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Number Theory behind Series for Powers of π

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Abstract and a Typical Example

One of the main contributions of Ramanujan is that he recorded 17 mysterious hypergeometric series for $1/\pi$ in 1914 which are related to elliptic integrals and modular forms and were not all proved until 1987. In this talk we will introduce the speaker's 160 conjectural series for powers of π discovered during 2010-2011 and explain number theory behind them. All the conjectural series came from **combinations of philosophy, intuition, inspiration, experience and computation!**

Abstract and a Typical Example

One of the main contributions of Ramanujan is that he recorded 17 mysterious hypergeometric series for $1/\pi$ in 1914 which are related to elliptic integrals and modular forms and were not all proved until 1987. In this talk we will introduce the speaker's 160 conjectural series for powers of π discovered during 2010-2011 and explain number theory behind them. All the conjectural series came from **combinations of philosophy, intuition, inspiration, experience and computation!**

A Typical Conjecture (Z. W. Sun, Jan. 2 (2:00 am), 2011).

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2 a_k=\frac{24}{\pi},$$

where a_k is the coefficient of x^k in $(x^2+x+16)^k$.

`Mathematica` Program:

```
T[n_]:=If[n>0,Coefficient[(x^2+x+16)^n,x^n],1]
S[n_]:=Sum[(30k+7)Binomial[2k,k]^2*T[k]/(-256)^k,{k,0,n}]
Print[N[S[200]Pi,20]]
```

Output: `24.000000000000000000`

Part A. My experience in 2010

related to series for π, π^2, π^3

Some classical π-series

Leibniz:

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Euler:

$$\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

π**-series involving the inverse sine function**:

$$\arcsin x = \sum_{k=0}^{\infty} \frac{\binom{2k}{k} x^{2k+1}}{(2k+1)4^k}, \qquad 2\arcsin^2 \frac{x}{2} = \sum_{k=1}^{\infty} \frac{x^{2k}}{k^2 \binom{2k}{k}}.$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)4^k} = \frac{\pi}{2}, \quad \sum_{k=1}^{\infty} \frac{4^k}{k^2 \binom{2k}{k}} = \frac{\pi^2}{2};$$

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} = \frac{\pi}{3}, \quad \sum_{k=1}^{\infty} \frac{1}{k^2 \binom{2k}{k}} = \frac{\pi^2}{18}.$$

Legendre symbols

Let p be an odd prime and $a \in \mathbb{Z}$. The Legendre symbol $(\frac{a}{p})$ is given by

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for some } x \in \mathbb{Z}, \\ -1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod{p} \text{ for no } x \in \mathbb{Z}. \end{cases}$$

It is well known that $(\frac{ab}{p}) = (\frac{a}{p})(\frac{b}{p})$ for any $a, b \in \mathbb{Z}$. Also,

$$\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv -1 \pmod{4}; \end{cases}$$

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = \begin{cases} 1 & \text{if } p \equiv \pm 1 \pmod{8}, \\ -1 & \text{if } p \equiv \pm 3 \pmod{8}. \end{cases}$$

The Law of Quadratic Reciprocity: If p and q are distinct odd primes, then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$

My joint work on congruences modulo prime powers

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d} \equiv \left(\frac{p-d}{3}\right) \pmod p \quad (d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k} \equiv 0 \pmod p \quad \text{for } p>3.$$

Sun & R. Tauraso [AAM 45(2010); IJNT 7(2011)].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k} \equiv \frac{8}{9}p^2B_{p-3} \pmod{p^3} \quad \text{for } p>3,$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers given by

$$B_0=1, \quad \sum_{k=0}^{n}\binom{n+1}{k}B_k=0 \quad (n=1,2,3,\ldots).$$

My result on $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod p^2

Sun [Sci. China Math. 53(2010)]: Let p be an odd prime and let $m \in \mathbb{Z}$ with $p \nmid m$. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m^2-4m}{p}\right) + u_{p-(\frac{m^2-4m}{p})} \pmod{p^2},$$

where $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0=0,\ u_1=1,\ \text{and}\ u_{n+1}=(m-2)u_n-u_{n-1}\ (n=1,2,3,\ldots).$$

In particular,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} \pmod{p^2}.$$

Remark. I only found two values of p such that the last congruence holds mod p^3: $p=149,\ 241$.

My unexpected discovery in Jan. 2010

Let p be an odd prime. I wanted to know $\sum_{k=1}^{(p-1)/2}\binom{2k}{k}/k$ mod p^2 and I found that $\sum_{k=1}^{(p-1)/2}\binom{2k}{k}/k \equiv 0 \pmod{p^3}$ for $p = 149, 241$.

A conjecture of Rodriguez-Villegas proved by Mortenson [JNT, 2003] states that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \pmod{p^2}.$$

I found that it holds mod p^3 for $p = 149, 241$.

A conjecture of van Hamme proved by Mortenson [PAMS, 2008] asserts that

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{-1}{p}\right)p \pmod{p^3}.$$

I found that it holds mod p^4 for $p = 149, 241$.

Connections to Euler numbers

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1, \ \sum_{2|k} \binom{n}{k} E_{n-k} = 0 \ (n = 1, 2, 3, \ldots).$$

It is known that $E_1 = E_3 = E_5 = \cdots = 0$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \ \left(|x| < \frac{\pi}{2}\right).$$

Z. W. Sun [Sci. China Math., 54(2011)]:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2} \ (p > 3),$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3},$$

Connections between series and congruences involving E_{p-3}

Series:

$$\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}=\frac{\pi^2}{18},\ \sum_{k=1}^{\infty}\frac{4^k}{k^2\binom{2k}{k}}=\frac{\pi^2}{2},\ \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

Corresponding congruences that I proved:

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)\frac{4}{3}E_{p-3}\pmod{p}\ (p>3),$$

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k^2\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)4E_{p-3}\pmod{p},$$

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\ (\mathrm{mod}\ p^4).$$

Connections between series and congruences

Known series involving $H_n = \sum_{k=1}^n 1/k$ **or** $H_n^{(2)} = \sum_{k=1}^n 1/k^2$:

$$\sum_{k=1}^\infty \frac{H_k}{k2^k} = \frac{\pi^2}{12},\ \sum_{k=1}^\infty \frac{H_k^{(2)}}{k2^k} = \frac{5}{8}\zeta(3),\ \sum_{k=1}^\infty \frac{H_k^2}{k^2} = \frac{17\pi^4}{360}.$$

Corresponding congruences for any prime $p > 5$:

$$\sum_{k=1}^{(p-1)/2} \frac{H_k}{k2^k} \equiv \sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}$$

[Z. W. Sun, Proc. AMS 140(2012), 415-428],

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \frac{7}{24} pB_{p-3} \pmod{p^2},\ \sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p}$$

[Conjectured by Sun and proved by Sun and Zhao (arXiv:0911.4433)],

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv \frac{4}{5} pB_{p-5} \pmod{p^2}$$

[Conjectured by Sun and proved by R. Meštrović (arXiv:1108.1171)].

The philosophy about regular series involving π or the ζ-function

In a message to `Number Theory List` on March 15, 2010, I expressed the following viewpoint:

Almost every series with summation related to $\pi = 3.14\ldots$ *or the Riemann zeta function corresponds to a congruence for Euler numbers or Bernoulli numbers. Conversely, many congruences for* E_{p-3} *or* B_{p-3} *modulo a prime* p *yield corresponding series related to* π *or the zeta function.*

An example illustrating my philosophy

Example. It is known that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k}=\frac{\pi}{3},\quad \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}=\frac{\pi^2}{10}.$$

I [JNT 131(2011)] proved that for any prime $p>3$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv 0 \pmod{p^2},$$

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv \frac{p}{3}E_{p-3} \pmod{p^2}.$$

And I conjectured that for any prime $p>5$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\equiv -\frac{p}{15}B_{p-3} \pmod{p^2},$$

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\equiv -\frac{p}{4}B_{p-3} \pmod{p^2}.$$

Find new series for π^3

There are very few interesting series for π^3. The only well-known series for π^3 is the following one:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=\frac{\pi^3}{32}.$$

I observed that for any prime $p>3$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}\equiv\frac{(-1)^{(p+1)/2}}{12}B_{p-3}\pmod{p}.$$

Motivated by this observation, I guessed that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}=\frac{7}{216}\pi^3.$$

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After I announced this conjecture, Olivier Gerard pointed out there is a computer proof via `Mathematica` (version 7).

Find new series for π^3

Let p be an odd prime. I proved that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv(-1)^{(p-1)/2}-p^2E_{p-3}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-2)^k}\equiv(-1)^{(p-1)/2}2^{p-1}\pmod{p^3}.$$

For $k=0,\ldots,p-1$, it is easy to see that

$$\binom{p-1}{k}(-1)^k\equiv 1+pH_k+\frac{p^2}{2}(H_k^2-H_k^{(2)})\pmod{p^3}.$$

So, it is natural to investigate $\sum_{k=0}^{p-1}\binom{2k}{k}H_k^{(2)}/2^k \bmod p$.
Theorem (Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}H_k^{(2)}\equiv -E_{p-3}\pmod{p}.$$

Note that $\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{2^k}H_k^{(2)}$ is divergent!

Find new series for π^3

A useful observation: Let p be an odd prime. Then, for any $k = 1, \ldots, p-1$ we have

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p\rfloor - 1} 2p \pmod{p^2}.$$

Note also that

$$H_{p-k}^{(2)} = H_{p-1}^{(2)} - \frac{1}{(p-k+1)^2} - \cdots - \frac{1}{(p-1)^2} \equiv -H_{k-1}^{(2)} \pmod{p}.$$

Thus via the transformation $k \to p-k$ we should investigate $\sum_k 2^k H_{k-1}^{(2)}/(k\binom{2k}{k})$ which cannot be evaluated via `Mathematica`.

Theorem (Sun, Sept. 2010). We have

$$\sum_{k=1}^{\infty} \frac{2^k H_{k-1}^{(2)}}{k\binom{2k}{k}} = \frac{\pi^3}{48}.$$

A sketch of the proof

Using the fact that

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{for any } a,b>0,$$

and the dilogarithm function $\mathrm{Li}_2(x)$ given by

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad (|x|<1),$$

I deduced that

$$\sum_{k=1}^{\infty} \frac{2^{k-1} H_{k-1}^{(2)}}{k\binom{2k}{k}} = \int_{-1}^{1} \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt$$
$$= \frac{\pi^3}{96} \quad \text{(by Mathematica 7).}$$

Remark. The indefinite integral

$$\int \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt$$

is **very very** complicated. It occurs more than two pages!

Six conjectured series for π^2 and other constants

Conjecture (Discovered in March-April 2010, see Z. W. Sun [Sci. China Math. 54(2011)]): We have

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},\ \sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2 \text{ (proved by Guillera in 2010)},$$

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=-27\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2}$$

(proved by K. Hessami Pilehrood and T. Hessami Pilehrood in 2011),

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

Part B. Series for $1/\pi$

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

Gaussian hypergeometric series

$$y = {}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right)$$

satisfies the differential equation:

$$\left(\theta \prod_{t=1}^{r}(\theta + \beta_t - 1) - x \prod_{s=0}^{r}(\theta + \alpha_s)\right) y = 0$$

where

$$\theta = x\frac{d}{dx}.$$

Clausen's Identity:

$$\begin{aligned} & {}_2F_1\left(2a, 2b; a + b + 1/2 \mid x\right)^2 \\ = & {}_3F_2\left(2a, 2b, a + b; a + b + 1/2, 2a + 2b \mid 4x(1 - x)\right). \end{aligned}$$

In the case $a = b = 1/4$, it gives the identity

$$\left(\sum_{k=0}^{\infty} \binom{2k}{k}^2 x^k\right)^2 = \sum_{k=0}^{\infty} \binom{2k}{k}^3 (x(1 - 16x))^k.$$

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England
Modular equations and approximations to π,
Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for* $1/\pi$". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

Elliptic integrals

Complete elliptic integrals (with the modulus $k \in (0,1)$):

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{(the first kind)},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \quad \text{(the seond kind)}.$$

Legendre's Relation: Let $k' = \sqrt{1-k^2}$. Then

$$E(k)K(k') + E(k')K(k) - K(k)K(k') = \frac{\pi}{2}.$$

A Central Result: Let $q = e^{-\pi K(k')/K(k)}$. Then

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \mid k^2\right) = \frac{2}{\pi} K(k) = \varphi^2(q)$$

where $\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2}$ (theta function).

Modular equation of degree n: A relation between k and $l \in (0,1)$ induced by $nK(k')/K(k) = K(l')/K(l)$, or an identity relating $\varphi(q)$ to $\varphi(q^n)$.

Series for $1/\pi$ given by Ramanujan

Two of the 17 series for $1/\pi$ recorded by Ramanujan:

$$\sum_{k=0}^{\infty} \frac{6k+1}{4^k} \cdot \frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{26390k+1103}{396^{4k}} \binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

In 1985 Jr. R. W. Gosper used the last series of Ramanujan to calculate $17,526,100$ digits of π (a world record at that time).

In 1987 J. Borwein and P. Borwein succeeded in proving all the 17 Ramanujan series. (S. Chowla proved the first one in 1928.)

In 1997 van Hamme investigated p-adic analogues of some hypergeometric series. For example, he conjectured that

$$\sum_{k=0}^{p-1} (6k+1) \frac{\binom{2k}{k}^3}{(-512)^k} \equiv p \left(\frac{-2}{p}\right) \pmod{p^3}.$$

while the infinite sum is known to equal $2\sqrt{2}/\pi$.

Ramanujan-type series for $1/\pi$

General forms:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are totally 36 known Ramanujan-type series for $1/\pi$ with a, b, m rational.

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Remark. This yielded the record for the calculation of π during 1989-1994.

Other known series for $1/\pi$

T. Sato (2002, announced):

$$\sum_{k=0}^{\infty}(20n+10-3\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{12n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\frac{20\sqrt{3}+9\sqrt{15}}{6\pi}.$$

Yifan Yang (2005, unpublished):

$$\sum_{n=0}^{\infty}\frac{4n+1}{36^n}\sum_{k=0}^{n}\binom{n}{k}^4=\frac{18}{\sqrt{15}\,\pi}.$$

H. H. Chan, S. H. Chan and Z. G. Liu (2004, Adv. in Math.)

$$\sum_{n=0}^{\infty}\frac{5n+1}{64^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}=\frac{8}{\sqrt{3}\,\pi}.$$

H. H. Chan and H. Verill (2009, Math. Res. Lett.), M. D. Rogers (2009, Ramanujan J.)

$$\sum_{n=0}^{\infty}\frac{3n+1}{(-32)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2n-2k}{n-k}=\frac{2}{\pi}.$$

Connections to modular forms

Many known series for $1/\pi$ have the form

$$\sum_{k=0}^{\infty} \frac{bk+c}{m^k} \binom{2k}{k} u_k = \frac{C}{\pi},$$

where $u_{-1} = 0,\ u_0 = 1$ and

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B)u_k + Ck^2 u_{k-1} \ (k = 1, 2, 3, \ldots),$$

and there are modular functions (i.e., meromorphic modular forms of weight 0) $x(\tau)$ and $\tilde{x}(\tau)$ such that

$$F(\tau) = \sum_{k=0}^{\infty} u_k (x(\tau))^k \quad \text{and} \quad \tilde{F}(\tau) = \sum_{k=0}^{\infty} \binom{2k}{k} u_k (\tilde{x}(\tau))^k$$

are modular forms of weights 1 and 2 respectively.

Zagier's contribution

Don Zagier (2009) investigated what integer sequence $\{u_n\}$ satisfies $u_{-1} = 0$, $u_0 = 1$, and the Apéry-like recurrence relation

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B)u_k + Ck^2 u_{k-1} \ (k = 1, 2, 3, \ldots).$$

For example, he noted that if $(A, B, C) = (7, 2, 8)$, then $u_n = \sum_{k=0}^{n} \binom{n}{k}^3$ and

$$\frac{\eta_2 \eta_3^6}{\eta_1^2 \eta_6^3} = \sum_{n=0}^{\infty} u_n \left(\frac{\eta_1^3 \eta_6^9}{\eta_2^3 \eta_3^9} \right)^n,$$

where

$$\eta_m := q^{m/24} \prod_{n=1}^{\infty} (1 - q^{mn}) = \eta(m\tau)$$

with $q = e^{2\pi i \tau}$ and $\mathrm{Im}(\tau) > 0$.

Connections to differential equations

Let $u_{-1} = 0$, $u_0 = 1$ and

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B)u_k + Ck^2 u_{k-1} \ (k = 1, 2, 3, \ldots).$$

Then $y = \sum_{k=0}^{\infty} \binom{2k}{k} u_k x^k$ satisfies the third-order differential equation

$$x^2(1 - 4Ax - 16Cx^2)y''' + 3x(1 - 6Ax - 32Cx^2)y''$$
$$+ (1 - (12A + 4B)x - 108Cx^2)y' - 2(b + 6Cx)y = 0.$$

For $f(x) = \sum_{k=0}^{\infty} u_k x^k$ and $\tilde{f}(x) = \sum_{k=0}^{\infty} \binom{2k}{k} u_k x^k$, H. H. Chan, Y. Tanigawa, Y. Yang and W. Zudilin [Adv. in Math. 228(2011), 1294-1314] gave a Clausen-type relation

$$(1 + Cx^2)f(x)^2 = \tilde{f}\left(\frac{x(1 - Ax - Cx^2)}{(1 + Cx^2)^2}\right)$$

and use it to derive some series for $1/\pi$ such as

$$\sum_{n=0}^{\infty} \frac{9n+2}{50^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^3 = \frac{25}{2\pi}.$$

Some Comments

S. Ramanujan attributed his mathematical discoveries to inspirations from the God. He once said: "**An equation for me has no meaning, unless it represents a thought of God**."

At the end of the article *Ramanujan's series for* $1/\pi$*: a survey* [Amer. Math. Monthly 116(2009)] by N. D. Baruah, B. C. Berndt and H. H. Chan, the authors wrote the following comments:

One test of "good" mathematics is that it should generate more "good" mathematics. Readers have undoubtedly concluded that Ramanujan's original series for $1/\pi$ *have shown the seeds for an abundant crop of "good" mathematics.*

In the abstract of the paper *"New analogues of Clausen's identities arising from the theory of modular forms"* [Adv. in Math., 2011)], by Chan, Tanigawa, Yang and Zudilin, the authors wrote:
Since Ramanujan's work in 1914, there were several attempts to find new analogues of Clausen's identities with the hope to derive new classes of series for $1/\pi$**. Unfortunately, none were successful.**

Generalized central trinomial coefficients

For real numbers b and c, we define

$$T_n(b,c) := [x^n](x^2+bx+c)^n$$

$$\text{(the coefficient of } x^n \text{ in } (x^2+bx+c)^n)$$

$$= \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k.$$

Recursion: Set $d = b^2 - 4c$. Then $T_0(b,c) = 1$, $T_1(b,c) = b$,

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c)\ (n>0).$$

Note that $T_n(2,1) = \binom{2n}{n}$. It is known that if $d \neq 0$ then

$$T_n(b,c) = \sqrt{d}^n P_n\left(\frac{b}{\sqrt{d}}\right)$$

where

$$P_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k$$

is the Legendre polynomial of degree n.

Asymptotic behavior of $T_n(b, c)$

By the Laplace-Heine formula, for $x \notin [-1, 1]$ we have

$$P_n(x) \sim \frac{(x + \sqrt{x^2 - 1})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{x^2 - 1}} \quad \text{as } n \to +\infty.$$

It follows that if $b > 0$ and $c > 0$ then

$$T_n(b, c) \sim f_n(b, c) := \frac{(b + 2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}.$$

as $n \to +\infty$. Note that $T_n(-b, c) = (-1)^n T_n(b, c)$.

I conjectured that if $c < 0$ and $b \in \mathbb{R}$ then

$$\lim_{n\to\infty} \sqrt[n]{|T_n(b, c)|} = \sqrt{b^2 - 4c}.$$

This was recently proved by S. Wagner.

My conjectural series involving $T_k(b,c)$ for $1/\pi$

In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a, b, c, d, m integers and $mbcd(b^2-4c)$ nonzero.

Type I. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

Type II. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

Type III. $\sum_{k=0}^{\infty}(a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

Type IV. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

Type V. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

In August I added 8 new series for $1/\pi$ of type III.

In October I found 10 conjectural series for $1/\pi$ of two new types:

Type VI. $\sum_{k=0}^{\infty}(a+dk)T_k^3(b,c)/m^k$.

Type VII. $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k} T_k^2(b,c)/m^k$.

Recall that a series $\sum_{k=0}^{\infty} a_k$ is said to *converge at a geometric rate with ratio* r if

$$\lim_{k\to+\infty}\frac{a_{k+1}}{a_k}=r\in(0,1).$$

My conjectural series of type I

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2 T_k(1,16)=\frac{24}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-1024)^k}\binom{2k}{k}^2 T_k(34,1)=\frac{12}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{30k-1}{4096^k}\binom{2k}{k}^2 T_k(194,1)=\frac{80}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{42k+5}{4096^k}\binom{2k}{k}^2 T_k(62,1)=\frac{16\sqrt{3}}{\pi}.$$

Remark. The first identity was found by me soon after I waked up in the early morning (about 2:00 am) of Jan. 2, 2011. This began my discovery of many new series for $1/\pi$. Note that $T_k(1,16) \sim \frac{3}{4}\cdot\frac{9^k}{\sqrt{k\pi}}$ while $\binom{2k}{k} \sim \frac{4^k}{\sqrt{k\pi}}$.

My conjectural series of type II

I have 12 conjectural series of type II. Here are five of them.

$$\sum_{k=0}^{\infty} \frac{15k+2}{972^k} \binom{2k}{k} \binom{3k}{k} T_k(18,6) = \frac{45\sqrt{3}}{4\pi},$$

$$\sum_{k=0}^{\infty} \frac{91k+12}{10^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(10,1) = \frac{75\sqrt{3}}{2\pi},$$

$$\sum_{k=0}^{\infty} \frac{6930k+559}{102^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(102,1) = \frac{1445\sqrt{6}}{2\pi},$$

$$\sum_{k=0}^{\infty} \frac{210k-7157}{198^{3k}} \binom{2k}{k} \binom{3k}{k} T_k(287298,1) = \frac{114345\sqrt{3}}{\pi},$$

$$\sum_{k=0}^{\infty} \frac{63k+11}{(-13500)^k} \binom{2k}{k} \binom{3k}{k} T_k(40,1458) = \frac{25}{12\pi}(3\sqrt{3}+4\sqrt{6}).$$

Remark. The 4th series converges very slow (with geometric ratio 71825/71874), even 2000 terms could not contribute one digit. Prof. G. Almkvist wondered how I could find the identity.

Some of my conjectural series of type III

$$\sum_{k=0}^{\infty}\frac{85k+2}{66^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(52,1)=\frac{33\sqrt{33}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{28k+5}{(-96^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(110,1)=\frac{3\sqrt{6}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{3080k-58871}{39216^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(23990402,1)=\frac{17974\sqrt{2451}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{80k+9}{264^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(257,256)=\frac{11\sqrt{66}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi}.$$

Remark. Some mathematicians (including my twin brother Z. H. Sun) wondered how I could find the last identity involving $14\sqrt{210}+21\sqrt{42}$.

My conjectural series of type IV

I have 18 conjectural series of type IV. Here are five of them.

$$\sum_{k=0}^{\infty}\frac{340k+59}{(-480^2)^k}\binom{2k}{k}^2T_{2k}(62,1)=\frac{120}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{13940k+1559}{(-5760^2)^k}\binom{2k}{k}^2T_{2k}(322,1)=\frac{4320}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{14280k+899}{39200^{2k}}\binom{2k}{k}^2T_{2k}(198,1)=\frac{1155\sqrt{6}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{57720k+3967}{439280^{2k}}\binom{2k}{k}^2T_{2k}(5778,1)=\frac{2890\sqrt{19}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{1615k-314}{243360^{2k}}\binom{2k}{k}^2T_{2k}(54758,1)=\frac{1989\sqrt{95}}{4\pi}.$$

Remark. I conjectured that my list of the 18 series of type IV with $c=1$ is complete! Prof. G. Almkvist asked me why I thought so.

My conjectural series of type V

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}.$$

$u_k=\binom{3k}{k}T_{3k}(62,1)$ satisfies a very complicated recursion:

$$\begin{aligned}&(n+2)^2(2n+1)(2n+3)(8652n^2+11536n+3525)u_{n+2}\\ =&372(2n+1)(6n+7)(25021584n^4+116767392n^3\\ &+188134216n^2+121113048n+25958565)u_{n+1}\\ &-127401984000(3n+1)^2(3n+2)^2(8652n^2+28840n+23713)u_n\\ &-9(n+2)P(n)62^n\binom{2n+2}{n}\binom{3n+2}{n}\binom{3n+2}{2n},\end{aligned}$$

where

$$\begin{aligned}P(n):=&31420906020n^5+136307337012n^4+127456779135n^3\\ &-126369328953n^2-174985958380n+705000.\end{aligned}$$

Latest Progress

Quite recently, some important progress on my conjectural series for $1/\pi$ of types I-V was made by H. H. Chan, J. Wan and W. Zudilin, see the preprints

1. H. H. Chan, J. Wan and W. Zudilin, *Legendre polynomials and Ramanujan-type series for* $1/\pi$, Israel J. Math., to appear.

2. J. Wan and W. Zudilin, *Generating functions of Legendre polynomials: a tribute to Fred Brafman*, preprint, June 2011.

Their work depends heavily on Brafman's identity [Proc. AMS 2(1951), 942-949] and its extensions.

Brafman's Identity. We have

$$\sum_{n=0}^{\infty}\frac{(s)_n(1-s)_n}{n!^2}P_n(x)z^n$$

$$={}_2F_1\left(s,1-s;1\ \Big|\ \frac{1-\rho-z}{2}\right){}_2F_1\left(s,1-s;1\ \Big|\ \frac{1-\rho+z}{2}\right),$$

where $\rho:=\sqrt{1-2xz+z^2}$.

Comments

The above two preprints provide theoretical interpretations to my conjectural series of types I-V, with all the details for proofs of 7 series conjectured by me. In my opinion, the crucial parts in such proofs are related to modular equations. Some of my conjectural series involve modular equations of higher degrees (e.g., 17, 19).

Bruce Berndt wrote (in his book *Number Theory in the Spirit of Ramanujan*): **There is no single method one can use to discover or construct modular equations. One needs to be resourceful and use a variety of tools. Generally, as the degree of the modular equation increases, the difficulty of establishing modular equations rises sharply.**

In October 2011 I introduced conjectural series of types VI and VII which could not be proved by using Brafman's identity.

Besides the 58 series involving $T_k(b, c)$, I have totally 155 conjectural series for $1/\pi$. For the full list, see my article *List of conjectural series for powers of π and other constants* `http://arxiv.org/abs/1102.5649`

My conjectural series of type VI

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

Remark. I would like to offer $300 as the prize for the person who can provide first rigorous proofs of all the above three identities.

My conjectural series of type VII

I have 7 conjectural series of type VII, here are five of them.

$$\sum_{k=0}^{\infty}\frac{221k+28}{450^k}\binom{2k}{k}T_k^2(6,2)=\frac{2700}{7\pi},$$

$$\sum_{k=0}^{\infty}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k^2(4,9)=\frac{49}{9\pi}(\sqrt{3}+\sqrt{6}),$$

$$\sum_{k=0}^{\infty}\frac{3696k+445}{46^{2k}}\binom{2k}{k}T_k^2(7,1)=\frac{1587\sqrt{7}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{450296k+53323}{(-5177196)^k}\binom{2k}{k}T_k^2(171,-171)=\frac{113535\sqrt{7}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k^2(73,576)=\frac{10406669}{2\sqrt{6}\,\pi}.$$

More examples of my conjectural series

$$\sum_{n=0}^{\infty}\frac{1054n+233}{480^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}(-1)^k 8^{2k-n}=\frac{520}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{20n-67}{(-3136)^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}(-192)^{n-k}=\frac{490}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{8851815n+1356374}{(-29584)^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}175^{2k-n}=\frac{1349770\sqrt{7}}{\pi},$$

$$\sum_{n=0}^{\infty}\frac{944607040n+86734691}{33385284^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}76^{2k}$$

$$=1071111195\frac{\sqrt{95}}{38\pi}.$$

Remark. I would like to offer \$520 for the first proof of the first identity given in 2012 since May 20 is the day for Nanjing University.

Part C. On x^2 mod p^2 with $4p = x^2 + dy^2$

Gauss' congruence

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

Determine x in $4p = x^2 + dy^2$ with $d = 7, 11, 19, 43, 67, 163$

It is well known that the only imaginary quadratic fields with class number one are those $\mathbb{Q}(\sqrt{-d})$ with

$$d = 1, 2, 3, 7, 11, 19, 43, 67, 163.$$

In 1977, A. R. Rajwade proved that for any odd prime p we have

$$\sum_{x=0}^{p-1} \left(\frac{x^3 + 21x^2 + 112x}{p}\right) = \begin{cases} -2x(\frac{x}{7}) & \text{if } (\frac{p}{7}) = 1 \ \& \ p = x^2 + 7y^2 \ (x, y \in \mathbb{Z}), \\ 0 & \text{if } (\frac{p}{7}) = -1, \text{ i.e., } p \equiv 3, 5, 6 \pmod 7). \end{cases}$$

There are similar results for $d = 11, 19, 43, 67, 163$ via elliptic curves with complex multiplication.

My problems for x^2 mod p^2 with $4p = x^2 + dy^2$

Problem 1. Given a squarefree positive integer d, find *integers* $a_0, a_1, a_2, \ldots$ such that for sufficiently large primes p we have

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1. \end{cases}$$

If one thinks that the integral condition of $a_0, a_1, a_2, \ldots$ in Problem 1 is too harsh, we may study the following easier problem.

Problem 2. Given a squarefree positive integer d, find *rational numbers* $a_0, a_1, a_2, \ldots$ with denominators not divisible by large primes such that for large primes p we have

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1. \end{cases}$$

We find that Problems 1 and 2 have affirmative answers for most of those $d \in \mathbb{Z}^+$ with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ having class number 1 or 2 or 4.

Apéry numbers

In his proof of the irrationality of $\zeta(3)$, Apéry introduced

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \ (n = 0, 1, 2, \ldots).$$

Conjecture (Z. W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}; \end{cases}$$

also,

$$\sum_{k=0}^{p-1} (-1)^k A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}. \end{cases}$$

Remark. In 2011 I proved the mod p version of both congruences and that

$$\sum_{k=0}^{p-1} (-1)^k A_k \equiv 0 \pmod{p^2} \ \text{for any prime } p \equiv 2 \pmod{3}.$$

Solution to Problem 1 for $d = 1$

Define Apéry polynomials by

$$A_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \ (n = 0, 1, 2, \ldots).$$

Theorem 1 (Z. W. Sun, 2011) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(-2) \equiv \sum_{k=0}^{p-1} (-1)^k A_k\left(\frac{1}{4}\right)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

Remark. A lemma states that for any odd prime p we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-8)^k} \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4}. \end{cases}$$

This was first conjectured by the author in 2009 and later confirmed by his twin brother Z.-H. Sun in 2010.

Apéry polynomials

Theorem (Z. W. Sun, 2011). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{16^k} x^k \pmod{p^2}.$$

Also, for any p-adic integer $x \not\equiv 0 \pmod{p}$ we have

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}.$$

A Key Lemma (Z. W. Sun, 2011). If x is a p-adic integer with $x \equiv 2k \pmod{p}$ where $k \in \{0, \ldots, (p-1)/2\}$, then we have

$$\sum_{r=0}^{p-1}(-1)^r \binom{x}{r}^2 \equiv (-1)^k \binom{x}{k} \pmod{p^2}.$$

Problem 1 for $d = 7$

Conjecture [Z. W. Sun, JNT 131(2011)]. Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod 7.\end{cases}$$

Remark. (1) Recently Z. H. Sun confirmed the second congruence in the case $(\frac{p}{7})=-1$ via Legendre polynomials.
(2) It is known that for any odd prime p we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Problem 1 for $d = 15$

For $k = 0, 1, 2, \ldots$ let $T_k = T_k(1,1) = [x^k](x^2 + x + 1)^k$.
Conjecture (Z. W. Sun, 2011). For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2 T_k$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \ \&\ p = x^2 + 15y^2, \\ 2p - 12x^2 \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \ \&\ p = 3x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1. \end{cases}$$

and

$$\sum_{k=0}^{p-1}(105k+44)(-1)^k\binom{2k}{k}^2 T_k \equiv p\left(20 + 24\left(\frac{p}{3}\right)(2 - 3^{p-1})\right) \pmod{p^3}.$$

Also,

$$\frac{1}{2n\binom{2n}{n}}\sum_{k=0}^{n-1}(-1)^{n-1-k}(105k+44)\binom{2k}{k}^2 T_k \in \mathbb{Z}^+ \quad \text{for all } n = 1, 2, \ldots.$$

Problem 1 for $d = 5$

Define polynomials

$$S_n(x) := \sum_{k=0}^{n} \binom{n}{k}^4 x^k \quad (n = 0, 1, 2, \ldots).$$

Conjecture (Sun) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(-4) \equiv \sum_{k=0}^{p-1} S_k(-64)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \&\ p = x^2 + 5y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \ \&\ 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-5}{p}) = -1. \end{cases}$$

$$\sum_{k=0}^{p-1} (8k+7) S_k(-64) \equiv p\left(\frac{p}{15}\right)\left(3 + 4\left(\frac{-1}{p}\right)\right) \pmod{p^2}.$$

$$\frac{1}{n}\sum_{k=0}^{n-1} (8k+7) S_k(-64) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Problem 1 for $d = 30$

Conjecture (Sun) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(36)$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = (\frac{p}{3}) = (\frac{p}{5}) = 1,\ p = x^2 + 30y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{3}) = 1,\ (\frac{2}{p}) = (\frac{p}{5}) = -1,\ p = 3x^2 + 10y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } (\frac{2}{p}) = 1,\ (\frac{p}{3}) = (\frac{p}{5}) = -1,\ p = 2x^2 + 15y^2, \\ 2p - 20x^2 \pmod{p^2} & \text{if } (\frac{p}{5}) = 1,\ (\frac{2}{p}) = (\frac{p}{3}) = -1,\ p = 5x^2 + 6y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-30}{p}) = -1. \end{cases}$$

$$\sum_{k=0}^{p-1} (8k+7) S_k(36) \equiv p \left(\frac{p}{15}\right) \left(3 + 4\left(\frac{-6}{p}\right)\right) \pmod{p^2}.$$

$$\frac{1}{n} \sum_{k=0}^{n-1} (8k+7) S_k(36) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Prize: \$300 for a complete proof.

Problem 2 for $d = 11, 35$

Conjecture [Sun, JNT 131(2011)]. Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}$$

$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{11})=-1,\ \text{i.e.,}\ p\equiv 2,6,7,8,10 \pmod{11}.\end{cases}$$

$$\frac{1}{p}\sum_{k=0}^{p-1}\frac{11k+3}{64^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 3+\frac{7}{2}p^3B_{p-3} \pmod{p^4}.$$

Conjecture (Sun, 2011). Let $p\neq 2,5,7$ be a prime. Then

$$\sum_{n=0}^{p-1}\frac{1}{(-8)^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{3k}{n}\binom{3(n-k)}{n}$$

$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=1\ \&\ 4p=x^2+35y^2,\\ 2p-5x^2 \pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ 4p=5x^2+7y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{35})=-1.\end{cases}$$

Philosophy about Series for $1/\pi$

I formulated the following viewpoint the initial version of which appeared in my message to Number Theory Mailing List sent on March 30, 2010.

Philosophy about Series for $1/\pi$. Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, 2^3, \ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k}\equiv\begin{cases}(\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p=x^2+dy^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p})=-1,\end{cases}$$

or $d=1$, $\gcd(15,m)>1$, and for any prime $p\equiv 3 \pmod 4$ with $p \nmid 3m$ we have $\sum_{k=0}^{p-1} a_k/m^k \equiv 0 \pmod{p^2}$.

Illustrating the Philosophy by an Example

Recall my following conjectural series

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}.$$

Actually this identity was motivated by the following conjecture.

Conjecture (Sun). Let $p>5$ be a prime. Then

$$\left(\frac{15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}$$

$$\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=1\ \&\ 4p=x^2+91y^2,\\ 2p-7x^2\pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=-1\ \&\ 4p=7x^2+13y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{91})=-1.\end{cases}$$

$$\sum_{k=0}^{p-1}(1638k+277)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}$$

$$\equiv\frac{p}{40}\left(8701\left(\frac{-\mathbf{105}}{p}\right)+2379\left(\frac{735}{p}\right)\right)\pmod{p^2}.$$

Another Example Illustrating the Philosophy

Recall my following conjectural series

$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi}.$$

Actually this identity was motivated by the following conjecture.

Conjecture (Sun). Let $p>3$ be a prime with $p\neq 7$. Then

$$\left(\frac{-42}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)}{(-168^2)^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\ \&\ p=3x^2+5y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{15})=-1.\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)$$

$$\equiv p\left(3\left(\frac{-\mathbf{42}}{p}\right)+10\left(\frac{-\mathbf{210}}{p}\right)\right)\pmod{p^2}.$$

The 3rd Example Illustrating the Philosophy

I would like to offer \$90 for the first proof of the identity in the following conjecture and \$1050 for the first proof of congruences in the conjecture.

Conjecture (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

The 3rd Example Illustrating the Philosophy (continued)

And

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } 2p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 2p-6x^2 \pmod{p^2} & \text{if } 2p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 10x^2-2p \pmod{p^2} & \text{if } 2p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 28x^2-2p \pmod{p^2} & \text{if } p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 14x^2-2p \pmod{p^2} & \text{if } 2p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p})=-1.\end{cases}$$

Remark. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

My secret criterion for existence of series for $1/\pi$ of type IV

Hypothesis (Sun, 2011). (i) Suppose that

$$\sum_{k=0}^{\infty}\frac{a_0+a_1k}{m^k}\binom{2k}{k}^2T_{2k}(b,1)=\frac{C}{\pi}$$

with $a_0,a_1,b,m\in\mathbb{Z}$, $b>0$ and $C^2\in\mathbb{Q}\setminus\{0\}$. Then $\sqrt{|m|}$ is an integer dividing $16(b^2-4)$. Also, $b=7$ or $b\equiv 2 \pmod 4$.
(ii) Let $\varepsilon\in\{\pm1\}$, $b,m\in\mathbb{Z}^+$ and $m\mid 16(b^2-4)$. Then, there are $a_0,a_1\in\mathbb{Z}$ such that

$$\sum_{k=0}^{\infty}\frac{a_0+a_1k}{(\varepsilon m^2)^k}\binom{2k}{k}^2T_{2k}(b,1)=\frac{C}{\pi}$$

for some $C\neq 0$ with C^2 rational, if and only if $m>4(b+2)$ and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(b,1)}{(\varepsilon m^2)^k}\equiv\left(\frac{\varepsilon(b^2-4)}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(b,1)}{(\varepsilon\bar{m}^2)^k}\pmod{p^2}$$

for all odd primes $p\nmid b^2-4$, where $\bar{m}=16(b^2-4)/m$.

Summary

Problem 1 for $d = 1$ already has a positive answer.

We suggest positive answers to Problem 1 for

$$d \in \{2, 3, 5, 6, 7, 10, 13, 15, 22, 30, 37, 58, 70, 85, 130, 190\}.$$

We also formulate many conjectures concerning Problem 2; in particular, we give explicit conjectural positive answers for those squarefree positive integers d with $\mathbb{Q}(\sqrt{-d})$ having class number at most two except for $d = 187, 403$.

Note that $\mathbb{Q}(\sqrt{-d})$ has class number two if and only if

$$d \in \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}.$$

Connections of Problems 1 and 2 to series for $1/\pi$ are very mysterious!
Some conjectures of mine might remain open for many years!
For more detailed survey, the reader may consult my preprint available from `http://arxiv.org/abs/1103.4325`

Thank you!