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Super congruences involving binomial coefficients and new series for famous constants

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Abstract

A p-adic congruence is called a super congruence if it not only holds mod p but also happens to hold modulo a higher power of p. The topic of super congruences is related to the p-adic Gamma function, Gauss and Jacobi sums, hypergeometric series, modular forms, Calabi-Yau manifolds, representations of p by certain quadratic forms, and some sophisticated combinatorial identities involving harmonic numbers. Recently the speaker formulated many conjectures on super congruences and revealed that super congruences are related to Euler numbers and series with summation related to π and other constants. In this talk we will analyze few typical conjectures of the speaker and introduce related progress.

Part A. Previous Work by Others

Rational p-adic integers

Let p be a prime. For $m \in \mathbb{Z}$ define the p-adic valuation (or order)

$$\nu_p(m) = \sup\{a \in \mathbb{N} = \{0, 1, 2, \dots\} : \ p^a \mid m\}.$$

For $x = m/n$ with $m \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \dots\}$ define the p-adic valuation $\nu_p(x)$ of x by

$$\nu_p(x) = \nu_p(m) - \nu_p(n).$$

The p-adic norm is given by

$$|x|_p = \frac{1}{p^{\nu_p(x)}}.$$

Those $x \in \mathbb{Q}$ with $|x|_p \leqslant 1$ (i.e., $\nu_p(x) \geqslant 0$) are called *rational p-adic integers* or *p-integers*, they form a ring.

An Example for Congruences involving p-Integers:

$$1 + \frac{1}{2} \equiv 1 - 4 = -3 \pmod{3^2}.$$

Classical congruences for central binomial coefficients

A central binomial coefficient has the form

$$\binom{2k}{k} \ (k = 0, 1, 2, \ldots).$$

If $p = 2n + 1$ is an odd prime, then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p} \ \text{ for every } k = \frac{p+1}{2}, \ldots, p-1.$$

Wolstenholme's Congruence. For any prime $p > 3$ we have

$$H_{p-1} = \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}$$

and

$$\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}.$$

Remark. In 1900 Glaiser proved that for any prime $p > 3$ we have

$$\binom{2p-1}{p-1} \equiv 1 - \frac{2}{3}p^3 B_{p-3} \pmod{p^4}.$$

Classical congruences for central binomial coefficients

Morley's Congruence. For any prime $p > 3$ we have

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3}.$$

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

Beukers' Conjecture for Apéry Numbers

In 1978 Apéry proved that $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is irrational! During his proof he used the sequence $\{B(n)/A(n)\}_{n=1}^{\infty}$ of rational numbers to approximate $\zeta(3)$, where

$$A(0) = 1,\ A(1) = 5,\ B(0) = 0,\ B(1) = 6,$$

and both $\{A(n)\}_{n\geqslant 0}$ and $\{B(n)\}_{n\geqslant 0}$ satisfy the recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}\ (n = 1, 2, \ldots).$$

In fact,

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

and these numbers are called *Apéry numbers*.

Dedekind eta function in the theory of modular forms:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau}$$

Note that $|q| < 1$ if $\tau \in \mathbb{H} = \{z \in \mathbb{C} :\ \mathrm{Im}(z) > 0\}$.

Beukers' Conjecture (1985). For any prime $p > 3$ we have

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2},$$

where $a(n)$ $(n = 1, 2, 3, \ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = \sum_{n=1}^{\infty} a(n)q^n.$$

A Simple Observation. Let $p = 2n + 1$ be an odd prime. Then

$$\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{n}{k}\binom{-n-1}{k}$$

$$=\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k} \equiv \binom{-1/2}{k}^2$$

$$=\left(\binom{2k}{k}/(-4)^k\right)^2 = \binom{2k}{k}^2/16^k \pmod{p^2}.$$

Thus Beukers' conjecture has the following equivalent form:

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^4}{256^k} \equiv a(p) \pmod{p^2}.$$

Ahlgren and Ono's Proof of the Beukers conjecture

Key steps in S. Ahlgren and Ken Ono's proof [2000].

(i) For an odd prime p let $N(p)$ denote the number of $\mathbb{F}_p$-points of the following Calabi-Yau threefold

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0.$$

Then

$$a(p) = p^3 - 2p^2 - 7 - N(p).$$

(ii) For any positive integer n we have

$$\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k) = 0,$$

where $H_k = \sum_{0<j\leqslant k} 1/j$.

T. Kilbourn [Acta Arith. 123(2006)]: For any odd prime p we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^4}{256^k} \equiv a(p) \pmod{p^3}.$$

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$ and $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$.

Legendre symbols

Let p be an odd prime and $a \in \mathbb{Z}$. The Legendre symbol $\left(\frac{a}{p}\right)$ is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for some } x \in \mathbb{Z}, \\ -1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for no } x \in \mathbb{Z}. \end{cases}$$

It is well known that $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ for any $a, b \in \mathbb{Z}$. Also,

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv -1 \pmod{4}; \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

The Law of Quadratic Reciprocity: If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$

Conjectures of Rodriguez-Villegas

In 2001 Rodriguez-Villegas conjectured 22 congruences which relate truncated hypergeometric series to the number of $\mathbb{F}_p$-points of some family of Calabi-Yau manifolds. Here we list some of them.

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) = \left(\frac{-3}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6 \pmod{p^2}.$$

Where the denominators and $\left(\frac{\cdot}{p}\right)$ come from?

By Stirling's formula,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to +\infty$$

It follows that

$$\begin{aligned}
\binom{2k}{k}^2 &\sim \frac{16^k}{k\pi},\\
\binom{2k}{k}^3 &\sim \frac{64^k}{(k\pi)^{3/2}},\\
\binom{2k}{k}\binom{3k}{k} &\sim \frac{\sqrt{3}\ 27^k}{2k\pi},\\
\binom{2k}{k}\binom{4k}{2k} &\sim \frac{64^k}{\sqrt{2}k\pi},\\
\binom{3k}{k}\binom{6k}{3k} &\sim \frac{432^k}{2k\pi}.
\end{aligned}$$

Progress on Rodriguez-Villegas conjectures

The congruences we list have been confirmed, see,

E. Motenson, J. Number Theory 99(2003); Trans. AMS 355(2003); Proc. AMS 133(2005).

Many of the 22 conjectures remain open.

Ramanujan's series for $1/\pi$

Here are 5 of the 17 Ramanujan series recorded by him in 1914:

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{6k+1}{4^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}=\frac{4}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{6k+1}{(-8)^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{42k+5}{64^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{20k+3}{(-4)^k}\cdot\frac{(1/2)_k(1/4)_k(3/4)_k}{(1)_k^3}=\sum_{k=0}^{\infty}(20k+3)\frac{\binom{4k}{k,k,k,k}}{(-1024)^k}=\frac{8}{\pi}.$$

Remark. The first one was actually proved by G. Bauer in 1859.

Hamme's Conjectures

L. Van Hamme [1997] conjectured the p-adic analogues of the above first 4 identities and W. Zudilin [JNT, 2009] obtained the p-adic analogue of the last identity.

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{-1}{p}\right) p \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{256^k} \equiv \left(\frac{-1}{p}\right) p \pmod{p^4},$$

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k} \equiv \left(\frac{-2}{p}\right) p \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv \left(\frac{-1}{p}\right) 5p \pmod{p^4},$$

$$\sum_{k=0}^{p-1}(20k+3)\frac{\binom{4k}{k,k,k,k}}{(-1024)^k} \equiv \left(\frac{-1}{p}\right) 3p \pmod{p^3}.$$

Progress on Hamme's conjectures

The first of the above congruence was proved by E. Mortenson [Proc. AMS 136(2008)] and the second one was recently shown by Ling Long, while the last was confirmed by Zudilin via the WZ method. The third and the fourth remain open.

The p-adic Gamma function plays an important role in Hamme's formulation of those conjectures. It is defined in the following way:

$$\Gamma_p(n) := (-1)^n \prod_{\substack{1<k<n \\ p\nmid k}} k \quad (n = 1, 2, 3, \ldots)$$

and

$$\Gamma_p(x) = \lim_{n \to x} \Gamma_p(n) \ \text{ for any } p\text{−adic integer } x.$$

Some Series for π

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty} \frac{545140134k + 13591409}{(-640320)^{3k}} \binom{6k}{3k} \binom{3k}{k, k, k} = \frac{3 \times 53360^2}{2\pi\sqrt{10005}}.$$

This yielded the record for the calculation of π during 1989-1994.

D. Zeilberger (1993):

$$\sum_{k=1}^{\infty} \frac{21k - 8}{k^3 \binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}.$$

T. Amdeberhan and D. Zeilberger (1997):

$$\sum_{k=1}^{\infty} \frac{(-1)^k (205k^2 - 160k + 32)}{k^5 \binom{2k}{k}^5} = -2\zeta(3).$$

A Conjecture of J. Guillera (2003):

$$\sum_{k=1}^{\infty} \frac{(21k^3 - 22k^2 + 8k - 1)256^k}{k^7 \binom{2k}{k}^7} = \frac{\pi^4}{8}.$$

Part B. My Results and Conjectures

Some Joint Work

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod{p}\quad(d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod{p}\quad\text{for } p>3.$$

Sun & R. Tauraso [arXiv:0709.1665, Adv. in Appl. Math.].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\quad\text{for } p>3,$$

L. L. Zhao, H. Pan and Z. W. Sun [Proc. AMS, 2010]

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv 0\pmod{p}.$$

My own results

Recall that if $p/2 < k < p$ then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p}.$$

Thus

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \pmod{p},$$

where m is an integer with $p \nmid m$.

In 2009 I [arXiv:0909.5648, arXiv:0911.3060, 0909.3808] determined

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \bmod p^2, \quad \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \bmod p^2, \quad \sum_{k=0}^{p-1} \frac{\binom{3k}{k}}{m^k} \bmod p$$

in terms of linear recurrences.

Some particular congruences due to me

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{8^k}\equiv\frac{3}{4}\left(\left(\frac{p}{5}\right)-1\right)\pmod p,$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{7^k}\equiv\begin{cases}-2\pmod p & \text{if } p\equiv\pm2\pmod 7,\\ 1\pmod p & \text{otherwise.}\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k}}{5^k}\equiv\begin{cases}1\pmod p & \text{if } p\equiv 1\pmod 5\ \&\ p\neq 11,\\ -1/11\pmod p & \text{if } p\equiv 2,3\pmod 5,\\ -9/11\pmod p & \text{if } p\equiv 4\pmod 5.\end{cases}$$

If $p\equiv 1\pmod 3$ then

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{6^k}\equiv 2^{(p-1)/3}\pmod p.$$

Connection between super congruences and Euler numbers

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1,\ \sum_{2|k} \binom{n}{k} E_{n-k} = 0\ (n = 1, 2, 3, \ldots).$$

It is known that $E_1 = E_3 = E_5 = \cdots = 0$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}\ \left(|x| < \frac{\pi}{2}\right).$$

Z. W. Sun [arXiv:1001.4453].

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{8^k} \equiv \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right) \frac{p^2}{4} E_{p-3} \pmod{p^3}.$$

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Connection between super congruences and Euler numbers

Theorem (Sun, 2010). For any prime $p > 3$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2}\frac{8}{3}pE_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2\binom{2k}{k}} \equiv (-1)^{(p-1)/2}\frac{4}{3}E_{p-3} \pmod{p},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2E_{p-3} \pmod{p^3}.$$

Remark. Note that

$$\lim_{k\to+\infty} \frac{k\binom{2k}{k}^2}{16^k} = \frac{1}{\pi} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} = \frac{\pi^2}{18}.$$

Some auxiliary results needed for the proof

A Lemma (Sun, 2010). (i) If $p = 2n+1$ is an odd prime, then

$$\binom{n}{k}\binom{n+k}{k}(-1)^k\left(1-\frac{p}{4}(H_{n+k}-H_{n-k})\right) \equiv \frac{\binom{2k}{k}^2}{16^k} \pmod{p^4}.$$

(ii) We have

$$(-1)^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k(H_{n+k}-H_{n-k}) = \frac{3}{2}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}.$$

Some auxiliary identities:

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k} = \frac{n+1}{3}\binom{2n+1}{n}\sum_{k=1}^{n}\frac{1}{k^2\binom{n}{k}^2} \text{ (Staver)},$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^2\binom{n}{k}\binom{n+k}{k}} = (-1)^{n-1}\left(3\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}} + 2\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right) \text{ (Apéry)}$$

$$\sum_{k=1}^{n}\frac{1}{k^2\binom{n+k}{k}} = 3\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}} - \sum_{k=1}^{n}\frac{1}{k^2} \text{ (Sun)}.$$

Six conjectured series for π^2 and other constants

Conjecture (Z. W. Sun, 2010): We have

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},$$

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=-27\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

Conjecture involving $x^2 + 7y^2$

Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1 \ \&\ p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1. \end{cases}$$

Moreover,

$$\sum_{k=0}^{(p-1)/2} (21k + 8) \binom{2k}{k}^3 \equiv 8p + \left(\frac{-1}{p}\right) 32p^3 E_{p-3} \pmod{p^4}.$$

Remark. M. Jameson and K. Ono are working on the first part of this conjecture but they have not yet got a proof.

Conjecture involving x^2+11y^2

Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}$$
$$\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{11})=-1,\ \text{i.e., } p\equiv 2,6,7,8,10\pmod{11}.\end{cases}$$

Furthermore,

$$\sum_{k=0}^{p-1}(11k+3)\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv 3p+\frac{7}{2}p^4B_{p-3}\pmod{p^5},$$

$$p\sum_{k=1}^{(p-1)/2}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}\equiv 32\frac{2^{p-1}-1}{p}-\frac{64}{3}p^2B_{p-3}\pmod{p^3}.$$

Remark. It is well-known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $(\frac{p}{11})=1$ we can write $4p=x^2+11y^2$ with $x,y\in\mathbb{Z}$.

A conjecture motivated by some series for $\zeta(3)$ and $\zeta(4)$

Conjecture (Sun, 2010). Let $p > 7$ be a prime and let $H_{p-1} = \sum_{k=1}^{p-1} 1/k \equiv -p^2 B_{p-3}/3 \pmod{p^3}$. Then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv -\frac{2}{p^2} H_{p-1} \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - \frac{H_{p-1}}{p^3} \equiv -\frac{7}{45} p B_{p-5} \pmod{p^2}.$$

Also,

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3 \binom{2k}{k}} \equiv -2B_{p-3} \pmod{p}.$$

Motivation.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5}\zeta(3) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} = \frac{17}{36}\zeta(4).$$

A conjecture on divisibility of binomial coefficients

Recall that

$$C_n = \frac{1}{n+1}\binom{2n}{n} \in \mathbb{Z} \text{ for all } n \in \mathbb{N}.$$

The author observed that for any $k, l \in \mathbb{Z}^+$ we have

$$\frac{ln+1}{(k, ln+1)} \;\Big|\; \binom{kn+ln}{kn} \text{ for any } n \in \mathbb{N}.$$

In particular, if all prime factors of k divides l then $(ln+1) \mid \binom{kn+ln}{kn}$ for every $n = 0, 1, 2, \ldots$.

Conjecture (Sun, 2010). Let k and l be positive integers. If $(ln+1) \mid \binom{kn+ln}{kn}$ for all sufficiently large positive integers n, then each prime factor of k divides l. In other words, if k has a prime factor not dividing l then there are infinitely many positive integers n such that $(ln+1) \nmid \binom{kn+ln}{kn}$.

Some numerical examples

Let k and l be positive integers such that not all prime factors of k divides l. Define $f(k,l)$ as the smallest positive integer n such that $(ln+1) \nmid \binom{kn+ln}{kn}$. Via `Mathematica` we obtained the following data:

$$f(7,36)=279,\ f(10,192)=362,\ f(11,100)=1187,$$
$$f(13,144)=2001,\ f(22,200)=6462,\ f(31,171)=1765;$$
$$f(43,26)=640,\ f(53,32)=790,\ f(67,56)=2004,$$
$$f(73,61)=2184,\ f(74,62)=885,\ f(97,81)=2904,$$
$$f(179,199)=28989,\ f(223,93)=13368,\ f(307,189)=31915,$$
$$f(277,254)=36552,\ f(313,287)=41307.$$

On Apéry numbers

$$A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2.$$

Theorem (Sun, 2010). $\sum_{k=0}^{n-1}(2k+1)A_k \equiv 0 \pmod{n}$ for any $n \in \mathbb{Z}^+$. If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p \pmod{p^4}.$$

Conjecture (Sun, 2010). For any positive integer n we have

$$n \mid \sum_{k=0}^{n-1}(2k+1)(-1)^k A_k.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k \equiv p\left(\frac{p}{3}\right) \pmod{p^3}.$$

On Apéry numbers

Conjecture (Sun, 2010) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} A_k$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,3 \pmod 8 \text{ and } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5,7 \pmod 8; \end{cases}$$

and

$$\sum_{k=0}^{p-1} (-1)^k A_k$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \text{ and } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

On central Delannoy numbers

$$D_n := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}.$$

In combinatorics, D_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$.

Theorem (Sun, 2010). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} D_k \equiv \left(\frac{-1}{p}\right) - p^2 E_{p-3} \pmod{p^3}.$$

When $p > 3$ we also have

$$\sum_{k=0}^{p-1} (2k+1)(-1)^k D_k \equiv p \pmod{p^4},$$

$$\sum_{k=0}^{p-1} (2k+1) D_k \equiv p + 2p^2 q_p(2) - p^3 q_p(2)^2 \pmod{p^4},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

On central Delannoy numbers

Conjecture (Sun, 2010). (i) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(2k+1)D_k^2 \equiv 0 \pmod{n^2}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)D_k^2 \equiv p^2 - 4p^3 q_p(2) - 2p^4 q_p(2)^2 \pmod{p^5}.$$

(ii) Let p be any odd prime. Then

$$\sum_{k=1}^{p-1}\frac{D_k}{k^2} \equiv 2\left(\frac{-1}{p}\right)E_{p-3} \pmod{p} \text{ and } \sum_{k=0}^{p-1}D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p}.$$

Remark. I can show that $n \mid \sum_{k=0}^{n-1}(2k+1)(-1)^k D_k^2$ for $n \in \mathbb{Z}^+$.

More Conjectures on Congruences

For more conjectures of mine on congruences, see

Z. W. Sun, *Open Conjectures on Congruences*,

`arXiv:0911.5665`.

You are welcome to solve my conjectures!

Thank you!