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Some Open Combinatorial Congruences

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Abstract

In this talk we mainly introduce some open conjectures on congruences involving various combinatorial quantities including central binomial coefficients and Catalan numbers, central trinomial coefficients and Motzkin numbers, Franel numbers and Apery numbers. We will also mention various related results and motivations.

Legendre symbols

Let p be an odd prime and $a \in \mathbb{Z}$. The Legendre symbol $(\frac{a}{p})$ is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for some } x \in \mathbb{Z}, \\ -1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for no } x \in \mathbb{Z}. \end{cases}$$

It is well known that $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$ for any $a, b \in \mathbb{Z}$. Also,

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv -1 \pmod{4}; \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

The Law of Quadratic Reciprocity: If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$

What are supercongruences ?

A *supercongruence* is a p-adic congruence which happens to hold not just modulo a prime p but a higher power of p.

Example. (Wolstenholme, 1862) For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}$$

and

$$\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}.$$

Remark. It is easy to see that

$$\sum_{k=1}^{p-1} \frac{1}{k} = \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k} + \frac{1}{p-k}\right) = \sum_{k=1}^{(p-1)/2} \frac{p}{k(p-k)} \equiv 0 \pmod{p}$$

and

$$\binom{2p-1}{p-1} = \prod_{k=1}^{p-1} \frac{p+k}{k} = \prod_{k=1}^{p-1} \left(1 + \frac{p}{k}\right) \equiv 1 + \sum_{k=1}^{p-1} \frac{p}{k} \equiv 1 \pmod{p^2}.$$

Some congruences involving one binomial coefficient

(i) (Conjectured by A. Adamchuk, and proved by Z. W. Sun and R. Tauraso [Int. J. Number Theory 2011])

$$\sum_{k=0}^{p-1}\binom{2k}{k}\equiv\left(\frac{p}{3}\right)\pmod{p^2}\quad\text{for any prime } p>3.$$

(ii) (Z. W. Sun and R. Tauraso [Adv. in Appl. Math. 2010])

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\quad\text{for any prime } p>3,$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.

(ii) (Z. W. Sun and R. Tauraso [Adv. in Appl. Math. 2010])

$$\sum_{k=1}^{p-1}(-1)^k\frac{\binom{2k}{k}}{k}\equiv-5\frac{F_{p-(\frac{p}{5})}}{p}\pmod{p}\quad\text{for any prime } p>5,$$

where $(F_n)_{n\geqslant 0}$ is the Fibonacci sequence.

(iv) (Conjectured by Sun-Tauraso, and proved by Pan-Sun) For any prime $p>5$, $\sum_{k=1}^{p-1}(-1)^k\binom{2k}{k}\equiv(\frac{p}{5})(1-2F_{p-(\frac{p}{5})})\pmod{p^3}$.

Some general supercongruences involving one binomial coefficient

Let p be an odd prime and let $m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod{p}$. Using linear recurrent sequences, Z.-W. Sun determined

(1) $\sum_{k=0}^{p-1} \binom{2k}{k}/m^k$ mod p^2 (Sci. China Math. 2010)

(2) $\sum_{k=0}^{(p-1)/2} \binom{2k}{k}/m^k$ mod p^2 (Taiwan. J. Math. 2013)

(3) $\sum_{k=0}^{p-1} \binom{p-1}{k}\binom{2k}{k}/m^k$ mod p^2 (Colloq. Math. 2012)

(4) $\sum_{k=0}^{p-1} \binom{3k}{k}/m^k$ mod p (preprint, 2009)

For example,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m^2-4m}{p}\right) + u_{p-(\frac{m^2-4m}{p})} \pmod{p^2},$$

where $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0 = 0,\ u_1 = 1,\ \text{and}\ u_{n+1} = (m-2)u_n - u_{n-1}\ (n = 1, 2, 3, \ldots).$$

In particular, $\sum_{k=0}^{p-1} \binom{2k}{k}/2^k \equiv (\frac{-1}{p}) \pmod{p^2}$.

Other supercongruences involving one binomial coefficient

Let p be any prime greater than 3.

(a) **L. L. Zhao, Z. W. Sun and H. Pan** [Proc. AMS 2010]:
$\sum_{k=1}^{p-1} 2^k\binom{3k}{k}/k \equiv 0 \pmod p$.

Conjecture (Z. W. Sun [Illinois J. Math., in press])

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k} \equiv -3p\,q_p(2)^2 \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{2^k}{k^2}\binom{3k}{k} \equiv 6\left(\frac{-1}{p}\right)E_{p-3} \pmod{p},$$

where $q_p(2) := (2^{p-1}-1)/p$ and $\frac{2}{e^x+e^{-x}} = \sum_{n=0}^{\infty} E_n\frac{x^n}{n!}$.

(b) **Z. W. Sun** [J. Number Theory 2011]:

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)16^k} \equiv 0 \pmod{p^2}.$$

Motivation. $\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k} = \frac{\pi}{3}$.

Other supercongruences involving one binomial coefficient

(c) **Conjecture** (Sun [J. Number Theory 2011]) For prime $p > 5$,

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} \equiv \left(\frac{-1}{p}\right)\left(\frac{H_{p-1}}{4p^2} + \frac{p^2}{36}B_{p-5}\right) \pmod{p^3}.$$

($H_{p-1} := \sum_{k=1}^{p-1}\frac{1}{k} \equiv -\frac{p^2}{3}B_{p-3} \pmod{p^3}$ for any prime $p > 3$.)

Corresponding Series.

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^3 16^k} = \frac{7\pi^3}{216}.$$

(d) **Conjecture** (Sun [Sci. Math. Sci. 2011]) For any prime $p > 5$,

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv -\frac{2}{p^2}H_{p-1}, \sum_{k=1}^{p-1} \frac{1}{k^4\binom{2k}{k}} - \frac{H_{p-1}}{p^3} \equiv -\frac{7}{45}pB_{p-5} \pmod{p^2}.$$

(The mod p version was confirmed by K. Hessami Pilehrood and T. Hessami Pilehrood [Adv. in Appl. Math. 2012].)

Motivation. $\sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}} = \frac{17}{36}\zeta(4)$, where $\zeta(4) = \sum_{n=1}^{\infty} 1/n^4$.

A p-adic analogue of Apéry's series

In 1979 R. Apéry proved the irrationality of $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$.
His stating point is the fast convergent series

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3\binom{2k}{k}}=-\frac{2}{5}\zeta(3).$$

R. Tauraso [J. Number Theory 2010]: For any prime $p>5$,

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3\binom{2k}{k}}\equiv-\frac{2}{5}\cdot\frac{H_{p-1}}{p^2}\pmod{p^3}.$$

This was obtained via putting $n=p$ in the known identity

$$\sum_{k=1}^{n}\binom{2k}{k}\frac{k^2}{4n^4+k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{4n^4+j^4}=\frac{2}{5n^2}.$$

Z. W. Sun [J. Number Theory, revised]: For any prime $p>5$,

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^3\binom{2k}{k}}\equiv-2B_{p-3}\pmod{p}.$$

Supercongruences involving two binomial coefficients

Conjecture of Rodriguez-Villegas (proved by E. Mortenson [J. Number Theory 2003; Trans. AMS 2005]). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{-3}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

Remark. Mortenson's proof involves Gauss and Jacobi sums and the p-adic Gamma function. In fact, now there are elementary proofs.

An elementary proof of $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k \equiv (\frac{-1}{p}) \pmod{p^2}$

Let $p = 2n+1$ be a prime. As observed by van Hammer, for $k = 0, \ldots, n$ we have

$$\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{n}{k}\binom{-n-1}{k}$$
$$=\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k}$$
$$\equiv\binom{-1/2}{k}^2 = \left(\frac{\binom{2k}{k}}{(-4)^k}\right)^2 = \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}.$$

It follows that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k$$
$$=\sum_{k=0}^{n}\binom{-n-1}{k}\binom{n}{n-k} = \binom{-1}{n} = (-1)^n \pmod{p^2}.$$

Congruences involving two binomial coefficients

Z. W. Sun [Acta Arith. 156(2012)]: Let $p\equiv 1 \pmod 4$ be a prime. Write $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $2\mid y$. Then

$$(-1)^{(p-1)/4}x\equiv\sum_{k=0}^{p-1}\frac{k+1}{8^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{p-1}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2 \pmod{p^2}.$$

Z. W. Sun [Finite Fields Appl. 22(2013)]: For any prime $p\equiv 3 \pmod 4$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv\frac{(-1)^{(p+1)/4}2p}{\binom{(p+1)/2}{(p+1)/4}} \pmod{p^2}.$$

Open Conjecture (Z. W. Sun [J. Number Theory 131(2011)]). Let $p\equiv 3 \pmod 4$ be a prime. Then

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}^2}{(-8)^k}\equiv 0 \pmod{p^2},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k} \pmod{p^3}.$$

Supercongruences involving two binomial coefficients

Second-order Catalan numbers:

$$C_k^{(2)} = \frac{1}{2k+1}\binom{3k}{k} = \binom{3k}{k} - 2\binom{3k}{k-1} \in \mathbb{Z}.$$

Conjecture (Sun, Sci. China Math. 2011) For any prime $p > 3$,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k+1}}{48^k} \equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{\lfloor \frac{3}{4}p \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^3} \quad \text{if } p \equiv 1 \pmod 4,$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} C_k^{(2)}}{27^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{6k}{3k} C_k^{(2)}}{432^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} C_{2k}}{64^k} \equiv \left(\frac{-1}{p}\right) - 3p^2 E_{p-3} \pmod{p^3},$$

where $E_0, E_1, \ldots$ are Euler numbers, and $C_n := \binom{2n}{n}/(n+1)$.

Congruences involving three binomial coefficients

Let $p > 3$ be a prime. In 2003 Rodriguez-Villegas conjectured congruences on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

modulo p^2. Via an advanced approach Mortenson [2005] provided a partial solution with some remaining things including

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 5 \pmod 6,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 7 \pmod 8,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{1728^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 11 \pmod{12}.$$

Z. W. Sun [Acta Arith. 2012]: Use Zeilberger's algorithm to prove all the remaining open parts of the above conjecture.

More congruences involving three binomial coefficients

van Hamme (1997): Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+4y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Z. W. Sun [Acta Arith. 156(2012)]: For any odd prime p and $d\in\{0,1,\ldots,p-1\}$ with $d\equiv\frac{p+1}{2} \pmod 2$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}\equiv 0 \pmod{p^2}.$$

Z. W. Sun [Acta Arith. 156(2012)]: Let $p\equiv 1 \pmod 4$ be a prime and write $p=x^2+y^2$ with x odd and y even. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k}\equiv 2p-2x^2 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv -2p \pmod{p^2}.$$

Open congruences involving three binomial coefficients

Conjecture (Z. W. Sun [J. Number Theory 2011]): Let p be any odd prime. Then

$$\sum_{k=0}^{(p-1)/2}\frac{kC_k^3}{16^k}\equiv 2p-2 \pmod{p^2}\ \text{ if } p\equiv 1 \pmod 3,$$

$$\sum_{k=0}^{(p-1)/2}\frac{C_k^3}{64^k}\equiv 8 \pmod{p^2}\ \text{ if } p\equiv 1 \pmod 4.$$

Conjecture (Z. W. Sun [Sci. China Math. 2011]): For any prime $p\equiv 3,5,6 \pmod 7$,

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}^2\binom{2k}{k}}{63^k}\equiv 0 \pmod{p}.$$

Delannoy polynomials and Legendre polynomials

$$D_n := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \ (n = 0, 1, 2, \ldots)$$

are called central Delannoy numbers. D_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$.

Theorem (Sun [J. Number Theory 2011]) We have the following congruences modulo any prime $p > 3$ with $q_p(2) = (2^{p-1} - 1)/p$:

$$\sum_{k=1}^{p-1} \frac{D_k}{k} \equiv -q_p(2), \ \sum_{k=1}^{p-1} \frac{D_k}{k^2} \equiv 2 \left(\frac{-1}{p}\right) E_{p-3}, \ \sum_{k=1}^{p-1} \frac{D_k^2}{k^2} \equiv -2q_p(2)^2.$$

We define

$$D_n(x) := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k.$$

The *Legendre polynomial* of degree n is given by

$$P_n(x) := \frac{1}{n! 2^n} \cdot \frac{d^n}{dx^n} (x^2 - 1)^n = D_n \left(\frac{x-1}{2}\right).$$

On central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](1+x+x^2)^n \text{ (the coefficient of } x^n \text{ in } (1+x+x^2)^n)$$
$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Theorem (i) (H. Q. Cao and Sun, 2010). For any prime $p > 3$ we have

$$T_{p-1} \equiv \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}.$$

(ii) (Z. W. Sun, 2010) For any odd prime p we have

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod{p}.$$

Conjecture on central trinomial coefficients

Conjecture (Sun, 2010) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8k+5)T_k^2 \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k} \equiv \frac{3+(\frac{p}{3})}{2} - p\left(1+\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

where H_k denotes the harmonic number $\sum_{0<j\leqslant k} 1/j$.

Mod p^2 congruences for Motzkin numbers

The nth Motzkin number

$$M_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

is the number of paths from $(0,0)$ to $(n,0)$ which never dip below the line $y=0$ and are made up only of the allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$.

Conjecture (Sun, 2010). Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} M_k T_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) + \frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Generalized central trinomial coefficients and generalized Motzkin numbers

Given $b, c \in \mathbb{Z}$, the *generalized central trinomial coefficients*

$$T_n(b,c) := [x^n](x^2+bx+c)^n = [x^0](b+x+cx^{-1})^n$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} b^{n-2k}c^k$$

and we introduce the *generalized Motzkin numbers*

$$M_n(b,c) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} \frac{b^{n-2k}c^k}{k+1}$$

$(n = 0, 1, 2, \ldots)$. Note that

$$T_n = T_n(1,1),\ M_n = M_n(1,1),\ T_n(2,1) = [x^n](x+1)^{2n} = \binom{2n}{n},$$

and

$$M_n(2,1) = \sum_{k=0}^{n} \binom{n}{2k} C_k 2^{n-2k} = C_{n+1}.$$

Generating functions

Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. H. S. Wilf observed that

$$\sum_{n=0}^{\infty} T_n(b,c)x^n = \frac{1}{\sqrt{1-2bx+dx^2}}$$

which implies the recursion

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c) \quad (n \in \mathbb{Z}^+).$$

By the Zeilberger algorithm we have

$$(n+3)M_{n+1}(b,c) = (2n+3)bM_n(b,c) - ndM_{n-1}(b,c) \quad (n \in \mathbb{Z}^+),$$

and hence

$$2cx^2\sum_{n=0}^{\infty} M_n(b,c)x^n = 1 - bx - \sqrt{1-2bx+dx^2}.$$

Relations between $T_n(b,c)$ and Legendre polynomials

For Legendre polynomials, it is known that

$$\sum_{n=0}^{\infty} P_n(t)x^n = \frac{1}{\sqrt{1-2tx+x^2}}.$$

Thus, if $d = b^2 - 4c \neq 0$ then

$$\sum_{n=0}^{\infty} T_n(b,c)\left(\frac{x}{\sqrt{d}}\right)^n = \frac{1}{\sqrt{1-2bx/\sqrt{d}+d(x/\sqrt{d})^2}} = \sum_{n=0}^{\infty} P_n(b)x^n$$

and hence

$$T_n(b,c) = (\sqrt{d})^n P_n\left(\frac{b}{\sqrt{d}}\right).$$

It follows that

$$T_n(2x+1, x^2+x) = P_n(2x+1) = D_n(x) \quad \text{for all } x \in \mathbb{Z};$$

in particular, $D_n = T_n(3,2)$.

On $\sum_{k=0}^{p-1} T_k(b,c)/m^k$ and $\sum_{k=0}^{p-1} M_k(b,c)/m^k \bmod p$

Theorem (Sun, 2010). Let b and c be integers.

(i) Let p be an odd prime not dividing $m \in \mathbb{Z}$. Then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)}{m^k} \equiv \left(\frac{(m-b)^2-4c}{p}\right) \pmod p$$

and

$$2c\sum_{k=0}^{p-1} \frac{M_k(b,c)}{m^k} \equiv (m-b)^2-((m-b)^2-4c)\left(\frac{(m-b)^2-4c}{p}\right) \pmod p.$$

(ii) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} \equiv 0 \pmod n$$

and

$$6\sum_{k=0}^{n-1} kT_k(b,c^2)(b-2c)^{n-1-k} \equiv 0 \pmod n.$$

Congruences modulo n

Theorem (Sun, 2010). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$.
(i) For any $n \in \mathbb{Z}^+$, we have

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} \equiv 0 \pmod{n},$$

and furthermore

$$b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} = nT_n(b,c)T_{n-1}(b,c).$$

(ii) Suppose that $b^2 - 4c = 1$, i.e., there is an $m \in \mathbb{Z}$ such that $b = 2m+1$, $c = m^2 + m$ and hence $T_k(b,c) = D_k(m)$. Then

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k(b,c) = \sum_{k=1}^{n}\binom{n}{k}\binom{n+k-1}{k-1}\left(\frac{b-1}{2}\right)^{k-1} \in \mathbb{Z}$$

for all $n \in \mathbb{Z}^+$.

On $\sum_{k=0}^{p-1} T_k(b,c)^2/m^k \bmod p$

Theorem (Sun, 2010) Let $b, c \in \mathbb{Z}$ with $d = b^2 - 4c$ and let p be an odd prime.

(i) If $p \nmid d$, then we have

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)^2}{d^k} \equiv \left(\frac{cd}{p}\right) \pmod{p}.$$

If $b \not\equiv 2c \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv \left(\frac{-c^2}{p}\right) \pmod{p}.$$

(ii) Assume $p \nmid c$. If $p \nmid d$, then $\sum_{k=0}^{p-1} T_k(b,c)M_k(b,c)/d^k \equiv 0 \pmod{p}$.
If $D = b^2 - 4c^2 \not\equiv 0 \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2)M_k(b,c^2)}{(b-2c)^{2k}} \equiv \frac{4b}{b+2c}\left(\frac{D}{p}\right) \pmod{p}.$$

A Corollary

Since $D_k(x) = T_k(2x+1, x^2+x)$ and $(2x+1)^2 - 4(x^2+x) = 1$, we have

Corollary Let p be an odd prime. For any integer x we have

$$\sum_{k=0}^{p-1} D_k(x)^2 \equiv \left(\frac{x(x+1)}{p}\right) \pmod{p}.$$

In particular,

$$\sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p}.$$

Congruences modulo n^2

Theorem (Sun, 2011). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2 d^{n-1-k} = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_k c^k d^{n-1-k}.$$

If c is nonzero and p is an odd prime not dividing d, then

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} \equiv 1 + \frac{b^2}{c}\cdot\frac{(\frac{d}{p})-1}{2} \pmod{p}.$$

Corollary. For each $n = 1, 2, 3, \ldots$ we have

$$\sum_{k=0}^{n-1}(2k+1)D_k^2 \equiv 0 \pmod{n^2}.$$

A conjecture involving $T_k(b, c^2)$

Conjecture (Sun, 2010) Let $b, c \in \mathbb{Z}$. For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8ck+4c+b)T_k(b,c^2)^2(b-2c)^{2(n-1-k)} \equiv 0 \pmod{n}.$$

If p is an odd prime not dividing $b(b-2c)$, then

$$\sum_{k=0}^{p-1}(8ck+4c+b)\frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv p(b+2c)\left(\frac{b^2-4c^2}{p}\right) \pmod{p^2}.$$

A conjecture involving $D_k(x)$

Conjecture (Sun, 2010). Let x be any integer. If p is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^3 \equiv p\left(\frac{-4x-3}{p}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^4 \equiv p \pmod{p^2}.$$

Conjectural congruences involving powers of $T_k(b,c)$

Conjecture (Sun, 2010). Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{4^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{36^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},$$

and

$$\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{8^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{(-64)^k}$$

$$\equiv\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{(-64)^k}\equiv\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{512^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\text{ and }p=x^2+6y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\text{ and }p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1.\end{cases}$$

Conjectural congruences involving powers of $T_k(b,c)$

Conjecture (Sun, 2011). Let $p>3$ be a prime. Then

$$\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{8^{3k}}\equiv\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{16^{3k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1\pmod 4\ \&\ p=x^2+4y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4;\end{cases}$$

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{(-8)^{3k}}\equiv\left(\frac{6}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{12^{3k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,3\pmod 8\ \&\ p=x^2+2y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-2}{p})=-1,\ \text{i.e., } p\equiv 5,7\pmod 8).\end{cases}$$

Seven series for $1/\pi$ involving powers of $T_k(b,c)$

Recall that $T_k(b,c)$ is the coefficient of x^k in $(x^2+bx+c)^k$.

Conjecture (Z. W. Sun, 2011).

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}),$$

$$\sum_{k=0}^{\infty}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k^2(4,9)=\frac{49}{9\pi}(\sqrt{3}+\sqrt{6}),$$

$$\sum_{k=0}^{\infty}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k^2(73,576)=\frac{10406669}{2\sqrt{6}\pi}.$$

Franel numbers

In 1894 J. Franel introduced the Franel numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots)$$

and noted the recurrence relation

$$(n+1)^2 f_{n+1} = (7n(n+1)+2)f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \ldots).$$

In 2008 D. Callan gave a combinatorial interpretation of the Franel numbers.

V. Strehl's Identity:

$$A_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} f_k \ \text{ where } A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Barrucand's Identity:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = g_n \quad \text{where } g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Connection with modular forms

Don Zagier (2009) investigated what integer sequence $\{u_n\}$ satisfies $u_{-1} = 0,\ u_0 = 1$, and the Apéry-like recurrence relation

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B)u_k + Ck^2 u_{k-1}\ (k = 1, 2, 3, \ldots).$$

When $(A, B, C) = (7, 2, 8)$, u_n is just the Franel number f_n, and Zagier noted that

$$\sum_{n=0}^{\infty} f_n \left(\frac{\eta(\tau)^3\eta(6\tau)^9}{\eta(2\tau)^3\eta(3\tau)^9}\right)^n = \frac{\eta(2\tau)\eta(3\tau)^6}{\eta(\tau)^2\eta(6\tau)^3}$$

for any complex number τ with $\mathrm{Im}(\tau) > 0$, where

$$\eta(\tau) := e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1 - e^{2\pi in\tau}).$$

Supercongruences involving Franel numbers

Wolstenholme's Congruence: For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p}.$$

Skula-Granville Congruence: For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv -\left(\frac{2^{p-1}-1}{p}\right)^2 \pmod{p}.$$

Theorem (Z. W. Sun, Adv. in Appl. Math., in press). For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_k \equiv 0 \pmod{p^2}, \quad \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} f_k \equiv 0 \pmod{p},$$

$$\sum_{k=0}^{p-1} (-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \quad \sum_{k=0}^{p-1} (-1)^k k f_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2}.$$

Connection between $p = x^2 + 3y^2$ and Franel numbers

Z.W. Sun [J. Number Theory 133(2013)]: Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

If $p \equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv -2 \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

Conjecture (Z. W. Sun): For any prime $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$, we have

$$x \equiv \frac{1}{4} \sum_{k=0}^{p-1} (3k+4) \frac{f_k}{2^k} \equiv \frac{1}{2} \sum_{k=0}^{p-1} (3k+2) \frac{f_k}{(-4)^k} \pmod{p^2}.$$

A conjecture on f_n and g_n

Conjecture [Z. W. Sun, JNT 133(2013)]. For each $n = 1, 2, 3, \ldots$,

$$\frac{1}{2n^2}\sum_{k=0}^{n-1}(3k+2)(-1)^k f_k \in \mathbb{Z} \quad\text{and}\quad \frac{1}{n^2}\sum_{k=0}^{n-1}(4k+1)g_k 9^{n-1-k} \in \mathbb{Z}.$$

Moreover, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}(3k+2)(-1)^k f_k \equiv 2p^2(2^p-1)^2 \pmod{p^5},$$

$$\sum_{k=0}^{p-1}(4k+1)\frac{g_k}{9^k} \equiv \frac{p^2}{2}\left(3-\left(\frac{p}{3}\right)\right) - p^2(3^p-3) \pmod{p^4}.$$

Remark. The part for Franel numbers has been confirmed by V. J. W. Guo.

More conjectures for f_n and g_n

Conjecture (Z. W. Sun) (i) For any integer $n > 1$, we have

$$\sum_{k=0}^{n-1}(9k^2+5k)(-1)^k f_k \equiv 0 \pmod{(n-1)n^2},$$

$$\sum_{k=0}^{n-1}(12k^4+25k^3+21k^2+6k)(-1)^k f_k \equiv 0 \pmod{4(n-1)n^3},$$

$$\sum_{k=0}^{n-1}(12k^3+34k^2+30k+9)g_k \equiv 0 \pmod{3n^3}.$$

(ii) For each odd prime p we have

$$\sum_{k=0}^{p-1}(9k^2+5k)(-1)^k f_k \equiv 3p^2(p-1)-16p^3q_p(2) \pmod{p^4},$$

$$\sum_{k=0}^{p-1}(12k^4+25k^3+21k^2+6k)(-1)^k f_k \equiv -4p^3 \pmod{p^4}.$$

A conjecture on Apéry numbers

In 1978 Apéry proved that $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is irrational! The following Apéry numbers play a central role in his proof.

$$A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k}^2 \binom{2k}{k}^2 \ (n = 0, 1, 2, \ldots).$$

Conjecture (Z. W. Sun [J. Number Theory 2012]). For any odd prime p, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Remark. I was able to prove the mod p version of this conjecture.

Apéry polynomials

Define Apéry polynomials by

$$A_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \ (n = 0, 1, 2, \ldots).$$

Z. W. Sun [J. Number Theory 132(2012)]. Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}.$$

Also, for any p-adic integer $x \not\equiv 0 \pmod{p}$ we have

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}.$$

Arithmetic means involving Apéry numbers

Theorem. Let n be a positive integer.

(i) (Z. W. Sun [J. Number Theory 132(2012)]) We have

$$\sum_{k=0}^{n-1}(2k+1)A_k \equiv 0 \pmod{n}.$$

For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers.
(ii) (Conjectured by Z. W. Sun proved by V.J.W. Guo and J. Zeng)

$$\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k \equiv 0 \pmod{n}.$$

More conjectures for Apéry numbers

Conjecture (Z. W. Sun) (i) For any positive integer n, we have

$$\sum_{k=0}^{n-1}(6k^3+9k^2+5k+1)(-1)^kA_k\equiv 0 \pmod{n^3},$$

$$\sum_{k=0}^{n-1}(18k^5+45k^4+46k^3+24k^2+7k+1)(-1)^kA_k\equiv 0 \pmod{n^4}.$$

(ii) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}(6k^3+9k^2+5k+1)A_k\equiv p^3+2p^4H_{p-1}-\frac{2}{5}p^8B_{p-5} \pmod{p^9},$$

where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers. If $p>5$, then

$$\sum_{k=0}^{p-1}(18k^5+45k^4+46k^3+24k^2+7k+1)(-1)^kA_k$$
$$\equiv -2p^4+3p^5+(6p-8)p^5H_{p-1}-\frac{12}{5}p^9B_{p-5} \pmod{p^{10}}.$$

A conjecture involving 3-adic valuations

For a prime p and a rational number a/b, its p-adic valuation (or p-adic order) $\nu_p(a/b)$ is defined as $\nu_p(a)-\nu_p(b)$, where $\nu_p(m):=\sup\{n\in\mathbb{N}:\ p^n\mid m\}$ for any nonzero integer m.

Conjecture (Z. W. Sun) Let n be any positive integer. Then

$$\nu_3\left(\sum_{k=0}^{n-1}(-1)^k f_k\right)=2\nu_3(n),\quad \nu_3\left(\sum_{k=0}^{n-1}(-1)^k kf_k\right)\geqslant 2\nu_3(n),$$

$$\nu_3\left(\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k\right)=3\nu_3(n)\leqslant \nu_3\left(\sum_{k=0}^{n-1}(2k+1)^3(-1)^k A_k\right).$$

If n is a positive multiple of 3, then

$$\nu_3\left(\sum_{k=0}^{n-1}(2k+1)^3(-1)^k A_k\right)=3\nu_3(n)+2.$$

A conjecture involving p-adic valuations

N. Strauss, J. Shallit and D. Zagier [Amer. Math. Monthly 1992]:

$$\nu_3\left(\sum_{k=0}^{n-1}\binom{2k}{k}\right)=\nu_3\left(n^2\binom{2n}{n}\right).$$

Theorem (Z. W. Sun [Acta Arith. 2011]). An integer $p>1$ is prime if and only if

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k}\equiv 0 \pmod p,\ \text{ where } \binom{(p-1)k}{k,\ldots,k}=\frac{((p-1)k)!}{(k!)^{p-1}}.$$

Conjecture (Z. W. Sun [Acta Arith. 2011]). For any odd prime p and positive integer n, we have

$$\nu_p\left(\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\right)\geqslant \nu_p\left(n\binom{2n}{n}\right),$$

and in particular the arithmetic mean $\frac{1}{n}\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}$ is always an p-adic integer.

Thank you!