860031964k^2 - 2k + 1)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}$$

vanishes.

Continue the proof

Note that

$$
\begin{aligned}
&2802461(10939058860032072k^3 + 10939058860031964k^2 - 2k + 1)\\
=&1864188626454(k+1)(16444841148k^2 - 1709536232k - 53241371)\\
&+ 5P(k).
\end{aligned}
$$

Therefore we get

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{P(k)\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(k+1)(2k-1)(6k-1)(-640320)^{3k}}\\
=&-\frac{1864188626454}{5} \times (-1672209)\frac{\sqrt{10005}}{\pi} = 18 \times 557403^3 \frac{\sqrt{10005}}{5\pi}.
\end{aligned}
$$

Series with binomial coefficients in the denominators

Theorem (Sun, 2019). We have the identities

$$\sum_{k=1}^{\infty}\frac{28k^2+31k+8}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{\pi^2-8}{2},$$

$$\sum_{k=1}^{\infty}\frac{42k^2+39k+8}{(2k+1)^3k^3\binom{2k}{k}^3}=\frac{9\pi^2-88}{2},$$

$$\sum_{k=1}^{\infty}\frac{(3k+1)16^k}{(2k+1)^2k^3\binom{2k}{k}^3}=\frac{\pi^2-8}{2},$$

$$\sum_{k=1}^{\infty}\frac{(4k+1)(-64)^k}{(2k+1)^3k^3\binom{2k}{k}^3}=16G-16,$$

$$\sum_{k=1}^{\infty}\frac{(5k+1)(-27)^k}{(2k+1)(3k+1)k^2\binom{2k}{k}^2\binom{3k}{k}}=6-9K,\quad \ldots\ldots$$

where $G:=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$ and $K:=\sum_{k=0}^{\infty}\frac{(\frac{k}{3})}{k^2}$.

q-integers

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, its q-analogue is given by

$$[n]_q := \frac{1 - q^n}{1 - q} = \sum_{0 \leqslant k < n} q^k \in \mathbb{Z}[q].$$

Note that $\lim_{q \to 1} [n]_q = n$.

For complex numbers a and q with $|q| < 1$, we adopt the standard notation:

$$(a; q)_n := \prod_{0 \leqslant k < n} (1 - aq^k) \quad \text{and} \quad (a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k).$$

Thus

$$(q; q)_\infty = \prod_{n=1}^{\infty} (1 - q^n) \quad \text{and} \quad (q; q^2)_\infty = \prod_{n=1}^{\infty} (1 - q^{2n-1}).$$

Wallis' formula

Wallis (1655):

$$\prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right)=\frac{\pi}{2}.$$

This is the special case $x=\pi/2$ of Euler's formula

$$\frac{\sin x}{x}=\prod_{n=1}^{\infty}\left(1-\frac{x^2}{(n\pi)^2}\right).$$

By Wallis' formula,

$$\lim_{\substack{q\to 1\\ |q|<1}}(1-q)\frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2}=\lim_{\substack{q\to 1\\ |q|<1}}\prod_{n=1}^{\infty}\frac{[2n]_q^2}{[2n-1]_q[2n+1]_q}=\frac{\pi}{2}.$$

A q-analogue of Euler's formula $\zeta(2) = \pi^2/6$

Many series for π^2 actually just give formulas for $\zeta(2)$. To get $\pi^2/6$ one has to quote Euler's basic formula $\zeta(2) = \pi^2/6$. So, it is essential and important to present q-analogues of Euler's formula $\zeta(2) = \pi^2/6$.

Note that

$$\zeta(2) = \frac{\pi^2}{6} \iff \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}$$

since

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{k=1}^{\infty} \frac{1}{(2k)^2} = \left(1 - \frac{1}{4}\right)\zeta(2),$$

Z.-W. Sun [Colloq. Math. 158(2019)] Euler's formula $\zeta(2) = \pi^2/6$ has the following q-analogue with $|q| < 1$:

$$\sum_{k=0}^{\infty} \frac{q^k(1+q^{2k+1})}{(1-q^{2k+1})^2} = \prod_{n=1}^{\infty} \frac{(1-q^{2n})^4}{(1-q^{2n-1})^4}.$$

Guo and Liu's q-analogues of two Ramanujan series

V.J.W. Guo and J.-C. Liu [J. Difference Equ. Appl. 24(2018), 1368-1373] used the WZ method to obtain that

$$\sum_{n=0}^{\infty} q^{n^2}[6n+1]_q \frac{(q;q^2)_n^2(q^2;q^4)_n}{(q^4;q^4)_n^3} = \frac{(1+q)(q^2;q^4)_\infty(q^6;q^4)_\infty}{(q^4;q^4)_\infty^2}$$

and

$$\sum_{n=0}^{\infty}(-1)^n q^{3n^2}[6n+1]_q \frac{(q;q^2)_n^3}{(q^4;q^4)_n^3} = \frac{(q^3;q^4)_\infty(q^5;q^4)_\infty}{(q^4;q^4)_\infty^2},$$

which are q-analogues of Ramanujan's formulas

$$\sum_{n=0}^{\infty}(6n+1)\frac{\binom{2n}{n}^3}{256^n} = \frac{4}{\pi} \text{ and } \sum_{n=0}^{\infty}(6n+1)\frac{\binom{2n}{n}^3}{(-512)^n} = \frac{2\sqrt{2}}{\pi}.$$

Four series for power of π

The following four identities:

$$\sum_{k=0}^{\infty} \frac{k(4k-1)\binom{2k}{k}^3}{(2k-1)^2(-64)^k} = -\frac{1}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{(4k-1)\binom{2k}{k}^3}{(2k-1)^3(-64)^k} = \frac{2}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{(12k^2-1)\binom{2k}{k}^3}{(2k-1)^2 256^k} = -\frac{2}{\pi},$$

$$\sum_{k=1}^{\infty} \frac{(3k+1)16^k}{(2k+1)^2 k^3 \binom{2k}{k}^3} = \frac{\pi^2-8}{2}.$$

are (1.1), (1.2), (1.3) and (1.77) of Sun [Electron. Res. Arch. 28(2020)] respectively.

Joint with Qing-Hu Hou, we give q-analogues of these four identities.

Joint work with Qing-Hu Hou

Theorem (Q.-H. Hou and Z.-W. Sun, arXiv:1808.04717v2) For $|q| < 1$ we have

$$\sum_{k=0}^{\infty}(-1)^k q^{k^2}\frac{[2k]_q([4k]_q-1)}{([2k]_q-1)^2}\cdot\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} = -\frac{(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2},$$

$$\sum_{k=0}^{\infty}(-1)^k q^{k^2+2k}\frac{[4k]_q-1}{([2k]_q-1)_q^3}\cdot\frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} = \frac{(q;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty^2},$$

$$\sum_{k=0}^{\infty}\frac{P_k(q)q^{k^2}(q;q^2)_k^2(q^2;q^4)_k}{(1-q)^3([2k]_q-1)^2(q^4;q^4)_k^3} = 2q(1+q)\frac{(q^2;q^4)_\infty(q^6;q^4)_\infty}{(q^4;q^4)_\infty^2},$$

$$q\sum_{k=0}^{\infty}\frac{[3k+4]_q}{[2k+3]_q^2}\cdot\frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3}q^{k(k+5)/2} = (1-q)^2\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4}-1-q,$$

where $P_k(q)$ denotes

$$q^{12k+1}-3q^{10k+2}+3(2q^2-1)q^{8k+1}-(3q^4-1)q^{6k} + 3q^{4k+1}-3q^{2k+2}+2q^3-q.$$

Proof of the last q-analogue

It is easy to verify that

$$\frac{q^{2k+1}[3k+4]_q}{[2k+3]_q^2}\cdot\frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3}q^{k(k+1)/2}$$
$$-[3k+2]_q\frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3}q^{k(k+1)/2}$$

$$=\Delta_k\left(\frac{(1+q^{k+1})(1-q^{2k+1})}{1-q}\cdot\frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3}q^{k(k+1)/2}\right)$$

for all $k\in\mathbb{N}$, where $\Delta_k(f(k))=f(k+1)-f(k)$. Therefore,

$$q\sum_{k=0}^{\infty}\frac{[3k+4]_q}{[2k+3]_q^2}\cdot\frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3}q^{k(k+5)/2}$$
$$-\sum_{k=0}^{\infty}[3k+2]_q\frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3}q^{k(k+1)/2}$$

coincides with $-1-q$.

Continue the proof

By Hou, Krattenthaler and Sun [Proc. Amer. Math. Soc. 147(2019)],

$$\sum_{k=0}^{\infty}[3k+2]_q \frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3} q^{k(k+1)/2} = (1-q)^2 \frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4},$$

which is a q-analogue of the identity $\sum_{k=1}^{\infty} \frac{(3k-1)16^k}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{2}$. So

$$q\sum_{k=0}^{\infty} \frac{[3k+4]_q}{[2k+3]_q^2} \cdot \frac{(q;q)_k^3(-q;q)_k}{(q^3;q^2)_k^3} q^{k(k+5)/2} = (1-q)^2 \frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4} - 1 - q,$$

which is a q-analogue of the identity

$$\sum_{k=1}^{\infty} \frac{(3k+1)16^k}{(2k+1)^2 k^3 \binom{2k}{k}^3} = \frac{\pi^2-8}{2}$$

which is (1.77) of Sun [Electron. Res. Arch. 28(2020)].

The first, second and third q-identities of Hou and Sun

The first, second and third q-analogues of Hou and Sun can be proved similarly, but we need the identity (which follows from Jackson's formula by taking a limit, see Guo [Ramanujan J. 52(2020)])

$$\sum_{k=0}^{\infty}(-1)^k q^{k^2}[4k+1]_q \frac{(q;q^2)_k^3}{(q^2;q^2)_k^3} = \frac{(q;q^2)_\infty (q^3;q^2)_\infty}{(q^2;q^2)_\infty^2},$$

which is the q-analogue of Bauer's formula $\sum_{k=0}^{\infty}(4k+1)\binom{2k}{k}^3/(-64)^k = \frac{2}{\pi}$, and the q-analogue

$$\sum_{k=0}^{\infty}[6k+1]_q \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^4;q^4)_k^3} q^{k^2} = (1+q)\frac{(q^2;q^4)_\infty (q^6;q^4)_\infty}{(q^4;q^4)_\infty^2}$$

(due to Guo and Liu [J. Difference Equ. Appl. 24(2018)]) of the Ramanujan series

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi}.$$

Series for $1/\pi$ involving Domb numbers

The *Domb numbers* are given by

$$D_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} \quad (n = 0, 1, 2, \ldots).$$

H. H. Chan, S. H. Chan and Z. G. Liu [Adv. in Math. 2004]:

$$\sum_{n=0}^{\infty} \frac{5n+1}{64^n} D_n = \frac{8}{\sqrt{3}\,\pi}.$$

H. H. Chan and H. Verill [Math. Res. Lett. 2009], **M. D. Rogers** [Ramanujan J. 2009]:

$$\sum_{n=0}^{\infty} \frac{3n+1}{(-32)^n} D_n = \frac{2}{\pi}.$$

Conjecture (Z.-W. Sun [Sci. China Math. 54(2011)]):

$$\sum_{n=0}^{\infty} \frac{40n^2+26n+5}{(-256)^n} \binom{2n}{n}^2 D_n = \frac{24}{\pi^2}.$$

Generalized central trinomial coefficients

For $b, c \in \mathbb{Z}$, we define

$$\begin{aligned} T_n(b,c) :=& [x^n](x^2+bx+c)^n \\ & \qquad \text{(the coefficient of } x^n \text{ in } (x^2+bx+c)^n) \\ =& \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k. \end{aligned}$$

Recursion: $T_0(b,c) = 1$, $T_1(b,c) = b$, and

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c) \ (n > 0),$$

where $d = b^2 - 4c$.

Remark. Note that $T_n(2,1) = \binom{2n}{n}$. Those $T_n := T_n(1,1)$ $(n \in \mathbb{N})$ are called *central trinomial coefficients*. We may view $T_n(b,c)$ as natural extensions of the central binomial coefficients.

A new kind of series for $1/\pi$

Motivated by the Domb numbers $D_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}\binom{2(n-k)}{n-k}$ and related series for $1/\pi$, for $b, c \in \mathbb{Z}$, Z.-W. Sun introduced in 2019

$$S_n(b,c) := \sum_{k=0}^{n} \binom{n}{k}^2 T_k(b,c) T_{n-k}(b,c) \quad (n = 0, 1, 2, \ldots).$$

Theorem (Z.-W. Sun [Electron. Res. Arch. 28(2020)]) We have

$$\sum_{k=0}^{\infty} \frac{7k+3}{24^k} S_k(1,-6) = \frac{15}{\sqrt{2}\,\pi},$$

$$\sum_{k=0}^{\infty} \frac{12k+5}{(-28)^k} S_k(1,7) = \frac{6\sqrt{7}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{84k+29}{80^k} S_k(1,-20) = \frac{24\sqrt{15}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{3k+1}{(-100)^k} S_k(1,25) = \frac{25}{8\pi},$$

A new kind of series for $1/\pi$ (continued)

$$\sum_{k=0}^{\infty} \frac{228k+67}{224^k} S_k(1,-56) = \frac{80\sqrt{7}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{399k+101}{(-676)^k} S_k(1,169) = \frac{2535}{8\pi},$$

$$\sum_{k=0}^{\infty} \frac{2604k+563}{2600^k} S_k(1,-650) = \frac{850\sqrt{39}}{3\pi},$$

$$\sum_{k=0}^{\infty} \frac{39468k+7817}{(-6076)^k} S_k(1,1519) = \frac{4410\sqrt{31}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{41667k+7879}{9800^k} S_k(1,-2450) = \frac{40425\sqrt{6}}{4\pi},$$

$$\sum_{k=0}^{\infty} \frac{74613k+10711}{(-530^2)^k} S_k(1,265^2) = \frac{1615175}{48\pi}.$$

Such series are related to $\mathbb{Q}(\sqrt{-d})$ with $3 \mid d$ and $h(-d) = 4$.

A mysterious connection between primes and π

Any prime $p \equiv 1 \pmod 4$ can be written uniquely as $x_p^2 + y_p^2$ with $x_p, y_p \in \mathbb{Z}$ and $0 < x_p < y_p$, this was first conjectured by Fermat and then proved by Euler.

Conjecture (Zhi-Wei Sun, Dec. 15, 2019). We have

$$\lim_{N\to\infty} \frac{\sum_{\substack{p\leqslant N \\ p\equiv 1 \pmod 4}} p}{\sum_{\substack{p\leqslant N \\ p\equiv 1 \pmod 4}} x_p y_p} = \pi. \qquad (*)$$

Furthermore,

$$\frac{\sum_{\substack{p\leqslant N \\ p\equiv 1 \pmod 4}} x_p y_p}{\sum_{\substack{p\leqslant N \\ p\equiv 1 \pmod 4}} p} = \frac{1}{\pi} + O\left(\frac{1}{\sqrt{N}}\right) \quad \text{for } N \geqslant 5.$$

I posted $(*)$ to `MathOverflow` on Dec. 15, 2019, it was later confirmed by the user `GH from MO` with the aid of Hecke's equidistribution theorem which states that the angles of the lattice points (x_p, y_p) are asymptotically equidistributed in $[\pi/4, \pi/2]$.

Main References:

1. Z.-W. Sun, *New series for powers of π and related congruences*, Electron. Res. Arch. **28** (2020), to appear. (See also `arXiv:1911.05456`.)
2. Q.-H. Hou and Z.-W. Sun, *q-analogues of some series for powers of π*, arXiv:1808.04717v2.

Thank you!