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OPEN CONJECTURES ON THE THREE TOPICS

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Abstract. We list some open conjectures on the three topics (covering systems, restricted sumsets and zero-sum problems). The choice reflects my own flavor and hence it is not comprehensive.

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, \dots\}$ we call

$$a(n) = a + n\mathbb{Z} = \{a + nx \colon x \in \mathbb{Z}\}$$

a *residue class* with modulus n. For a finite system $A = \{a_s(n_s)\}_{s=1}^k$ of residue classes, the *covering function* $w_A \colon \mathbb{Z} \to \mathbb{N} = \{0, 1, 2, \dots\}$ is given by

$$w_A(x) = |\{1 \leqslant s \leqslant k \colon x \in a_s(n_s)\}|.$$

Let m be a positive integer. If $w_A(x) \geqslant m$ for all $x \in \mathbb{Z}$ then we call A an *m-cover* of $\mathbb{Z}$; if $w_A(x) = m$ for all $x \in \mathbb{Z}$ then we call A an *exact m-cover* of $\mathbb{Z}$. We use the term *cover* instead of 1-cover (of $\mathbb{Z}$), and the term *disjoint cover* instead of exact 1-cover (of $\mathbb{Z}$).

Conjecture 1 (P. Erdős). *For any $c > 0$ there is a cover $A = \{a_s(n_s)\}_{s=1}^k$ of $\mathbb{Z}$ with $c \leqslant n_1 < \cdots < n_k$.*

Remark 1. The best record in this direction is that Conjecture 1.1 holds for $c = 24$, this was obtained by R. Morikawa [Bull. Fac. Liberal Arts Nagasaki Univ. 21(1981), MR 84j:10064]. I'm afraid that the paper is not available to many researchers including me.

Conjecture 2 (P. Erdős and J. L. Selfridge). *There is no cover $A = \{a_s(n_s)\}_{s=1}^k$ with all the moduli distinct, odd and greater than one.*

Remark 2. In contrast with the Erdős-Selfridge conjecture, Z. W. Sun [J. Number Theory, to appear, `arXiv:math.NT/0409279`] showed that if $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$ then it cannot cover every integer odd times! Quite recently S. Guo and Z. W. Sun [`arXiv:math.NT/0412217`] proved that if $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with all the moduli distinct, odd, squarefree and greater than one, then the least common multiple $N_A = [n_1, \ldots, n_k]$ has at least 22 (distinct) prime factors. A. Schinzel [Acta Arith. 13(1967)] showed that if Conjecture 2 is true then for any $P(x) \in \mathbb{Z}[x]$ with $P(0) \neq 0$, $P(1) \neq -1$ and $P(x) \not\equiv 1$ there are infinitely many $n \in \mathbb{Z}^+$ such that $x^n + P(x) \in \mathbb{Q}[x]$ is irreducible.

Conjecture 3 (A. Schinzel). *If $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ with $k > 1$, then $n_s \mid n_t$ for some $s, t = 1, \ldots, k$ with $s \neq t$.*

Remark 3. Š. Porubský showed that if n_s does not divide a period of the covering function $w_A(x)$ then $n_s \mid n_t$ for some $t \neq s$.

Conjecture 4 (Z. W. Sun). *Let $A = \{a_s(n_s)\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$. Then $\sum_{s\in I} 1/n_s \in m\mathbb{Z}^+$ for some $I \subseteq \{1, \ldots, k\}$. Moreover, for any $m_1, \ldots, m_k \in \mathbb{Z}$ and $J \subseteq \{1, \ldots, k\}$ there is an $I \subseteq \{1, \ldots, k\}$ with $I \neq J$ such that $\sum_{s\in I} m_s/n_s - \sum_{s\in J} m_s/n_s \in m\mathbb{Z}$.*

Remark 4. Recently Z. W. Sun [Electron. Res. Announc. Amer. Math. Soc. 9(2003), 51-60; `arXiv:math.NT/0305369`] confirmed this in the case where m is a prime power. The conjecture also holds when A is an exact m-cover of $\mathbb{Z}$ (see [Z. W. Sun, Acta Arith. 72(1995)]).

The following conjecture arose from Z. W. Sun's solutions to two problems of A. P. Huhn and L. Megyesi.

Conjecture 5. *If the residue classes $a_1(n_1), \ldots, a_k(n_k)$ $(k > 1)$ are pairwise disjoint, then $\gcd(n_s, n_t) \geqslant k$ for some $1 \leqslant s < t \leqslant k$.*

Remark 5. A finite sequence $\{n_s\}_{s=1}^k$ of positive integers is said to be harmonic if there are integers $a_1, \ldots, a_k$ such that $a_1(n_1), \ldots, a_k(n_k)$ are pairwise disjoint. A. P. Huhn and L. Megyesi [Discrete Math. 41(1982)] asked whether $\{n_s\}_{s=1}^k$ is harmonic if and only if $\max_{s,t\in I,\ s\neq t} \gcd(n_s, n_t) \geqslant |I|$ for all $I \subseteq \{1, \ldots, k\}$ with $|I| \geqslant 2$. Z. W. Sun [Chinese Ann. Math. Ser. A 13(1992)] proved that the answer is positive for $k \leqslant 3$ and that the condition is not sufficient for $k \geqslant 4$ but necessary for $k = 4$. Clearly the necessity of the condition is equivalent to Conjecture 5.

Conjecture 6 (Z. W. Sun, Internat. J. Math., `arXiv:math.GR/0501451`). *Let G be a group and $a_1G_1, \ldots, a_kG_k$ $(k > 1)$ be finitely many left cosets of subgroups of G with the indices $[G : G_1], \ldots, [G : G_k]$ finite. If the k cosets are pairwise disjoint, then $\gcd([G : G_i], [G : G_j]) \geqslant k$ for some $1 \leqslant i < j \leqslant k$.*

Remark 6. When G is the infinite cyclic group $\mathbb{Z}$, Conjecture 6 reduces to Conjecture 5. It is easy to show that Conjecture 6 holds for $k = 2$ and for p-groups.

Conjecture 7 (Z. W. Sun, 1992). *Let $n_1, \ldots, n_k$ be positive integers. If*

$$|\{\{s,t\}\colon 1 \leqslant s < t \leqslant k \ \&\ \gcd(n_s, n_t) = d\}| < 2d - 1$$

for each positive integer $d \leqslant 2^{k-2}$, then there are integers $a_1, \ldots, a_k$ such that the residue classes $a_1(n_1), \ldots, a_k(n_k)$ are pairwise disjoint.

Remark 7. Z. W. Sun [Discrete Math. 104(1992)] proved that the answer is positive if we replace $2d-1$ by $\sqrt{(d+7)/8}$. He also verified the conjecture for $k \leqslant 6$.

Conjecture 8 (Alon, Jaeger, Tarsi). *Let F be a finite field with $|F| > 3$, and let M be a nonsingular k by k matrix over F. Then there exists a vector $\vec{x} \in F^k$ such that both $\vec{x}$ and $M\vec{x}$ have no zero component.*

Remark 8. Alon and Tarsi [Combinatorica 9(1989)] confirmed the conjecture in the case where $|F|$ is not a prime.

Conjecture 9. *Let G be the direct sum of l copies of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$. Then for any $c_1, \ldots, c_{l(n-1)+1} \in G$ there is a nonempty $I \subseteq \{1, \ldots, l(n-1)+1\}$ such that $\sum_{s\in I} c_s = 0$.*

Remark 9. The conjecture is well known but we don't know who posed it first. The case $l = 1$ is trivial. J. E. Olson [J. Number Theory 1(1969)] confirmed the conjecture in the case where n is a prime power as well as the case $l = 2$. N. Alon, S. Friedland and G. Kalai [J. Combin. Theory Ser. B 37(1984)] mentioned the conjecture in their paper.

Conjecture 10 (Z. W. Sun, 2003). *Let n be a positive integer. If $A = \{a_s(n_s)\}_{s=1}^k$ covers every integer either exactly $2n-1$ times or exactly $2n$ times (i.e. $w_A(x) \in \{2n-1, 2n\}$ for all $x \in \mathbb{Z}$), then for any $c_1, \ldots, c_k \in \mathbb{Z}/n\mathbb{Z}$ there is a nonempty $I \subseteq \{1, \ldots, k\}$ such that $\sum_{s\in I} 1/n_s = n$ and $\sum_{s\in I} c_s = 0$.*

Remark 10. In the case $n_1 = \cdots = n_k = 1$, Conjecture 10 reduces to the well-known Erdős-Ginzburg-Ziv theorem. Z. W. Sun [Electron. Res. Announc. Amer. Math. Soc. 9(2003), 51-60; `arXiv:math.NT/0305369`] was able to show the conjecture for any prime power n.

Conjecture 11 [H. S. Snevily, Amer. Math. Monthly 106(1999)]**.** *Let G be an additive abelian group with $|G|$ odd. Let A and B be subsets of G with cardinality $n > 0$. Then there is a numbering $\{a_i\}_{i=1}^n$ of the elements of A and a numbering $\{b_i\}_{i=1}^n$ of the elements of B such that $a_1+b_1, \ldots, a_n+b_n$ are pairwise distinct.*

Remark 11. When G is cyclic, the conjecture was confirmed by S. Dasgupta, G. Károlyi, O. Serra and B. Szegedy [Israel J. Math. 126(2001)]. Their result was recently extended by Z. W. Sun [J. Combin. Theory Ser. A, 103(2003)] via the polynomial method of N. Alon, M.B. Nathanson and I.Z. Ruzsa [J. Number Theory 56(1996)].

Conjecture 12 [H. S. Snevily, Amer. Math. Monthly 106(1999)]**.** *Let m and n be positive integers with $n < m$. Then, for any $a_1, \ldots, a_n \in \mathbb{Z}$, there exists a permutation σ on $\{1, \ldots, n\}$ such that $a_1 + \sigma(1), \ldots, a_n + \sigma(n)$ are pairwise distinct modulo m.*

Remark 12. In 2002 A. E. Kézdy and H. S. Snevily [Combin. Probab. Comput. 11(2002)] confirmed the conjecture for $n \leqslant (m + 1)/2$.