Preprint, arXiv:0912.1280

CONGRUENCES INVOLVING BINOMIAL COEFFICIENTS AND LUCAS SEQUENCES

Zhi-Wei Sun

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Abstract. In this paper we obtain some congruences involving central binomial coefficients and Lucas sequences. For example, we show that if $p>5$ is a prime then

$$\sum_{k=0}^{p-1}\frac{F_k}{12^k}\binom{2k}{k}\equiv\begin{cases}0\ (\mathrm{mod}\ p)&\text{if }p\equiv\pm1\ (\mathrm{mod}\ 5),\\1\ (\mathrm{mod}\ p)&\text{if }p\equiv\pm13\ (\mathrm{mod}\ 30),\\-1\ (\mathrm{mod}\ p)&\text{if }p\equiv\pm7\ (\mathrm{mod}\ 30),\end{cases}$$

where $\{F_n\}_{n\geqslant0}$ is the Fibonacci sequence. We also raise several conjectures.

1. Introduction

Let p be an odd prime. In 2003 Roderiguez-Villeags [RV] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^{(p-1)/2}\ (\mathrm{mod}\ p^2)$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv a(p)\ (\mathrm{mod}\ p^2),$$

where the sequence $\{a(n)\}_{n\geqslant1}$ is defined by

$$\sum_{n=1}^{\infty}a(n)q^n=q\prod_{n=1}^{\infty}(1-q^{4n})^6.$$

2010 *Mathematics Subject Classification.* Primary 11B65; Secondary 05A10, 11A07, 11B39, 11E25.

Keywords. Central binomial coefficients, Lucas sequences, congruences modulo prime powers.

Supported by the National Natural Science Foundation (grant 10871087) and the Overseas Cooperation Fund (grant 10928101) of China.

This was later confirmed by E. Mortenson [M1, M2] via the p-adic Γ-function and the Gross-Koblitz formula. The reader may also consult [M3] and Ono [O] for more such "super" congruences.

In a series of recent papers, the author [S09a-S09e] investigated congruences related to central binomial congruences by using recurrences and combinatorial identities. (See also [PS] and [ST1, ST2].)

Let $A,B\in\mathbb{Z}$. The Lucas sequences $u_n=u_n(A,B)$ $(n\in\mathbb{N})$ and $v_n=v_n(A,B)$ $(n\in\mathbb{N})$ are defined by

$$u_0=0,\ u_1=1,\ \text{and}\ u_{n+1}=Au_n-Bu_{n-1}\ (n=1,2,3,\ldots)$$

and

$$v_0=2,\ v_1=A,\ \text{and}\ v_{n+1}=Av_n-Bv_{n-1}\ (n=1,2,3,\ldots).$$

The characteristic equation $x^2-Ax+B=0$ has two roots

$$\alpha=\frac{A+\sqrt{\Delta}}{2}\quad\text{and}\quad\beta=\frac{A+\sqrt{\Delta}}{2},$$

where $\Delta=A^2-4B$. It is well known that for any $n\in\mathbb{N}$ we have

$$u_n=\sum_{0\leqslant k<n}\alpha^k\beta^{n-1-k}\quad\text{and}\quad v_n=\alpha^n+\beta^n.$$

Note that $F_n=u_n(1,-1)$ and $L_n=v_n(1,-1)$ are Fibonacci numbers and Lucas numbers respectively. The sequences $P_n=u_n(2,-1)$ and $Q_n=v_n(2,-1)$ are called the Pell sequence and its companion. We also set $S_n=u_n(4,1)$ and $T_n=v_n(4,1)$ for $n\in\mathbb{N}$; the sequences $\{S_n\}_{n\geqslant0}$ and its companion $\{T_n\}_{n\geqslant0}$ are also useful (see, e.g., [S02]).

In this paper we study congruences involving both central binomial coefficients and Lucas sequences. Now we state our main results.

Theorem 1.1. *Let $A,m\in\mathbb{Z}$ and let p be an odd prime not dividing m. Suppose that $\delta^2\equiv A^2-4m^2\not\equiv0\pmod p$ where $\delta\in\mathbb{Z}$. Let $a,h\in\mathbb{Z}^+$. If $(\frac{A+\delta}{p^a})=(\frac{2m}{p^a})$, then*

$$\sum_{k=0}^{p^a-1}\frac{u_k(A,m^2)\binom{2k}{k}^h}{m^k(-4)^{hk}}\equiv0\pmod p.$$

If $(\frac{A+\delta}{p^a})=-(\frac{2m}{p^a})$, then

$$\sum_{k=0}^{p^a-1}\frac{v_k(A,m^2)\binom{2k}{k}^h}{m^k(-4)^{hk}}\equiv0\pmod p.$$

Corollary 1.1. *Let $p \equiv 1 \pmod 3$ be a prime and let $a \in \mathbb{Z}^+$. Then*

$$\sum_{k=0}^{p^a-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}}{(-4)^k} \equiv \sum_{k=0}^{p^a-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^2}{16^k} \equiv \sum_{k=0}^{p^a-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod p.$$

When $p^a \equiv 1 \pmod{12}$, we have

$$\sum_{k=0}^{p^a-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}}{4^k} \equiv \sum_{k=0}^{p^a-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^2}{(-16)^k} \equiv \sum_{k=0}^{p^a-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^3}{64^k} \equiv 0 \pmod p.$$

If $p^a \equiv 7 \pmod{12}$, then

$$\sum_{k=0}^{(p^a-1)/3}\frac{\binom{6k}{3k}}{64^k} \equiv 0 \pmod p \ \text{ and } \ \sum_{k=0}^{(p^a-1)/3}(-1)^k\frac{\binom{6k}{3k}^2}{2^{12k}} \equiv 0 \pmod p.$$

Corollary 1.2. *Let $p \equiv \pm 1 \pmod 5$ be a prime and let $a \in \mathbb{Z}^+$. Then*

$$\sum_{k=0}^{p^a-1}F_{2k}\frac{\binom{2k}{k}}{(-4)^k} \equiv \sum_{k=0}^{p^a-1}F_{2k}\frac{\binom{2k}{k}^2}{16^k} \equiv \sum_{k=0}^{p^a-1}F_{2k}\frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod p.$$

If $p^a \equiv 1, 9 \pmod{20}$, then

$$\sum_{k=0}^{p^a-1}F_{2k}\frac{\binom{2k}{k}}{4^k} \equiv \sum_{k=0}^{p^a-1}F_{2k}\frac{\binom{2k}{k}^2}{(-16)^k} \equiv \sum_{k=0}^{p^a-1}F_{2k}\frac{\binom{2k}{k}^3}{64^k} \equiv 0 \pmod p.$$

If $p^a \equiv 11, 19 \pmod{20}$, then

$$\sum_{k=0}^{p^a-1}L_{2k}\frac{\binom{2k}{k}}{4^k} \equiv \sum_{k=0}^{p^a-1}L_{2k}\frac{\binom{2k}{k}^2}{(-16)^k} \equiv \sum_{k=0}^{p^a-1}L_{2k}\frac{\binom{2k}{k}^3}{64^k} \equiv 0 \pmod p.$$

Theorem 1.2. *Let p be an odd prime and let $A, B \in \mathbb{Z}$ and $p \nmid AB\Delta$, where $\Delta = A^2 - 4B$.*

(i) *If $p \equiv 1 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{u_k(A,B)}{(16A)^k}\binom{2k}{k}^2 \equiv 0 \pmod{p^2}.$$

If $p \equiv 3 \pmod 4$, then

$$\sum_{k=0}^{p-1}\frac{v_k(A,B)}{(16A)^k}\binom{2k}{k}^2 \equiv 0 \pmod{p^2}.$$

(ii) *Suppose that* $(\frac{\Delta}{p}) = 1$. *If* $(\frac{-B}{p}) = 1$, *then*

$$\sum_{k=0}^{p-1} \frac{A^k u_k(A,B)}{(16B)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

If $(\frac{-B}{p}) = -1$, *then*

$$\sum_{k=0}^{p-1} \frac{A^k v_k(A,B)}{(16B)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

Remark. Theorem 1.2(i) with $A = 1$ and $B = -1$ was first noted by R. Tauraso [T].

Corollary 1.3. *Let* $p \equiv 1 \pmod 4$ *be a prime. Then*

$$\sum_{k=0}^{p-1} \left(\frac{k}{3}\right) \frac{\binom{2k}{k}^2}{(-16)^k} \equiv 0 \pmod{p^2}.$$

Corollary 1.4. *Let* $p > 5$ *be a prime.*

(i) *If* $p \equiv 1, 4 \pmod 5$ *then*

$$\sum_{k=0}^{p-1} \frac{F_k}{(-16)^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

(ii) *If* $p \equiv 1 \pmod 4$, *then*

$$\sum_{k=0}^{p-1} \frac{F_{2k}}{48^k} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}.$$

If $p \equiv 3 \pmod 4$ *then*

$$\sum_{k=0}^{p-1} \frac{L_{2k}}{48^k} \binom{2k}{k}^2 \equiv 0 \pmod{p^2}.$$

(iii) *If* $p \equiv 1, 9 \pmod{20}$, *then*

$$\sum_{k=0}^{p-1} \frac{3^k F_{2k}}{16^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

If $p \equiv 11, 19 \pmod{20}$, *then*

$$\sum_{k=0}^{p-1} \frac{3^k L_{2k}}{16^k} \binom{2k}{k}^2 \equiv 0 \pmod{p}.$$

Corollary 1.5. *Let p be an odd prime. If $p\equiv 1 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{P_k}{32^k}\binom{2k}{k}^2\equiv 0 \pmod{p^2}.$$

If $p\equiv 3 \pmod 4$ then

$$\sum_{k=0}^{p-1}\frac{Q_k}{32^k}\binom{2k}{k}^2\equiv 0 \pmod{p^2}.$$

If $p\equiv \pm1 \pmod 8$, then

$$\sum_{k=0}^{p-1}\frac{P_k}{(-8)^k}\binom{2k}{k}^2\equiv 0 \pmod{p}.$$

Corollary 1.6. *Let $p>3$ be a prime. If $p\equiv 1 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{S_k}{64^k}\binom{2k}{k}^2\equiv 0 \pmod{p^2}.$$

If $p\equiv 3 \pmod 4$ then

$$\sum_{k=0}^{p-1}\frac{T_k}{64^k}\binom{2k}{k}^2\equiv 0 \pmod{p^2}.$$

If $p\equiv 1 \pmod{12}$, then

$$\sum_{k=0}^{p-1}\frac{S_k}{4^k}\binom{2k}{k}^2\equiv 0 \pmod{p}.$$

If $p\equiv 11 \pmod{12}$, then

$$\sum_{k=0}^{p-1}\frac{T_k}{4^k}\binom{2k}{k}^2\equiv 0 \pmod{p}.$$

Theorem 1.3. *Let $A,B\in\mathbb{Z}$ and $\Delta=A^2-4B$. Let p be an odd prime and let $m\in\mathbb{Z}$ with $p\nmid m$. Suppose that $p\nmid\Delta$ and $d^2\equiv m^2-4Am+16B\not\equiv 0 \pmod p$ where $d\in\mathbb{Z}$. Then*

$$\sum_{k=0}^{p-1}\frac{u_k(A,B)}{m^k}\binom{2k}{k}\equiv\begin{cases}0 \pmod p & \text{if } (\frac{\Delta}{p})=1,\\ -\frac{4}{d}(\frac{2m}{p})(\frac{m-d-2A}{p}) \pmod p & \text{if } (\frac{\Delta}{p})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{v_k(A,B)}{m^k}\binom{2k}{k}\equiv\begin{cases}2(\frac{2m}{p})(\frac{m-d-2A}{p}) \pmod p & \text{if } (\frac{\Delta}{p})=1,\\ \frac{4A-2m}{d}(\frac{2m}{p})(\frac{m-d-2A}{p}) \pmod p & \text{if } (\frac{\Delta}{p})=-1.\end{cases}$$

Theorem 1.3 in the case $A=-1,\ B=1,\ m=-4$ and $d=1$ gives the following consequence.

Corollary 1.7. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{(\frac{k}{3})}{(-4)^k}\binom{2k}{k}\equiv\frac{(\frac{-1}{p})-(\frac{3}{p})}{2}\pmod p.$$

Applying Theorem 1.3 with $A=1,\ B=-1,\ m\in\{4,8\}$ and $d=4$, we immediately get the following corollary.

Corollary 1.8. *Let p be an odd prime. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{F_k}{(-4)^k}\binom{2k}{k}\equiv&\frac{1-(\frac{p}{5})}{2}\pmod p,\\ \sum_{k=0}^{p-1}\frac{L_k}{(-4)^k}\binom{2k}{k}\equiv&\frac{5(\frac{p}{5})-1}{2}\pmod p,\\ \sum_{k=0}^{p-1}\frac{F_k}{8^k}\binom{2k}{k}\equiv&\left(\frac{2}{p}\right)\frac{(\frac{p}{5})-1}{2}\pmod p,\\ \sum_{k=0}^{p-1}\frac{L_k}{8^k}\binom{2k}{k}\equiv&\left(\frac{2}{p}\right)\frac{5(\frac{p}{5})-1}{2}\pmod p.\end{aligned}$$

Theorem 1.3 in the case $A=2,\ B=-1,\ m\in\{-2,10\}$ and $d=2$, yields the following result.

Corollary 1.9. *Let p be an odd prime. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{P_k}{(-2)^k}\binom{2k}{k}\equiv&1-\left(\frac{2}{p}\right)\pmod p,\\ \sum_{k=0}^{p-1}\frac{Q_k}{(-2)^k}\binom{2k}{k}\equiv&4\left(\frac{2}{p}\right)-2\pmod p.\end{aligned}$$

If $p\neq 5$, then

$$\sum_{k=0}^{p-1}\frac{P_k}{10^k}\binom{2k}{k}\equiv\left(\frac{p}{5}\right)\left(\left(\frac{2}{p}\right)-1\right)\pmod p$$

and

$$\sum_{k=0}^{p-1}\frac{Q_k}{10^k}\binom{2k}{k}\equiv\left(\frac{p}{5}\right)\left(4\left(\frac{2}{p}\right)-2\right)\pmod p.$$

Theorem 1.3 in the case $A=4,\ B=1,\ m\in\{1,15,16\}$, leads the following corollary.

Corollary 1.10. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1} S_k\binom{2k}{k} \equiv 2\left(\left(\frac{p}{3}\right)-\left(\frac{-1}{p}\right)\right) \pmod p,$$
$$\sum_{k=0}^{p-1} T_k\binom{2k}{k} \equiv 8\left(\frac{-1}{p}\right)-6\left(\frac{p}{3}\right) \pmod p,$$
$$\sum_{k=0}^{p-1} \frac{S_k}{16^k}\binom{2k}{k} \equiv \frac{(\frac{6}{p})-(\frac{2}{p})}{2} \pmod p,$$
$$\sum_{k=0}^{p-1} \frac{T_k}{16^k}\binom{2k}{k} \equiv 3\left(\frac{6}{p}\right)-\left(\frac{2}{p}\right) \pmod p.$$

When $p>5$, we also have

$$\sum_{k=0}^{p-1} \frac{S_k}{15^k}\binom{2k}{k} \equiv 2\left(\frac{p}{5}\right)\left(\left(\frac{3}{p}\right)-1\right) \pmod p$$

and

$$\sum_{k=0}^{p-1} \frac{T_k}{15^k}\binom{2k}{k} \equiv 2\left(\frac{p}{5}\right)\left(8\left(\frac{3}{p}\right)-6\right) \pmod p.$$

Theorem 1.4. *Let $A,B\in\mathbb{Z}$ and $\Delta=A^2-4B$. Let p be an odd prime with $p\nmid AB$ and $(\frac{\Delta}{p})=1$. Then*

$$\sum_{k=0}^{p-1} \frac{A^k v_k(A,B)}{(4B)^k}\binom{2k}{k} \equiv 2\left(\frac{-B}{p}\right) \pmod{p^2}.$$

Theorem 1.5. *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1} \frac{F_k}{12^k}\binom{2k}{k} \equiv \begin{cases} 0 \pmod p & \text{if } p\equiv \pm1 \pmod 5,\\ 1 \pmod p & \text{if } p\equiv \pm13 \pmod{30},\\ -1 \pmod p & \text{if } p\equiv \pm7 \pmod{30}.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1} \frac{L_k}{12^k}\binom{2k}{k} \equiv \begin{cases} -1 \pmod p & \text{if } p\equiv \pm7 \pmod{30},\\ 1 \pmod p & \text{if } p\equiv \pm13 \pmod{30},\\ 2 \pmod p & \text{if } p\equiv \pm1 \pmod{30},\\ -2 \pmod p & \text{if } p\equiv \pm11 \pmod{30}.\end{cases}$$

Let $p>5$ be a prime. By the method we prove Theorem 1.5, we can also determine the following sums modulo p.

$$\sum_{k=0}^{p-1}\frac{F_k}{m^k}\binom{2k}{k}\ \text{and}\ \sum_{k=0}^{p-1}\frac{L_k}{m^k}\binom{2k}{k}\ (m=-3,\ 7,\ -8),$$

$$\sum_{k=0}^{p-1}\frac{P_k}{m^k}\binom{2k}{k}\ \text{and}\ \sum_{k=0}^{p-1}\frac{Q_k}{m^k}\binom{2k}{k}\ (m=-4,\ 12)$$

For example,

$$\sum_{k=0}^{p-1}\frac{F_k}{(-3)^k}\binom{2k}{k}\equiv\left(\frac{p}{5}\right)-1\pmod p,$$

and

$$\sum_{k=0}^{p-1}\frac{L_k}{(-3)^k}\binom{2k}{k}\equiv\begin{cases}2\pmod p&\text{if}\ p\equiv\pm1\pmod{30},\\-2\pmod p&\text{if}\ p\equiv\pm11\pmod{30},\\4\pmod p&\text{if}\ p\equiv\pm7\pmod{30},\\-4\pmod p&\text{if}\ p\equiv\pm13\pmod{30}.\end{cases}$$

Modifying the method slightly, we can also prove the following congruences.

$$\sum_{k=0}^{p-1}\frac{C_kP_k}{(-2)^k}\equiv2\left(\frac{2}{p}\right)-2\pmod p,\quad\sum_{k=0}^{(p-1)/2}C_kS_k\equiv\frac{(\frac{p}{3})-1}{2}\pmod p,$$

$$\sum_{k=0}^{(p-1)/2}\frac{C_kF_k}{(-4)^k}\equiv2\left(\frac{p}{5}\right)-2\pmod p,$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{C_kF_k}{12^k}\equiv\begin{cases}0\pmod p&\text{if}\ p\equiv\pm1\pmod{30},\\4\pmod p&\text{if}\ p\equiv\pm7\pmod{30},\\8\pmod p&\text{if}\ p\equiv\pm13\pmod{30},\\12\pmod p&\text{if}\ p\equiv\pm11\pmod{30}.\end{cases}$$

In the next section we will prove Theorem 1.1 and Corollaries 1.1 and 1.2. Theorems 1.2-1.3 and Corollaries 1.3-1.6 will be proved in Section 3. Section 4 is devoted to the proof of Theorems 1.4 and 1.5. We are going to raise some challenging conjectures in Section 5.

2. Proofs of Theorem 1.1 and Corollaries 1.1-1.2

Lemma 2.1. *Let $A, B \in \mathbb{Z}$ and let $\Delta = A^2 - 4B$. Suppose that p is an odd prime and $\delta^2 \equiv \Delta \pmod p$ with $\delta \in \mathbb{Z}$. Then, for any $n \in \mathbb{N}$ we have*

$$\delta u_n(A,B) \equiv \left(\frac{A+\delta}{2}\right)^n - \left(\frac{A-\delta}{2}\right)^n \pmod p \tag{2.1}$$

and

$$v_n(A,B) \equiv \left(\frac{A+\delta}{2}\right)^n + \left(\frac{A-\delta}{2}\right)^n \pmod p. \tag{2.2}$$

Proof. Observe that

$$\begin{aligned} v_n(A,B) &= \left(\frac{A+\sqrt{\Delta}}{2}\right)^n + \left(\frac{A-\sqrt{\Delta}}{2}\right)^n \\ &= \frac{1}{2^n}\sum_{k=0}^n \binom{n}{k} A^{n-k}\left(\sqrt{\Delta}^k + (-\sqrt{\Delta})^k\right) = \frac{2}{2^n}\sum_{\substack{k=0\\ 2\mid k}}^n \binom{n}{k} A^{n-k}\Delta^{k/2} \\ &\equiv \frac{2}{2^n}\sum_{\substack{k=0\\ 2\mid k}}^n \binom{n}{k} A^{n-k}\delta^k = \left(\frac{A+\delta}{2}\right)^n + \left(\frac{A-\delta}{2}\right)^n \pmod p. \end{aligned}$$

If $p \mid \Delta$, then both sides of (2.1) are multiples of p. When $p \nmid \Delta$, we have

$$\begin{aligned} u_n(A,B) &= \frac{1}{\sqrt{\Delta}}\left(\left(\frac{A+\sqrt{\Delta}}{2}\right)^n - \left(\frac{A-\sqrt{\Delta}}{2}\right)^n\right) \\ &= \frac{1}{2^n\sqrt{\Delta}}\sum_{k=0}^n \binom{n}{k} A^{n-k}\left(\sqrt{\Delta}^k - (-\sqrt{\Delta})^k\right) \\ &= \frac{2}{2^n}\sum_{\substack{k=0\\ 2\nmid k}}^n \binom{n}{k} A^{n-k}\Delta^{(k-1)/2} \\ &\equiv \frac{2}{2^n}\sum_{\substack{k=0\\ 2\nmid k}}^n \binom{n}{k} A^{n-k}\delta^{k-1} \\ &\equiv \frac{1}{\delta}\left(\left(\frac{A+\delta}{2}\right)^n - \left(\frac{A-\delta}{2}\right)^n\right) \pmod p. \end{aligned}$$

Thus both (2.1) and (2.2) hold. □

Lemma 2.2. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. Then,for every $k = 0, \ldots, p^a - 1$ we have*

$$\binom{(p^a-1)/2}{k} \equiv \frac{\binom{2k}{k}}{(-4)^k} \pmod{p}.$$

Proof. The congruence appeared as [S09e, (2.3)]. □

Theorem 2.1. *Let $A, B \in \mathbb{Z}$ and let $\Delta = A^2 - 4B$. Let p be an odd prime with $(\frac{\Delta}{p}) = 1$. Suppose that $\delta^2 \equiv \Delta \not\equiv 0 \pmod{p}$ where $\delta \in \mathbb{Z}$. Let $a, h \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod{p}$. If $(\frac{B}{p^a}) = 1$, then*

$$\sum_{k=0}^{p^a-1} \frac{u_k(A,B)}{m^k} \cdot \frac{\binom{2k}{k}^h}{(-4)^{hk}} \equiv -\left(\frac{2m(A+\delta)}{p^a}\right) \sum_{k=0}^{p^a-1} \frac{m^k u_k(A,B)}{B^k} \cdot \frac{\binom{2k}{k}^h}{(-4)^{hk}} \pmod{p}$$

and

$$\sum_{k=0}^{p^a-1} \frac{v_k(A,B)}{m^k} \cdot \frac{\binom{2k}{k}^h}{(-4)^{hk}} \equiv \left(\frac{2m(A+\delta)}{p^a}\right) \sum_{k=0}^{p^a-1} \frac{m^k v_k(A,B)}{B^k} \cdot \frac{\binom{2k}{k}^h}{(-4)^{hk}} \pmod{p}.$$

If $(\frac{B}{p^a}) = -1$, then

$$\sum_{k=0}^{p^a-1} \frac{u_k(A,B)}{m^k} \cdot \frac{\binom{2k}{k}^h}{(-4)^{hk}} \equiv \frac{1}{\delta} \left(\frac{2m(A+\delta)}{p^a}\right) \sum_{k=0}^{p^a-1} \frac{m^k v_k(A,B)}{B^k} \cdot \frac{\binom{2k}{k}^h}{(-4)^{hk}} \pmod{p}$$

and

$$\sum_{k=0}^{p^a-1} \frac{v_k(A,B)}{m^k} \cdot \frac{\binom{2k}{k}^h}{(-4)^{hk}} \equiv -\delta \left(\frac{2m(A+\delta)}{p^a}\right) \sum_{k=0}^{p^a-1} \frac{m^k u_k(A,B)}{B^k} \cdot \frac{\binom{2k}{k}^h}{(-4)^{hk}} \pmod{p}.$$

Proof. Set

$$n = \frac{p^a-1}{2}, \quad \alpha = \frac{A+\delta}{2} \quad \text{and} \quad \beta = \frac{A-\delta}{2}.$$

Clearly,

$$(2\alpha)^{(p-1)/2} = (A+\delta)^{(p-1)/2} \equiv \left(\frac{A+\delta}{p}\right) \pmod{p}$$

and hence

$$\alpha^n = \left(\alpha^{(p-1)/2}\right)^{\sum_{r=0}^{a-1} p^r} \equiv \left(\frac{2(A+\delta)}{p}\right)^{\sum_{r=0}^{a-1} p^r} = \left(\frac{2(A+\delta)}{p^a}\right) \pmod{p}.$$

Similarly,

$$\beta^n \equiv \left(\frac{2(A-\delta)}{p^a}\right) \pmod p$$

and hence

$$\alpha^n\beta^n \equiv \left(\frac{A^2-\delta^2}{p^a}\right) = \left(\frac{4B}{p^a}\right) = \left(\frac{B}{p^a}\right) \pmod p.$$

By Lemmas 2.1 and 2.2,

$$\begin{aligned}
&\sum_{k=0}^{p-1}\frac{u_k(A,B)}{m^k}\cdot\frac{\binom{2k}{k}^h}{(-4)^{hk}}\\
\equiv&\sum_{k=0}^{n}\binom{n}{k}^h\frac{\alpha^k-\beta^k}{m^k\delta} = \sum_{k=0}^{n}\binom{n}{k}^h\frac{\alpha^{n-k}-\beta^{n-k}}{m^{n-k}\delta}\\
\equiv&\frac{1}{m^n\delta}\sum_{k=0}^{n}\binom{n}{k}^h m^k(\alpha^{n-k}-\beta^{n-k})\\
\equiv&\frac{(\frac{m}{p^a})}{\delta}\sum_{k=0}^{n}\binom{n}{k}^h\frac{m^k}{B^k}(\alpha^n\beta^k-\beta^n\alpha^k)\\
\equiv&\left(\frac{m}{p^a}\right)\left(\frac{2(A+\delta)}{p}\right)\sum_{k=0}^{n}\frac{\binom{2k}{k}^h}{(-4)^{hk}}\cdot\frac{m^k}{B^k}\cdot\frac{\beta^k-(\frac{B}{p^a})\alpha^k}{\delta} \pmod p.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{k=0}^{p-1}\frac{v_k(A,B)}{m^k}\cdot\frac{\binom{2k}{k}^h}{(-4)^{hk}}\\
\equiv&\sum_{k=0}^{n}\binom{n}{k}^h\frac{\alpha^k+\beta^k}{m^k} = \sum_{k=0}^{n}\binom{n}{k}^h\frac{\alpha^{n-k}+\beta^{n-k}}{m^{n-k}}\\
\equiv&\frac{1}{m^n}\sum_{k=0}^{n}\binom{n}{k}^h m^k(\alpha^{n-k}+\beta^{n-k})\\
\equiv&\left(\frac{m}{p^a}\right)\sum_{k=0}^{n}\binom{n}{k}^h\frac{m^k}{B^k}(\alpha^n\beta^k+\beta^n\alpha^k)\\
\equiv&\left(\frac{m}{p^a}\right)\left(\frac{2(A+\delta)}{p^a}\right)\sum_{k=0}^{n}\frac{\binom{2k}{k}^h}{(-4)^{hk}}\cdot\frac{m^k}{B^k}\left(\beta^k+\left(\frac{B}{p^a}\right)\alpha^k\right) \pmod p.
\end{aligned}$$

Note that

$$\frac{\beta^k-(\frac{B}{p^a})\alpha^k}{\delta} \equiv \begin{cases} (\beta^k-\alpha^k)/\delta \equiv -u_k(A,B) \pmod p & \text{if } (\frac{B}{p^a})=1,\\ (\alpha^k+\beta^k)/\delta \equiv v_k(A,B)/\delta \pmod p & \text{if } (\frac{B}{p^a})=-1.\end{cases}$$

Also,

$$\beta^k+\left(\frac{B}{p^a}\right)\alpha^k\equiv\begin{cases}\alpha^k+\beta^k\equiv v_k(A,B)\pmod p & \text{if } (\frac{B}{p^a})=1,\\ \beta^k-\alpha^k\equiv -\delta u_k(A,B)\pmod p & \text{if } (\frac{B}{p^a})=-1.\end{cases}$$

So the desired results follow from the above. □

Proof of Theorem 1.1. Simply apply Theorem 2.1 with $B=m^2$. □

Proof of Corollary 1.1. Let ω be the primitive cubic root $(-1+\sqrt{-3})/2$ of unity. It is easy to see that

$$\left(\frac{k}{3}\right)=\frac{\omega^k-\bar\omega^k}{\sqrt{-3}}=u_k(\omega+\bar\omega,\omega\bar\omega)=u_k(-1,1)$$

for all $k\in\mathbb{N}$. Since $p\equiv 1\pmod 3$, we have $(\frac{-3}{p})=(\frac{p}{3})=1$ and hence $\delta^2\equiv -3\pmod p$ for some $\delta\in\mathbb{Z}$. Observe that

$$\left(\frac{-1+\delta}{p}\right)^3=\left(\frac{\delta(\delta^2+3)-(3\delta^2+1)}{p}\right)=\left(\frac{(-3)^2-1}{p}\right)=\left(\frac{2}{p}\right).$$

By Theorem 1.1,

$$\sum_{k=0}^{p^a-1}\frac{u_k(-1,1)\binom{2k}{k}^h}{(-4)^{hk}}\equiv 0\pmod p\quad\text{for all } h\in\mathbb{Z}^+.$$

If $p^a\equiv 1\pmod{12}$, then $(\frac{-1}{p^a})=1$ and hence by Theorem 1.1 with $A=m=-1$ we have

$$\sum_{k=0}^{p^a-1}\frac{u_k(-1,1)\binom{2k}{k}^h}{(-1)^k(-4)^{hk}}\equiv 0\pmod p\quad\text{for all } h\in\mathbb{Z}^+.$$

Now assume that $p^a\equiv 7\pmod{12}$. Then $(\frac{-1+\delta}{p^a})=-(\frac{-2}{p^a})$ and hence by Theorem 1.1 we have

$$\sum_{k=0}^{p^a-1}\frac{v_k(-1,1)\binom{2k}{k}^h}{(-1)^k(-4)^{hk}}\equiv 0\pmod p\quad\text{for all } h\in\mathbb{Z}^+.$$

Note that

$$v_k(-1,1)=\omega^k+\bar\omega^k=\begin{cases}2 & \text{if } 3\mid k,\\ -1 & \text{if } 3\nmid k.\end{cases}$$

Thus

$$3\sum_{\substack{k=0\\ 3\mid k}}^{p^a-1}\frac{\binom{2k}{k}}{(-1)^k(-4)^k}\equiv\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}}{4^k}\equiv 0\pmod p.$$

(We apply [ST2, Corollary 1.1] in the last step.) Also,

$$3\sum_{\substack{k=0\\3\mid k}}^{p^a-1}\frac{\binom{2k}{k}^2}{(-1)^k(-4)^{2k}}\equiv\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 \pmod p.$$

where $n=(p^a-1)/2$. Note that

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{n}{n-k}$$

coincides with the coefficient of x^n in $(1-x)^n(1+x)^n=(1-x^2)^n$ which is zero since n is odd. Therefore we also have the last two congruences in Corollary 1.1.

The proof of Corollary 1.1 is now complete. □

Proof of Corollary 1.2. As $p\equiv\pm1\pmod 5$, we have $(\frac{5}{p})=(\frac{p}{5})=1$. Thus $\delta^2\equiv5\pmod p$ for some $\delta\in\mathbb{Z}$. Note that

$$\left(\frac{2}{p}\right)\left(\frac{3+\delta}{p}\right)=\left(\frac{6+2\delta}{p}\right)=\left(\frac{(1+\delta)^2}{p}\right)=1.$$

Since $u_k(3,1)=F_{2k}$ and $v_k(3,1)=L_{2k}$, applying Theorem 1.1 with $A=3$ and $m=\pm1$ we immediately obtain the desired results. □

3. Proofs of Theorems 1.2-1.3 and Corollaries 1.3-1.6

Lemma 3.1. *Let p be an odd prime and let x be any algebraic p-adic integer. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\left(x^k-(-1)^{(p-1)/2}(1-x)^k\right)\equiv0 \pmod{p^2}.$$

Proof. This is a result recently obtained by Zhi-Hong Sun [S2] and R. Tauraso [T] independently. □

Proof of Theorem 1.2. (i) Let α and β be the two roots of the equation $x^2-Ax+B=0$. Then

$$\begin{aligned}&(-1)^{(p-1)/2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(16A)^k}v_k(A,B)\\ =&(-1)^{(p-1)/2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\left(\left(\frac{\alpha}{A}\right)^k+\left(\frac{\beta}{A}\right)^k\right)\\ \equiv&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\left(\left(1-\frac{\alpha}{A}\right)^k+\left(1-\frac{\beta}{A}\right)^k\right)\\ \equiv&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\left(\left(\frac{\beta}{A}\right)^k+\left(\frac{\alpha}{A}\right)^k\right)=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(16A)^k}v_k(A,B) \pmod{p^2}.\end{aligned}$$

Hence
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(16A)^k}v_k(A,B)\equiv 0 \pmod{p^2}$$
if $p\equiv 3\pmod 4$. Similarly,
$$\begin{aligned}&(-1)^{(p-1)/2}(\alpha-\beta)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(16A)^k}u_k(A,B)\\
=&(-1)^{(p-1)/2}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\left(\left(\frac{\alpha}{A}\right)^k-\left(\frac{\beta}{A}\right)^k\right)\\
\equiv&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\left(\left(1-\frac{\alpha}{A}\right)^k-\left(1-\frac{\beta}{A}\right)^k\right)\\
\equiv&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\left(\left(\frac{\beta}{A}\right)^k-\left(\frac{\alpha}{A}\right)^k\right)=(\beta-\alpha)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(16A)^k}u_k(A,B)\pmod{p^2}.\end{aligned}$$
If $p\equiv 1\pmod 4$ and $\Delta=(\alpha-\beta)^2\not\equiv 0\pmod p$, then
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(16A)^k}u_k(A,B)\equiv 0 \pmod{p^2}.$$
So part (i) holds.

(ii) Below we assume that $(\frac{\Delta}{p})=1$. Choose $\delta\in\mathbb{Z}$ such that $\delta^2\equiv\Delta\pmod p$. Combining part (i) with Theorem 2.1 in the case $m=A$ and $h=2$, we obtain the second part of Theorem 1.2.

The proof of Theorem 1.2 is now complete. □

Proofs of Corollaries 1.3-1.6. Recall that
$$\left(\frac{k}{3}\right)=u_k(-1,1),\quad F_{2k}=u_k(3,1),\quad L_{2k}=v_k(3,1),$$
and
$$P_k=u_k(2,-1),\ Q_k=v_k(2,-1),\ S_k=u_k(4,1)\ T_k=v_k(4,1).$$
In view of this, we immediately obtain the desired results from Theorem 1.2. □

Lemma 3.2. *Let $A,B\in\mathbb{Z}$. Let p be an odd prime with $(\frac{B}{p})=1$. Suppose that $b^2\equiv B\pmod p$ where $b\in\mathbb{Z}$. Then*
$$u_{(p-1)/2}(A,B)\equiv\begin{cases}0\pmod p & \text{if } (\frac{A^2-4B}{p})=1,\\ \frac{1}{b}(\frac{A-2b}{p})\pmod p & \text{if } (\frac{A^2-4B}{p})=-1;\end{cases}$$

$$u_{(p+1)/2}(A,B) \equiv \begin{cases} (\frac{A-2b}{p}) \pmod p & \text{if } (\frac{A^2-4B}{p}) = 1, \\ 0 \pmod p & \text{if } (\frac{A^2-4B}{p}) = -1. \end{cases}$$

Also,

$$v_{(p-1)/2}(A,B) \equiv \begin{cases} 2(\frac{A-2b}{p}) \pmod p & \text{if } (\frac{A^2-4B}{p}) = 1, \\ -\frac{A}{b}(\frac{A-2b}{p}) \pmod p & \text{if } (\frac{A^2-4B}{p}) = -1. \end{cases}$$

Proof. The first two congruences are known results, see, e.g., [S1]. The last one follows from the first two since $v_n = 2u_{n+1} - Au_n$ for $n \in \mathbb{N}$. □

Proof of Theorem 1.3. Let $n = (p-1)/2$, and

$$\alpha = \frac{A+\sqrt{\Delta}}{2} \quad \text{and} \quad \beta = \frac{A-\sqrt{\Delta}}{2}.$$

By Lemma 2.2,

$$\binom{2k}{k} \equiv \binom{n}{k}(-4)^k \pmod p \quad \text{for all } k = 0, \dots, p-1.$$

So we have

$$\begin{aligned}&(\alpha-\beta)\sum_{k=0}^{p-1}\frac{u_k(A,B)}{m^k}\binom{2k}{k}\\ \equiv&\sum_{k=0}^{n}\binom{n}{k}\left(\frac{(-4\alpha)^k}{m^k}-\frac{(-4\beta)^k}{m^k}\right)=\left(1-\frac{4\alpha}{m}\right)^n-\left(1-\frac{4\beta}{m}\right)^n\end{aligned}$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{v_k(A,B)}{m^k}\binom{2k}{k}\\ \equiv&\sum_{k=0}^{n}\binom{n}{k}\left(\frac{(-4\alpha)^k}{m^k}+\frac{(-4\beta)^k}{m^k}\right)=\left(1-\frac{4\alpha}{m}\right)^n+\left(1-\frac{4\beta}{m}\right)^n.\end{aligned}$$

Observe that

$$(m-4\alpha)+(m-4\beta) = 2m-4A \text{ and } (m-4\alpha)(m-4\beta) = m^2-4mA+16B.$$

Thus

$$\begin{aligned}\left(\frac{m}{p}\right)\sum_{k=0}^{p-1}\frac{u_k(A,B)}{m^k}\binom{2k}{k} &\equiv -4\times\frac{(m-4\alpha)^n-(m-4\beta)^n}{4\beta-4\alpha}\\ &\equiv -4u_n(2m-4A, m^2-4Am+16B) \pmod p\end{aligned}$$

and

$$\left(\frac{m}{p}\right)\sum_{k=0}^{p-1}\frac{v_k(A,B)}{m^k}\binom{2k}{k}\equiv v_n(2m-4A,m^2-4Am+16B)\pmod p.$$

Note that

$$(2m-4A)^2-4(m^2-4Am+16B)=16(A^2-4B)=16\Delta.$$

Via Lemma 3.2 we are able to determine $u_n(2m-4A,m^2-4Am+16B)$ and $v_n(2m-4A,m^2-4Am+16B)$ modulo p and hence the desired congruences follow. □

4. Proofs of Theorems 1.4 and 1.5

Proof of Theorem 1.4. As $(\frac{\Delta}{p})=1$, there is an integer δ such that $\delta^2\equiv\Delta\pmod{p^2}$. Set $\alpha=(A+\delta)/2$ and $\beta=(A-\delta)/2$. Then

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{A^kv_k(A,B)}{(4B)^k}\binom{2k}{k}\\ \equiv&\sum_{k=0}^{p-1}\binom{2k}{k}\left(\frac{(A\alpha)^k}{(4B)^k}+\frac{(A\beta)^k}{(4B)^k}\right)=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(4\beta/A)^k}+\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(4\alpha/A)^k}\pmod{p^2}.\end{aligned}$$

Note that

$$\frac{4\alpha}{A}\left(\frac{4\alpha}{A}-4\right)\equiv-\frac{4^2}{A^2}B\equiv\frac{4\beta}{A}\left(\frac{4\beta}{A}-4\right)\pmod{p^2}.$$

Hence by the main result of [S09b] we have

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(4\alpha/A)^k}+\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{(4\beta/A)^k}\\ \equiv&\left(\frac{-B}{p}\right)+u_{p-(\frac{-B}{p})}\left(\frac{4\alpha}{A}-2,1\right)+\left(\frac{-B}{p}\right)+u_{p-(\frac{-B}{p})}\left(\frac{4\beta}{A}-2,1\right)\pmod{p^2}.\end{aligned}$$

Since

$$\frac{4\alpha}{A}-2+\frac{4\beta}{A}-2=0$$

and $u_n(-x,1)=(-1)^{n-1}u_n(x,1)$ for any $n\in\mathbb{N}$, the desired result follows from the above. □

Lemma 4.1. *Let $p \neq 2, 5$ be a prime. Then*

$$F_{(p-(\frac{p}{5}))/2} \equiv \begin{cases} 0 \pmod p & \text{if } p \equiv 1 \pmod 4, \\ 2(-1)^{\lfloor (p+5)/10 \rfloor}(\frac{5}{p})5^{(p-3)/4} \pmod p & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and

$$F_{(p+(\frac{p}{5}))/2} \equiv \begin{cases} (-1)^{\lfloor (p+5)/10 \rfloor}(\frac{5}{p})5^{(p-1)/4} \pmod p & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{\lfloor (p+5)/10 \rfloor}(\frac{5}{p})5^{(p-3)/4} \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Also,

$$L_{(p-(\frac{p}{5}))/2} \equiv \begin{cases} 2(-1)^{\lfloor (p+5)/10 \rfloor}(\frac{5}{p})5^{(p-1)/4} \pmod p & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod p & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

and

$$L_{(p+(\frac{p}{5}))/2} \equiv \begin{cases} (-1)^{\lfloor (p+5)/10 \rfloor}5^{(p-1)/4} \pmod p & \text{if } p \equiv 1 \pmod 4, \\ (-1)^{\lfloor (p+5)/10 \rfloor}(\frac{5}{p})5^{(p+1)/4} \pmod p & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Proof. This follows from Z. H. Sun and Z. W. Sun [SS, Corollaries 1 and 2]. □

Proof of Theorem 1.5. As in the proof of Theorem 1.3, we have

$$\begin{aligned}\sum_{k=0}^{p-1} \frac{F_k}{12^k}\binom{2k}{k} &\equiv -4\left(\frac{12}{p}\right)u_{(p-1)/2}(2\times 12-4, 12^2-4\times 1\times 12+16(-1)) \\ &\equiv -4\left(\frac{3}{p}\right)u_{(p-1)/2}(20,80) \pmod p.\end{aligned}$$

Set $n = (p-1)/2$. As the equations $x^2 - 20x + 80 = 0$ has two roots $10 \pm 2\sqrt{5}$, we have

$$\begin{aligned}u_n(20,80) &= \frac{(10+2\sqrt{5})^n - (10-2\sqrt{5})^n}{4\sqrt{5}} \\ &= (4\sqrt{5})^{n-1}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - (-1)^n\left(\frac{1-\sqrt{5}}{2}\right)^n\right) \\ &= \begin{cases} 2^{p-3}5^{(p-1)/4}F_n & \text{if } 2 \mid n, \text{ i.e., } p \equiv 1 \pmod 4, \\ 2^{p-3}5^{(p-3)/4}L_n & \text{if } 2 \nmid n, \text{ i.e., } p \equiv 3 \pmod 4. \end{cases}\end{aligned}$$

Therefore

$$-\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{F_k}{12^k}\binom{2k}{k} \equiv \begin{cases} 5^{(p-1)/4}F_{(p-1)/2} & \text{if } p \equiv 1 \pmod 4, \\ 5^{(p-3)/4}L_{(p-1)/2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

Case 1. $(\frac{p}{5})=1$. By Lemma 4.1,

$$F_{(p-1)/2}=F_{(p-(\frac{p}{5}))/2}\equiv 0 \pmod p \quad \text{if } p\equiv 1 \pmod 4,$$

and

$$L_{(p-1)/2}=L_{(p-(\frac{p}{5}))/2}\equiv 0 \pmod p \quad \text{if } p\equiv 3 \pmod 4.$$

It follows that

$$\sum_{k=0}^{p-1}\frac{F_k}{12^k}\binom{2k}{k}\equiv 0 \pmod p.$$

Case 2. $(\frac{p}{5})=-1$. If $p\equiv 1 \pmod 4$, then by Lemma 4.1 we have

$$\begin{aligned}&5^{(p-1)/4}F_{(p-1)/2}=5^{(p-1)/4}F_{(p+(\frac{p}{5}))/2}\\ \equiv&5^{(p-1)/4}(-1)^{\lfloor (p+5)/10\rfloor}\left(\frac{5}{p}\right)5^{(p-1)/4}\equiv(-1)^{\lfloor (p+5)/10\rfloor} \pmod p.\end{aligned}$$

If $p\equiv 3 \pmod 4$, then by Lemma 4.1 we have

$$\begin{aligned}&5^{(p-3)/4}L_{(p-1)/2}=5^{(p-3)/4}L_{(p+(\frac{p}{5}))/2}\\ \equiv&5^{(p-3)/4}(-1)^{\lfloor (p+5)/10\rfloor}\left(\frac{5}{p}\right)5^{(p+1)/4}\equiv(-1)^{\lfloor (p+5)/10\rfloor} \pmod p.\end{aligned}$$

Therefore

$$\sum_{k=0}^{p-1}\frac{F_k}{12^k}\binom{2k}{k}\equiv-\left(\frac{p}{3}\right)(-1)^{\lfloor (p+5)/10\rfloor} \pmod p$$

and hence the first congruence in Theorem 1.5 follows.

The second congruence in Theorem 1.5 can be proved in a similar way. We omit the details. □

5. Some conjectures

Our following conjectures involve representations of primes by binary quadratic forms. The reader may consult [C] and [BEW, Chapter 9] for basic knowledge and background.

Conjecture 5.1. *Let $p>3$ be a prime. If $p\equiv 7 \pmod{12}$ and $p=x^2+3y^2$ with $y\equiv 1 \pmod 4$, then*

$$\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-3)/4}\left(4y-\frac{p}{3y}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\frac{k\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p+1)/4}y \pmod{p^2}.$$

If $p\equiv 11 \pmod{12}$, *then*

$$\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^2}{(-16)^k}\equiv 0 \pmod{p}.$$

If $p\equiv 1 \pmod{12}$, *then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^2}{16^k}\equiv 0 \pmod{p^2}.$$

Conjecture 5.2. (i) *Let* p *be a prime with* $p\equiv 1,3 \pmod 8$. *Write* $p=x^2+2y^2$ *with* $x,y\in\mathbb{Z}$ *and* $x\equiv 1,3 \pmod 8$. *Then*

$$\sum_{k=0}^{p-1}\frac{P_k}{(-8)^k}\binom{2k}{k}^2\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 8,\\ (-1)^{(p-3)/8}(p/(2x)-2x) \pmod{p^2} & \text{if } p\equiv 3 \pmod 8.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{kP_k}{(-8)^k}\binom{2k}{k}^2\equiv\frac{(-1)^{(x+1)/2}}{2}\left(x+\frac{p}{2x}\right) \pmod{p^2}.$$

(ii) *If* $p\equiv 5 \pmod 8$ *is a prime, then*

$$\sum_{k=0}^{p-1}\frac{P_k}{(-8)^k}\binom{2k}{k}^2\equiv 0 \pmod{p}.$$

If $p\equiv 7 \pmod 8$ *is a prime, then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{P_k}{8^k}\binom{2k}{k}^2\equiv 0 \pmod{p^2}.$$

Conjecture 5.3. *Let* p *be an odd prime.*

(i) *If* $p\equiv 3 \pmod 8$ *and* $p=x^2+2y^2$ *with* $y\equiv 1,3 \pmod p$, *then*

$$\sum_{k=0}^{p-1}\frac{P_k}{32^k}\binom{2k}{k}^2\equiv(-1)^{(y-1)/2}\left(2y-\frac{p}{4y}\right) \pmod{p^2}.$$

If $p\equiv 7 \pmod 8$, *then*

$$\sum_{k=0}^{p-1}\frac{P_k}{32^k}\binom{2k}{k}^2\equiv 0 \pmod p.$$

(ii) *Suppose that* $p\equiv 1,3 \pmod 8$, $p=x^2+2y^2$ *with* $x\equiv 1,3 \pmod 8$ *and also* $y\equiv 1,3 \pmod 8$ *when* $p\equiv 3 \pmod 8$. *Then*

$$\sum_{k=0}^{p-1}\frac{kP_k}{32^k}\binom{2k}{k}^2\equiv\begin{cases}(-1)^{(p-1)/8}(p/(4x)-x/2) \pmod{p^2} & \text{if } p\equiv 1 \pmod 8,\\ (-1)^{(y+1)/2}y \pmod{p^2} & \text{if } p\equiv 3 \pmod 8.\end{cases}$$

Conjecture 5.4. *Let* p *be an odd prime.*

(i) *When* $p\equiv 1,3 \pmod 8$ *and* $p=x^2+2y^2$ *with* $x,y\in\mathbb{Z}$ *and* $x\equiv 1,3 \pmod 8$, *we have*

$$\sum_{k=0}^{p-1}\frac{Q_k}{(-8)^k}\binom{2k}{k}^2\equiv(-1)^{(x-1)/2}\left(4x-\frac{p}{x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{kQ_k}{(-8)^k}\binom{2k}{k}^2\equiv\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 8,\\ (-1)^{(p-3)/8}2(x+p/x) \pmod{p^2} & \text{if } p\equiv 3 \pmod 8.\end{cases}$$

(ii) *When* $p\equiv 5,7 \pmod 8$, *we have*

$$\sum_{k=0}^{p-1}\frac{Q_k}{(-8)^k}\binom{2k}{k}^2\equiv 0 \pmod p.$$

Conjecture 5.5. *Let* p *be an odd prime.*

(i) *When* $p\equiv 1 \pmod 8$ *and* $p=x^2+2y^2$ *with* $x,y\in\mathbb{Z}$ *and* $x\equiv 1,3 \pmod 8$, *we have*

$$\sum_{k=0}^{p-1}\frac{Q_k}{32^k}\binom{2k}{k}^2\equiv(-1)^{(p-1)/8}\left(4x-\frac{p}{x}\right) \pmod{p^2}.$$

If $p\equiv 5 \pmod 8$, *then*

$$\sum_{k=0}^{p-1}\frac{Q_k}{32^k}\binom{2k}{k}^2\equiv 0 \pmod p.$$

(ii) *If* $p\equiv 1,3 \pmod 8$ *and* $p=x^2+2y^2$ *with* $x\equiv 1,3 \pmod 8$ *and also* $y\equiv 1,3 \pmod 8$ *when* $p\equiv 3 \pmod 8$, *then*

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{32^k}Q_k\equiv\begin{cases}(-1)^{(p-1)/8}(p/x-2x) \pmod{p^2} & \text{if } p\equiv 1 \pmod 8,\\ (-1)^{(y+1)/2}2y \pmod{p^2} & \text{if } p\equiv 3 \pmod 8.\end{cases}$$

Conjecture 5.6. *Let $p>3$ be a prime. If $p\equiv 7\pmod{12}$ and $p=x^2+3y^2$ with $y\equiv 1\pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{S_k}{4^k}\binom{2k}{k}^2\equiv(-1)^{(p+1)/4}\left(4y-\frac{p}{3y}\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{kS_k}{4^k}\binom{2k}{k}^2\equiv(-1)^{(p-3)/4}\left(6y-\frac{7p}{3y}\right)\pmod{p^2}.$$

We also have

$$\sum_{k=0}^{p-1}\frac{S_k}{4^k}\binom{2k}{k}^2\equiv\begin{cases}0\pmod{p^2} & \text{if } p\equiv 1\pmod{12},\\ 0\pmod{p} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Conjecture 5.7. *Let $p>3$ be a prime. If $p\equiv 7\pmod{12}$ and $p=x^2+3y^2$ with $y\equiv 1\pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{S_k}{64^k}\binom{2k}{k}^2\equiv 2y-\frac{p}{6y}\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{kS_k}{64^k}\binom{2k}{k}^2\equiv y\pmod{p^2}.$$

If $p\equiv 11\pmod{12}$, then

$$\sum_{k=0}^{p-1}\frac{S_k}{64^k}\binom{2k}{k}^2\equiv 0\pmod{p}.$$

Conjecture 5.8. *Let $p>3$ be a prime.*

(i) *If $p\equiv 1\pmod{12}$ and $p=x^2+3y^2$ with $x\equiv 1\pmod 3$, then*

$$\sum_{k=0}^{p-1}\frac{T_k}{4^k}\binom{2k}{k}^2\equiv(-1)^{(p-1)/4+(x-1)/2}\left(4x-\frac{p}{x}\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{T_k}{64^k}\binom{2k}{k}^2\equiv(-1)^{(x-1)/2}\left(4x-\frac{p}{x}\right)\pmod{p^2};$$

also

$$\sum_{k=0}^{p-1}\frac{kT_k}{4^k}\binom{2k}{k}^2\equiv(-1)^{(p-1)/4+(x+1)/2}\left(4x-\frac{2p}{x}\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{kT_k}{64^k}\binom{2k}{k}^2\equiv(-1)^{(x-1)/2}\left(2x-\frac{p}{x}\right)\pmod{p^2}.$$

(ii) *If* $p\equiv 7\pmod{12}$ *and* $p=x^2+3y^2$ *with* $y\equiv 1\pmod 4$, *then*

$$\sum_{k=0}^{p-1}\frac{T_k}{4^k}\binom{2k}{k}^2\equiv(-1)^{(p-3)/4}\left(12y-\frac{p}{y}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{kT_k}{4^k}\binom{2k}{k}^2\equiv(-1)^{(p+1)/4}\left(20y-\frac{8p}{y}\right)\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{kT_k}{64^k}\binom{2k}{k}^2\equiv 4y\pmod{p^2}.$$

(iii) *If* $p\equiv 5\pmod{12}$, *then*

$$\sum_{k=0}^{p-1}\frac{T_k}{4^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{p-1}\frac{T_k}{64^k}\binom{2k}{k}^2\equiv 0\pmod p.$$

If $p\equiv 11\pmod{12}$, *then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{T_k}{(-4)^k}\binom{2k}{k}^2\equiv 0\pmod{p^2}.$$

References

[BEW] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, John Wiley & Sons, 1998.

[C] D. A. Cox, *Primes of the Form x^2+ny^2*, John Wiley & Sons, 1989.

[M1] E. Mortenson, *A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function*, J. Number Theory **99** (2003), 139–147.

[M2] E. Mortenson, *Supercongruences between truncated ${}_2F_1$ by geometric functions and their Gaussian analogs*, Trans. Amer. Math. Soc. **355** (2003), 987–1007.

[M3] E. Mortenson, *Supercongruences for truncated ${}_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms*, Proc. Amer. Math. Soc. **133** (2005), 321–330.

[O] K. Ono, *Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, Amer. Math. Soc., Providence, R.I., 2003.

[PS] H. Pan and Z. W. Sun, *A combinatorial identity with application to Catalan numbers*, Discrete Math. **306** (2006), 1921–1940.

[RV] F. Rodriguez-Villegas, *Hypergeometric families of Calabi-Yau manifolds*, in: Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), pp. 223-231, Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003.

[St1] R. P. Stanley, *Enumerative Combinatorics*, Vol. 1, Cambridge Univ. Press, Cambridge, 1999.

[St2] R. P. Stanley, *Enumerative Combinatorics*, Vol. 2, Cambridge Univ. Press, Cambridge, 1999.

[S1] Z. H. Sun, *Values of Lucas sequences modulo primes*, Rocky Mount. J. Math. **33** (2003), 1123–1145.

[S2] Z. H. Sun, *Congruences concerning Legendre polynomials*, preprint, 2009.

[SS] Z. H. Sun and Z. W. Sun, *Fibonacci numbers and Fermat's last theorem*, Acta Arith. **60** (1992), 371–388.

[S02] Z. W. Sun, *On the sum* $\sum_{k\equiv r\,(\mathrm{mod}\ m)}\binom{n}{k}$ *and related congruences*, Israel J. Math. **128** (2002), 135–156.

[S09a] Z. W. Sun, *Various congruences involving binomial coefficients and higher-order Catalan numbers*, arXiv:0909.3808. `http://arxiv.org/abs/0909.3808`.

[S09b] Z. W. Sun, *Binomial coefficients, Catalan numbers and Lucas quotients*, preprint, arXiv:0909.5648. `http://arxiv.org/abs/0909.5648`.

[S09c] Z. W. Sun, *p-adic valuations of some sums of multinomial coefficients*, preprint, arXiv:0910.3892. `http://arxiv.org/abs/0910.3892`.

[S09d] Z. W. Sun, *On sums of binomial coefficients modulo* p^2, preprint, arXiv:0910.5667. `http://arxiv.org/abs/0910.5667`.

[S09e] Z. W. Sun, *Binomial coefficients, Catalan numbers and Lucas quotients (II)*, preprint, arXiv:0911.3060. `http://arxiv.org/abs/0911.3060`.

[S09f] Z. W. Sun, *On congruences related to central binomial coefficients*, preprint, arXiv:0911.2415. `http://arxiv.org/abs/0911.2415`.

[ST1] Z. W. Sun and R. Tauraso, *On some new congruences for binomial coefficients*, Acta Arith., to appear. `http://arxiv.org/abs/0709.1665`.

[ST2] Z. W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math., to appear. `http://arxiv.org/abs/0805.0563`.

[T] R. Tauraso, *An elementary proof of a Rodriguez-Villegas supercongruence*, preprint, arXiv:0911.4261. `http://arxiv.org/abs/0911.4261`.