Mysterious π-Series (I)

– Classical Ramanujan-type Series for $\frac{1}{\pi}$ and the Congruence-Reversing Technique

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Abstract

In this talk I'll introduce the classical Ramanujan-type series for $\frac{1}{\pi}$ and their p-adic analogues. I'll also tell how I found new π-series via the congruence-reversing technique from divergent Ramanujan-type series.

Part I. Ramanujan Series for $\frac{1}{\pi}$ and related p-adic Congruences

My initial contact with π-series (1984-87)

When I was an undergraduate at Nanjing University, I learned from calculus the following classical results :

Leibniz:

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Euler:

$$\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

But I did not know any other π-series then.

Saw a report on Ramanujan

In my diary dated Sept. 16, 1987, I saw a report on the Indian mathematician S. Ramanujan in a Chinese newspaper in which it mentions a quick converging series for $1/\pi$ discovered by Ramanujan:

$$\frac{1}{\pi} = 2\sqrt{2}\sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n(\frac{1}{4})_n(\frac{3}{4})_n}{(1)_n(1)_n n!}(1103 + 26390n)\left(\frac{1}{99}\right)^{4n+2},$$

where $(\alpha)_n$ denotes $\alpha(\alpha+1)\cdots(\alpha+n-1)$.

At that time, I had no special impression on this complicated formula.

Related work in 1988

In July-August 1988, I and my twin brother Zhi-Hong Sun studied the sum $\sum_{k\equiv r \pmod m}\binom{n}{k}$ and related congruences.

Z.-H. Sun determined $\sum_{k\equiv r \pmod 8}\binom{n}{k}$.

Z.-H. Sun and Z.-W. Sun [Acta Arith. 60 (1992)] determined $\sum_{k\equiv r \pmod{10}}\binom{n}{k}$ in terms of Fibonacci numbers and Lucas numbers, and gave an application to Fermat's Last Theorem.

Z.-W. Sun [Israel J. Math. 128(2002)] determined $\sum_{k\equiv r \pmod{12}}\binom{n}{k}$ and $\sum_{\substack{0<k<p\\ k\equiv r \pmod{12}}}\frac{1}{k} \pmod p$.

Later I realized that E. Lehmer determined $\sum_{k\equiv r \pmod m}\binom{n}{k}$ for $m=3,4$ in 1938. Consequently, for any odd prime p we have

$$\sum_{k=0}^{(p-3)/2}\frac{(-1)^k}{2k+1}\equiv\frac{(-1)^{(p-1)/2}}{2}\cdot\frac{2^{p-1}-1}{p} \pmod p.$$

In 1988 I compared this with the Leibniz series $\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}=\frac{\pi}{4}$, and thought that it is interesting to look at congruences for truncated series for π. But at that time I knew little series for π.

The Gamma function

The Classical Gamma Function:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \ (x > 0), \quad \Gamma(n) = (n-1)! \text{ for } n \in \mathbb{Z}^+.$$

Euler's Formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}.$$

In particular,

$$\Gamma\left(\frac{1}{2}\right)^2 = \pi, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

Gaussian hypergeometric series

$$y = {}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right)$$

satisfies the differential equation:

$$\left(\theta \prod_{t=1}^{r}(\theta + \beta_t - 1) - x \prod_{s=0}^{r}(\theta + \alpha_s)\right) y = 0$$

where

$$\theta = x\frac{d}{dx}.$$

Clausen's Identity:

$$\begin{aligned}&{}_2F_1\left(2a, 2b; a+b+1/2 \mid x\right)^2 \\ =&{}_3F_2\left(2a, 2b, a+b; a+b+1/2, 2a+2b \mid 4x(1-x)\right).\end{aligned}$$

In the case $a = b = 1/4$, it gives the identity

$$\left(\sum_{k=0}^{\infty}\binom{2k}{k}^2 x^k\right)^2 = \sum_{k=0}^{\infty}\binom{2k}{k}^3 (x(1-16x))^k.$$

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England
Modular equations and approximations to π,
Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for* $1/\pi$". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

Elliptic integrals

Complete elliptic integrals (with $0 < k < 1$):

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \quad \text{(the first kind)},$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 \theta}\, d\theta \quad \text{(the seond kind)}.$$

Legendre's Relation: If $0 < k < 1$ and $k' = \sqrt{1 - k^2}$, then

$$E(k)K(k') + E(k')K(k) - K(k)K(k') = \frac{\pi}{2}.$$

A Central Result:

$${}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1 \mid k^2\right) = \frac{2}{\pi} K(k) = \varphi^2(q)$$

where $q = e^{-\pi K(k')/K(k)}$ and

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{(theta function)}.$$

Modular equations

Modular equation of degree n: A relation between k and l in the interval $(0,1)$ induced by

$$n\frac{K(k')}{K(k)} = \frac{K(l')}{K(l)},$$

or an identity relating $\varphi(q)$ to $\varphi(q^n)$.

Bruce Berndt wrote (in his book *Number Theory in the Spirit of Ramanujan*): **There is no single method one can use to discover or construct modular equations. One needs to be resourceful and use a variety of tools. Generally, as the degree of the modular equation increases, the difficulty of establishing modular equations rises sharply.**

Series for $1/\pi$ given by Ramanujan

Two of the 17 series for $1/\pi$ recorded by Ramanujan:

$$\sum_{k=0}^{\infty} \frac{6k+1}{4^k} \cdot \frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty} (6k+1) \frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi},$$

(proved by S. Chowla in 1928)

$$\sum_{k=0}^{\infty} \frac{26390k+1103}{99^{4k}} \cdot \frac{(1/2)_k(1/4)_k(3/4)_k}{(1)_k^3}$$

$$= \sum_{k=0}^{\infty} \frac{26390k+1103}{396^{4k}} \binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

In 1985 Jr. R. W. Gosper used the last series of Ramanujan to calculate $17,526,100$ digits of π (a world record at that time).

In 1987 Jonathan Borwein and Peter Borwein succeeded in proving all the 17 Ramanujan series for $1/\pi$.

What happened in 2003

In 2003, I happened to see a paper on Ramanujan-type series. Here is one of Ramanujan series for $1/\pi$:

$$\sum_{k=0}^{\infty}(28k+3)\left(-\frac{27}{512}\right)^k \frac{(1/2)_k(1/6)_k(5/6)_k}{(1)_k^3}=\frac{32\sqrt{2}}{\pi}.$$

At that time I did not like this at all since it is too complicated! I only enjoy simple and beautiful results! Thus this paper gave me almost no impression and I could not remember what paper it is.

During Nov. 16-22, 2003 I attended the Second East Asian Conference on Algebra and Combinatorics held at Fukuoka in Japan. On the conference I met Prof. Jiang Zeng (a combinatorist) from Uinv. Lyon I in France.

Ramanujan-type series for $1/\pi$

General forms of Classical Ramanujan-type Series for $1/\pi$:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are totally 36 known Ramanujan-type series for $1/\pi$ with a, b, m rational. I prefer their forms in terms of binomial coefficients.

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Remark. This yielded the record for the calculation of π during 1989-1994.

What is needed for proving $\sum_{n=0}^{\infty}(6n+1)\binom{2n}{n}^3/256^n = 4/\pi$

The proofs of Ramanujan series involve lots of things such as modulo forms, elliptic integrals, theta functions, hypergeometric series, modular equations and symbolic computation.

$$P(q) := 1 - 24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j} \quad \text{(Eisenstein series)},$$

$$\varphi(q) := \sum_{j=-\infty}^{\infty} q^{j^2} \quad \text{(theta function)},$$

$$X = X(q) = q\prod_{j=1}^{\infty}\frac{(1-q^j)^{24}(1-q^{4j})^{24}}{(1-q^{2j})^{48}}.$$

$$\varphi(q)^4 = \sum_{n=0}^{\infty}\binom{2n}{n}X^n,\ P(q^2) = \sqrt{1-64X}\sum_{n=0}^{\infty}(3n+1)\binom{2n}{n}^3 X^n.$$

$$X(e^{-\pi\sqrt{3}}) = \frac{1}{256} \text{ and } P(e^{-2\pi\sqrt{3}}) = \frac{\sqrt{3}}{\pi} + \frac{\sqrt{3}}{4}\varphi(e^{-\pi\sqrt{3}})^4.$$

What happened in 2005-2006

During Jan. 11-March 10, 2005, I visited Prof. Jiang Zeng at Univ. Lyon-I. At that time, Dr. Victor Junwei Guo was a postdoctor there. Guo told me his following conjectural identity:

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l}\binom{2k}{k-2l+m}=\begin{cases}\binom{2m/3}{m/3}\binom{m/3}{l-m/3} & \text{if } 3\mid m,\\ 0 & \text{otherwise.}\end{cases}$$

When I returned to China, I asked my PhD student Hao Pan to prove this conjecture. At first, Pan had no idea.

During May 2005-May 2006, I visited Prof. Daqing Wan at Univ. of California at Irvine.

In 2005, H. Pan and I finally established the following result which extends the conjectural identity of Guo.

Theorem (H. Pan and Z.-W. Sun [Discrete Math. 306(2006)]). If $l,m,n\in\{0,1,2,\ldots\}$ then

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{n}\binom{2k}{k-2l+m}=\sum_{k=0}^{l}\binom{l}{k}\binom{2k}{n}\binom{n-l}{m+n-3k-l}.$$

My joint work on congruences modulo prime powers

H. Pan and Z. W. Sun [Discrete Math. 306(2006)].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\ (\mathrm{mod}\ p)\quad (d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\ (\mathrm{mod}\ p)\quad \text{for } p>3.$$

Sun & R. Tauraso [AAM 45(2010); IJNT 7(2011)].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\ (\mathrm{mod}\ p^2),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\ (\mathrm{mod}\ p^3)\quad \text{for } p>3,$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers given by

$$B_0=1,\quad \sum_{k=0}^{n}\binom{n+1}{k}B_k=0\quad (n=1,2,3,\ldots).$$

My result on $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod p^2

Z.-W. Sun [Sci. China Math. 53(2010)]: Let p be an odd prime and let $m \in \mathbb{Z}$ with $p \nmid m$. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m^2-4m}{p}\right) + u_{p-(\frac{m^2-4m}{p})} \pmod{p^2},$$

where $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0 = 0,\ u_1 = 1,\ \text{and}\ u_{n+1} = (m-2)u_n - u_{n-1}\ (n = 1, 2, 3, \ldots).$$

In particular,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} \pmod{p^2}. \qquad (*)$$

Remark. *Remark*. I only found two values of p such that the last congruence holds mod p^3: $p = 149,\ 241$.

Multinomial coefficients

Multinomial coefficients:

$$\binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n} = \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}.$$

Note that $\binom{2k}{k} = \binom{2k}{k,k}$. So, a natural extension of $\binom{2k}{k}$ is

$$\binom{kn}{k, k, \ldots, k} = \frac{(kn)!}{(k!)^n}.$$

Clearly,

$$\binom{3k}{k, k, k} = \binom{2k}{k}\binom{3k}{k}$$

and

$$\binom{4k}{k, k, k, k} = \binom{2k}{k}^2\binom{4k}{2k}.$$

My result and conjecture on multinomial coefficients

Theorem (Sun [Acta Arith. 148(2011)]). An integer $p > 1$ is a prime if and only if

$$\sum_{k=0}^{p-1} \binom{(p-1)k}{k, \ldots, k} \equiv 0 \pmod{p}.$$

Conjecture (Sun [Acta Arith. 148(2011)]). For any odd prime p and positive integer n,

$$\frac{1}{n\binom{2n}{n}} \sum_{k=0}^{n-1} \binom{(p-1)k}{k, \ldots, k}$$

is always a p-adic integer.

Remark. When $p = 3$, Strauss, Shallit and Zagier [Amer. Math. Monthly 99(1992)] show that $\sum_{k=0}^{n-1} \binom{2k}{k} / (n^2 \binom{2n}{n})$ is a 3-adic integer for any $n = 1, 2, 3, \ldots$.

Conjectures of Rodriguez-Villegas

It is easy to see that for any $k \in \mathbb{N} = \{0, 1, 2, \ldots\}$ we have

$$\binom{-1/2}{k} = \prod_{j=1}^{k} \frac{1/2 - j}{j} = (-1)^k \frac{(2k-1)!!}{k!2^k} = \frac{(-1)^k (2k)!}{(k!2^k)^2} = \frac{\binom{2k}{k}}{(-4)^k}.$$

In 2003 Rodriguez-Villegas conjectured the following congruences for primes $p > 3$ (which were soon confirmed by E. Mortenson):

$$\sum_{k=0}^{p-1} \binom{-1/2}{k}^2 = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \binom{-1/3}{k}\binom{-2/3}{k} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \binom{-1/4}{k}\binom{-3/4}{k} = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \binom{-1/6}{k}\binom{-5/6}{k} = \sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

On $a(p), b(p), c(p)$

For a power series $f(q)$ in q, we let $[q^n]f(q)$ denote the coefficient of q^n in $f(q)$.

For any prime $p > 3$, it is known that

$$a(p) := [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6 = \begin{cases} 4x^2-2p & \text{if } p = x^2+y^2 \ (2 \nmid x), \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

$$\begin{aligned} b(p) :&= [q^p]q\prod_{n=1}^{\infty}(1-q^{6n})^3(1-q^{2n})^3 \\ &= \begin{cases} 4x^2-2p & \text{if } p = x^2+3y^2 \text{ with } x,y \in \mathbb{Z}, \\ 0 & \text{if } p \equiv 2 \pmod 3, \end{cases} \end{aligned}$$

$$\begin{aligned} c(p) :=& [q^p]q\prod_{n=1}^{\infty}(1-q^{n})^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2 \\ =& \begin{cases} 4x^2-2p & \text{if } \left(\frac{-2}{p}\right) = 1 \text{ and } p = x^2+2y^2 \text{ with } x,y \in \mathbb{Z}, \\ 0 & \text{if } \left(\frac{-2}{p}\right) = -1, \text{ i.e., } p \equiv 5,7 \pmod 8. \end{cases} \end{aligned}$$

Conjectures of Rodriguez-Villegas

Let $p > 3$ be a prime. In 2003 Rodriguez-Villegas conjectured that

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}^3 = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/3}{k}\binom{-2/3}{k} = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/4}{k}\binom{-3/4}{k} = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{-1/2}{k}\binom{-1/6}{k}\binom{-5/6}{k} = \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

are congruent to $a(p), b(p), c(p)$ and $(\frac{p}{3})a(p)$ mod p^2 respectively. Actually the first one was proved by Ishikawa [Nagoya Math. J. 118(1990)]. E. Mortenson [Proc. AMS 133(2005)] provided partial solutions to the last three and the remaining thing were proved by Z.-W. Sun [156(2012)].

What happened in November, 2009

During Nov. 6-7, 2009 both Zhi-Hong and I attended the 1st National Conference on Combinatorial Number Theory held at Nanjing Normal University. After the conference, Zhi-Hong did not return home and came to our univ. to copy some books.

On Nov. 10, 2009 I had a supper with Zhi-Hong who brought a copy of Ken Ono's book *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series* (Amer. Math. Soc., 2004). I had a glance at the last page of the book and found a list of few supercongruences conjectured by F. Rodriguez-Villegas and proved by Mortenson including

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k,k,k}}{27^k}\equiv\left(\frac{-3}{p}\right)\pmod{p^2}.$$

I knew such things before. But, on that day, as I had determined $\sum_{k=0}^{p-1}\binom{2k}{k,k}/m^k \bmod p^2$, I suddenly realized that I should check $\sum_{k=0}^{p-1}\binom{3k}{k,k,k}/m^k \bmod p^2$ via `Mathematica` which I just began to learn. I wished to go home immediately (for secret computation).

What happened in November, 2009

But Zhi-Hong insisted that I should live with him in the guest room at the New Era Hotel. So, on Nov. 10, 2009 I brought my computer to the hotel and found that $\sum_{k=0}^{p-1}\binom{3k}{k,k,k}/24^k \equiv 0 \pmod{p}$ for any odd prime $p\equiv 2 \pmod 3$. Later I figured out the pattern and conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k,k,k}}{24^k} \equiv \begin{cases} \binom{2(p-1)/3}{(p-1)/3} \pmod{p^2} & \text{if } p\equiv 1 \pmod 3, \\ p/\binom{2(p+1)/3}{(p+1)/3} \pmod{p^2} & \text{if } p\equiv 2 \pmod 3. \end{cases}$$

(This was recently confirmed by C. Wang and me [JMAA 505(2022)].)

I also noted that

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{81^k} \equiv \sum_{k=0}^{p-1}\binom{2k}{k}^3 \equiv 0 \pmod{p^2}$$

for half of the primes:

$p = 5, 13, 17, 19, 31, 41, 59, 61, 73, 83, 89, 97, 101, 103, 131, 139, 157, \ldots$

But I could not find the pattern for these primes.

What happened in November, 2009

In the afternoon of Nov. 11 (Wednesday) we had a seminar. First I reported my discovery and asked if anybody (my students and Zhi-Hong) can figure out the pattern of those primes. Nobody gave an answer. Then I left for a meeting in our dept and Zhi-Hong gave a talk on his results on Euler numbers. When I got to the dept there are still few minutes left, so I opened my computer and searched the sequence 5,13,17,19,31,41 via google and this led me to find that these primes are quadratic nonresidues modulo 7. I immediately called my student Yong Zhang or Hao Pan in the seminar to inform this news.

On Nov. 11 I wrote a draft and posted it to arXiv. The results in the paper include

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p \pmod p & \text{if } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

Several days later I learned that this is not new, it has been proved to hold mod p^2.

What happened in November, 2009

If $(\frac{p}{7}) = 1$, what about $\sum_{k=0}^{p-1} \binom{2k}{k}^3$ mod p^2 ? I was puzzled by this. On Friday afternoon (Nov. 13), I attended another meeting in out dept, and suddenly remembered that $\mathbb{Q}(\sqrt{-7})$ is an Euclidean domain and hence an PID as I often taught undergraduates in the course Modern Algebra. If $(\frac{p}{7}) = 1$, i.e., $(\frac{-7}{p}) = 1$, then p splits by algebraic number theory and thus p can be written in the form

$$\frac{x + y\sqrt{-7}}{2} \times \frac{x - y\sqrt{-7}}{2} = \frac{x^2 + 7y^2}{4},$$

as both x and y must be even we have $p = (x/2)^2 + 7(y/2)^2$. After the meeting I immediately went back and verified this observation from Cox's book *Primes of the Form* $x^2 + ny^2$. Thus this led me to find that if $(\frac{p}{7}) = 1$ and $p = x^2 + 7y^2$ then $\sum_{k=0}^{p-1} \binom{2k}{k}^2 \equiv 4x^2 - 2p \pmod{p^2}$. I updated my arXiv article to add this immediately.

I also found patterns for $\sum_{k=0}^{p-1} \binom{2k}{k}^2 / m^k$ mod p^2 with $m = 8, -16, 32$.

What happened in November, 2009

On Nov. 14 (Saturday) I called Zhi-Hong and informed my discovery. He said that he just wanted to make computations to determine $\sum_{k=0}^{p-1}\binom{2k}{k}^3$ mod p^2 in the case $(\frac{p}{7})=1$, and he complained that his student was too lazy and did not compute for him.

Lesson. If one has not yet formulated a complete conjecture, better not inform others to avoid potential competition.

On Nov. 11 I also conjectured that if $(\frac{p}{7})=-1$ then $\sum_{k=0}^{p-1}k\binom{2k}{k}^3\equiv 0 \pmod p$. On Nov. 27, 2009 I posted *Open Conjectures on Congruences* to collect my conjectural congruences. After reading this material, on Nov. 28 Bilgin Ali and Bruno Mishutka guessed that if $p=x^2+7y^2$ then

$$\sum_{k=0}^{p-1}k\binom{2k}{k}^3\equiv\begin{cases}11y^2/3-x^2 \pmod p & \text{if } 3\mid y,\\ 4(y^2-x^2)/3 \pmod p & \text{if } 3\nmid y.\end{cases}$$

What happened in November, 2009

Inspired this I immediately realized that

$$\sum_{k=0}^{p-1} k\binom{2k}{k}^3 \equiv \begin{cases} \frac{8}{21}(3-4x^2) \pmod{p^2} & \text{if } p = x^2+7y^2, \\ \frac{8}{21}p \pmod{p^2}. \end{cases}$$

and circulated this via a message to Number Theory Mailing List.

Thus, in Nov. 2009 I formulated complete conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3 \bmod p^2$ and $\sum_{k=0}^{p-1} k\binom{2k}{k}^3 \bmod p^2$.

Prof. Ken Ono was very interested in this and he and one of his students worked on my conjecture. They claimed that they had a proof but in Jan. 2010 they replied me that they met real difficulties.

My unexpected discovery in Jan. 2010

Let p be an odd prime. I wanted to know $\sum_{k=1}^{(p-1)/2}\binom{2k}{k}/k \bmod p^2$ and I found that $\sum_{k=1}^{(p-1)/2}\binom{2k}{k}/k\equiv 0 \pmod{p^3}$ for $p=149,\,241$.

A conjecture of Rodriguez-Villegas proved by Mortenson [JNT, 2003] states that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)=(-1)^{(p-1)/2}\pmod{p^2}.$$

I found that it holds mod p^3 for $p=149,\,241$.

A conjecture of van Hamme proved by Mortenson [PAMS, 2008] asserts that

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\left(\frac{-1}{p}\right)p\pmod{p^3}.$$

I found that it holds mod p^4 for $p=149,\,241$.

Connections to Euler numbers

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1, \ \sum_{2|k} \binom{n}{k} E_{n-k} = 0 \ (n = 1, 2, 3, \ldots).$$

It is known that $E_1 = E_3 = E_5 = \cdots = 0$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \ \left(|x| < \frac{\pi}{2}\right).$$

Z. W. Sun [Sci. China Math., 54(2011)]:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2} \ (p > 3),$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3},$$

Connections between series and congruences involving E_{p-3}

Series:

$$\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}=\frac{\pi^2}{18},\ \sum_{k=1}^{\infty}\frac{4^k}{k^2\binom{2k}{k}}=\frac{\pi^2}{2},\ \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

Corresponding congruences that I proved:

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)\frac{4}{3}E_{p-3}\pmod p\ (p>3),$$

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k^2\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)4E_{p-3}\pmod p,$$

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\ (\text{mod}\ p^4).$$

Connections between series and congruences

Known series involving $H_n=\sum_{k=1}^n 1/k$ **or** $H_n^{(2)}=\sum_{k=1}^n 1/k^2$:

$$\sum_{k=1}^\infty\frac{H_k}{k2^k}=\frac{\pi^2}{12},\ \sum_{k=1}^\infty\frac{H_k^{(2)}}{k2^k}=\frac{5}{8}\zeta(3),\ \sum_{k=1}^\infty\frac{H_k^2}{k^2}=\frac{17\pi^4}{360}.$$

Corresponding congruences for any prime $p>5$:

$$\sum_{k=1}^{(p-1)/2}\frac{H_k}{k2^k}\equiv\sum_{k=1}^{p-1}\frac{H_k^2}{k^2}\equiv 0 \pmod p$$

[Z. W. Sun, Proc. AMS 140(2012), 415-428],

$$\sum_{k=1}^{p-1}\frac{H_k}{k2^k}\equiv\frac{7}{24}pB_{p-3} \pmod{p^2},\ \sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k2^k}\equiv-\frac{3}{8}B_{p-3} \pmod p$$

[Conjectured by Sun and proved by Sun and Zhao (arXiv:0911.4433)],

$$\sum_{k=1}^{p-1}\frac{H_k^2}{k^2}\equiv\frac{4}{5}pB_{p-5} \pmod{p^2}$$

[Conjectured by Sun and proved by R. Meštrović (arXiv:1108.1171)].

The philosophy about regular series involving π or the ζ-function

As Euler proved, for each $m \in \mathbb{Z}^+$ we have

$$2\zeta(2m) = (-1)^{m-1} B_{2m} \frac{(2\pi)^{2m}}{(2m)!}, \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2m+1}} = \frac{(-1)^m E_{2m} \pi^{2m+1}}{4^{m+1}(2m)!}.$$

J.W.L. Glaisher (1900): Let $p > 3$ be a prime. Then

$$H_{p-1}^{(m)} = \sum_{k=1}^{p-1} \frac{1}{k^m} \equiv \begin{cases} \frac{pm}{m+1} B_{p-1-m} \pmod{p^2} & \text{if } m \in \{2, 4, \ldots, p-3\}, \\ -\frac{p^2 m(m+1)}{2(m+2)} B_{p-2-m} \pmod{p^3} & \text{if } m \in \{1, 3, \ldots, p-4\}, \end{cases}.$$

In a message to `Number Theory List` on March 15, 2010, I expressed the following viewpoint:

Almost every series with summation related to $\pi = 3.14\ldots$ or the Riemann zeta function corresponds to a congruence for Euler numbers or Bernoulli numbers. Conversely, many congruences for E_{p-3} or B_{p-3} modulo a prime p yield corresponding series related to π or the zeta function.

An example illustrating my philosophy

Example. It is known that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k}=\frac{\pi}{3},\quad \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}=\frac{\pi^2}{10}.$$

I [JNT 131(2011)] proved that for any prime $p>3$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv 0 \pmod{p^2},$$

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv \frac{p}{3}E_{p-3} \pmod{p^2}.$$

And I conjectured that for any prime $p>5$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\equiv -\frac{p}{15}B_{p-3} \pmod{p^2},$$

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)^2(-16)^k}\equiv -\frac{p}{4}B_{p-3} \pmod{p^2}.$$

Find new series for π^3

There are very few interesting series for π^3. The only well-known series for π^3 is the following one:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=\frac{\pi^3}{32}.$$

I observed that for any prime $p>3$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}\equiv\frac{(-1)^{(p+1)/2}}{12}B_{p-3} \pmod{p}.$$

Motivated by this observation, I guessed that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}=\frac{7}{216}\pi^3.$$

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$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}=\frac{7}{216}\pi^3.$$

After I announced this conjecture, Olivier Gerard pointed out there is a computer proof via `Mathematica` (version 7).

What happened in Jan.-Feb., 2010

I visited India during Jan.-Feb. 2010. On Jan. 23 I suddenly realized that I should combine the congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ and $\sum_{k=0}^{p-1}k\binom{2k}{k}^3/m^k$ mod p^2. This led me to conjecture that

$$\frac{1}{p}\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv 8+16p^3B_{p-3} \pmod{p^4} \qquad (*)$$

and that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3 \in \mathbb{Z}.$$

After reading my message to Number Theory List on Feb. 10, Kasper Andersen found on Feb. 11 that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3 = \sum_{k=0}^{n-1}\binom{n+k-1}{k}^2$$

via Sloane's OEIS (Online Encyclopedia of Integer Sequences). Inspired by this I finally proved $(*)$.

van Hamme's conjecture

After I found $\sum_{k=0}^{p-1}\binom{2k}{k}^3/4096^k \bmod p^2$ and conjectured the congruence

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv 5p\left(\frac{-1}{p}\right) - p^3E_{p-3} \pmod{p^4},$$

I got to know that van Hamme had the conjecture

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv 5p\left(\frac{-1}{p}\right) \pmod{p^3}$$

motivated by Ramanujan's identity

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} = \frac{16}{\pi}.$$

Thus I became interested in Ramanujan-type series and wrote to several mathematicians to get Hamme's paper.

The p-adic Gamma function

Let p be a prime and let $\mathbb{Z}_p$ be the ring of p-adic integers. Any p-adic integer x has a unique p-adic series representation

$$x = a_0 + a_1 p + a_2 p^2 + \ldots \quad \text{with } a_0, a_1, a_2, \ldots \in \{0, \ldots, p-1\}$$

which converges according to the p-adic norm $|\cdot|_p$. Note that

$$x \equiv \sum_{k=0}^{n-1} a_k p^k \pmod{p^n} \text{ and } \left| x - \sum_{k=0}^{n-1} a_k p^k \right|_p \leqslant p^{-n} \to 0.$$

So each p-adic integer is the limit of a sequence of natural numbers which converges p-adically.

The p-adic Gamma function: For $n \in \mathbb{Z}^+$ define

$$\Gamma_p(n) := (-1)^n \prod_{\substack{0<k<n \\ p \nmid k}} k.$$

Also set $\Gamma_p(0) = 1$. For $x \in \mathbb{Z}_p$, choose a sequence of natural numbers $(x_n)_{n \geqslant 0}$ whose p-adic limit is x, and then define

$$\Gamma_p(x) = \lim_{n \to \infty} \Gamma_p(x_n).$$

van Hamme's idea

Similar to Euler's formula

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x},$$

for any $x \in \mathbb{Z}_p$ we have

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{\{x\}_p},$$

where $\{x\}_p$ is the unique $r \in \{1, \ldots, p\}$ with $x \equiv r \pmod{p}$. In particular,

$$\Gamma_p\left(\frac{1}{2}\right)^2 = (-1)^{\{1/2\}_p} = (-1)^{(p+1)/2} = -\left(\frac{-1}{p}\right).$$

Using this idea, in 1997 van Hamme posed p-adic analogues of many series for powers of π.

van Hamme's conjectures

For the two Ramanujan series

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi} \text{ and } \sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi},$$

in 1997 van Hamme conjectured their following p-adic analogues:

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\equiv p\left(\frac{-2}{p}\right) \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right) \pmod{p^4},$$

where p is an odd prime.

All the p-adic analogue conjectures of van Hamme were proved before 2017. Following van Hamme's idea, Zudilin [JNT, 2009] proposed more p-adic analogues for Ramanujan-type series.

Part II. Throwing the Linear Part in Ramanujan Series for $\frac{1}{\pi}$

My Philosophy about Series for $1/\pi$

Part I of the Philosophy (2010). Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k}\equiv 0 \pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1,2,2^2,2^3,\ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d>1$ and for any prime $p>3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k}\equiv\begin{cases}(\frac{m'}{p})(x^2-2p)\pmod{p^2} & \text{if } 4p=x^2+dy^2,\\ 0\pmod{p^2} & \text{if } (\frac{-d}{p})=-1,\end{cases}$$

or $d=1$, $\gcd(15,m)>1$, and for any prime $p\equiv 3 \pmod{4}$ with $p\nmid 3m$ we have $\sum_{k=0}^{p-1}a_k/m^k\equiv 0 \pmod{p^2}$.

Philosophy about Series for $1/\pi$ (continued)

Part II of the Philosophy (2011). Let $b, c, m, a_0, a_1, \ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k} \equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right) \pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

Remark. Almost all identities of the stated form are *regular*.

An Example Illustrating the Philosophy

Ramanujan Series:

$$\sum_{k=0}^{\infty}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}=\frac{16}{\sqrt{3}\,\pi}.$$

Conjecture (Sun [Sci. China Math. 54(2011)]). For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{12}3)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } 12\mid p-1,\ p=x^2+y^2,\ 3\nmid x \text{ and } 3\mid y,\\ -(\frac{xy}{3})4xy \pmod{p^2} & \text{if } 12\mid p-5 \text{ and } p=x^2+y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod{4},\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3p\left(\frac{p}{3}\right)+\frac{5}{24}p^3B_{p-2}\left(\frac{1}{3}\right)\pmod{p^4}.$$

Another Example Illustrating the Philosophy

In 1987 D. V. Chudnovsky and G. V. Chudnovsky got the formula

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k,k,k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Conjecture (Sun, 2010). Let $p>5$ be a prime with $p\neq 23,29$. Then

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{(-640320)^{3k}}$$

$$\equiv\begin{cases}(\frac{-10005}{p})(x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{163})=1\ \&\ 4p=x^2+163y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{163})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k,k,k}$$

$$\equiv 13591409p\left(\frac{-10005}{p}\right)\pmod{p^3}.$$

Part III. Techniques to Find Series via Transforms of Congruences

Zeilberger-type series

In 1993, D. Zeilberger used the Wilf-Zeilberger method to obtain the new identity

$$\sum_{k=1}^{\infty} \frac{21k-8}{k^3\binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}.$$

Define

$$F(n,k) = \frac{1}{\binom{2n}{n}(n+1)^2\binom{2n+k+1}{n+1}^2}$$

and

$$G(n,k) = \frac{n!^4(n+k)!^2}{2(2n+1)!(2n+k+2)!^2}P(n,k),$$

where $P(n,k)$ denotes

$$(n+1)^2(21n+13) + 2k^3 + k^2(13n+11) + k(28n^2+48n+20).$$

Then $\langle F, G\rangle$ is a **WZ pair** in the sense that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

Zeilberger's proof

$$\sum_{k=0}^{N-1}(F(n+1,k)-F(n,k)) = \sum_{k=0}^{N-1}(G(n,k+1)-G(n,k)) = G(n,N)-G(n,0).$$

$$\sum_{n=0}^{N}\left(\sum_{k=0}^{N-1}F(n+1,k)-\sum_{k=0}^{N-1}F(n,k)\right) = \sum_{n=0}^{N}G(n,N)-\sum_{n=0}^{N}G(n,0).$$

$$\sum_{k=0}^{N-1}F(N+1,k)-\sum_{k=0}^{N-1}F(0,k) = \sum_{n=0}^{N}(G(n,N)-G(n,0)).$$

$$F(0,k)=\frac{1}{(k+1)^2},\ \ G(n,0)=\frac{21(n+1)-8}{(n+1)^3\binom{2n+2}{n+1}^3}.$$

$$\sum_{k=0}^{N-1}F(N+1,k)-\sum_{n=1}^{N}\frac{1}{n^2} = \sum_{n=0}^{N}G(n,N)-\sum_{n=1}^{N+1}\frac{21n-8}{n^3\binom{2n}{n}^3}$$

and hence $\sum_{n=1}^{\infty}\frac{21n-8}{n^3\binom{2n}{n}^3}=\zeta(2)=\frac{\pi^2}{6}$ since $\sum_{k=0}^{N-1}F(N+1,k)\to 0$ and $\sum_{n=0}^{N}G(n,N)\to 0$.

Other Zeilberger-type series

J. Guillera [Ramanujan J. 15(2008)] used the WZ method to give three new Zeilberger-type series:

$$\sum_{k=1}^{\infty}\frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3}=-16G,$$

$$\sum_{k=1}^{\infty}\frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3}=-2G,$$

$$\sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}=\frac{\pi^2}{2},$$

where G denotes the Catalan constant $\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$.

Q.-H. Hou, C. Krattenthaler and Z.-W. Sun [Proc. Amer. Math. Soc. 147(2019)] provided a q-analogue of the last identity with $|q|<1$:

$$\sum_{n=0}^{\infty}q^{n(n+1)/2}\frac{1-q^{3n+2}}{1-q}\cdot\frac{(q;q)_n^3(-q;q)_n}{(q^3;q^2)_n^3}=(1-q)^2\frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4}.$$

A Useful Lemma

Lemma (Z.-W. Sun [Sci. China Math. 54(2011)]) Let p be an odd prime and let $k \in \{0, \ldots, p-1\}$. Then

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2}.$$

Thus,

$$\binom{2(p-k)}{p-k} \equiv \begin{cases} \frac{2p}{k\binom{2k}{k}} \pmod{p} & \text{if } k \in \{\frac{p+1}{2}, \ldots, p-1\}, \\ \frac{-2p}{k\binom{2k}{k}} \pmod{p^2} & \text{if } k \in \{1, \ldots, \frac{p-1}{2}\}. \end{cases}$$

Remark. R. Tauraso [J. Number Theory 130(2010)] realized that

$$\binom{2(p-k)}{p-k} \equiv \frac{2p}{k\binom{2k}{k}} \pmod{p} \text{ for all } k = 1, \ldots, p-1.$$

We have similar lemmas involving $\binom{3k}{k}$ or $\binom{4k}{2k}$.

Rediscover Zeilberger's series $\sum_{k=1}^\infty \frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6}$

In 2010 I proved that for any odd prime p we have

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv 8p + 16p^4 B_{p-3} \pmod{p^5}.$$

As the series $\sum_{k=0}^\infty (21k+8)\binom{2k}{k}^3$ diverges, it does not provide a Ramanujan-type series for $1/\pi$. However, I observe that

$$\begin{aligned}\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 =& 8 + \sum_{k=(p+1)/2}^{p-1}(21(p-k)+8)\binom{2(p-k)}{p-k}^3 \\ \equiv& 8 - \sum_{k=(p+1)/2}^{p-1}(21k-8)\left(\frac{2p}{k\binom{2k}{k}}\right)^3 \pmod{p}\end{aligned}$$

and this led me to rediscover that

$$\sum_{k=1}^\infty \frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6} \quad \text{(D. Zeilberger, 1993).}$$

Conjecture: $\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2$

Conjecture (Z.-W. Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{11})=-1,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{11k+3}{64^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 3p+\frac{7}{2}p^4B_{p-3} \pmod{p^5},$$

$$p\sum_{k=1}^{(p-1)/2}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}\equiv 32\frac{2^{p-1}-1}{p}-\frac{64}{3}p^2B_{p-3} \pmod{p^3}.$$

Also,

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2 \quad \text{(confirmed by J. Guillera in 2013).}$$

Conjecture: $\sum_{k=1}^\infty \frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}} = K$

Conjecture (Z.-W. Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2,\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1;\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 4p\left(\frac{p}{3}\right)+\frac{4}{3}p^3B_{p-2}\left(\frac{1}{3}\right) \pmod{p^4}.$$

Also,

$$\sum_{k=1}^\infty\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K:=\sum_{k=1}^\infty\frac{(\frac{k}{3})}{k^2}$$ (confirmed by Kh. Hessami Pilehrood and T. Hessami Pilehrood in 2012).

More such conjectural series

Conjecture (Z.-W. Sun, 2010; Sci. China Math. 54(2011))

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} &= \frac{\pi^2}{2},\\ \sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} &= 12\pi^2,\\ \sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} &= -\frac{45}{2}K.\end{aligned}$$

The three conjectural identities were finally confirmed by J. Guillera and M. Rogers [J. Austral. Math. Soc. 97(2014)].

A curious identity with \$480 prize for the solution

Conjecture (Z.-W. Sun) (i) (2009-11-29) For any prime $p > 3$,

$$\sum_{k=1}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv 0 \pmod{p^2}.$$

(Confirmed by Chen Wang and Z.-W. Sun [JMAA 306(2022)].)

(ii) (2014-07-07) For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv \frac{5}{12} p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3},$$

$$p^2 \sum_{k=1}^{p-1} \frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}} \equiv 4\left(\frac{p}{3}\right) + 4p \pmod{p^2}.$$

(iii) (2014-08-12, **\$480 prize for the solution**) We have

$$\sum_{k=1}^{\infty} \frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}} = \frac{15}{2} \sum_{k=1}^{\infty} \frac{\left(\frac{k}{3}\right)}{k^2}.$$

Three more conjectural series

Motivated by corresponding congruences, I made the following conjecture in 2010-2011.

Conjecture (Z.-W. Sun) (i) [Sci. China Math. 54(2011)] We have

$$\sum_{n=0}^{\infty}\frac{18n^2+7n+1}{(-128)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2=\frac{4\sqrt{2}}{\pi^2}$$

$$\sum_{n=0}^{\infty}\frac{40n^2+26n+5}{(-256)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{24}{\pi^2}.$$

(In 2004 H.H. Chan, S.H. Chan and Z. Liu [Adv. Math.] proved that $\sum_{n=0}^{\infty}\frac{5n+1}{64^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{8}{\sqrt{3}\pi}$.)

(ii) [Electron. J. Combin. 20(2013)] We have

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

Another transform of congruences

Z.-W. Sun [Nanjing Univ. Math. Biquarterly 32(2015)]: Let $p = 2n+1$ be an odd prime. Then, for each $k = 0, \ldots, n$ we have

$$\frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{-1}{p}\right)\binom{2(n-k)}{n-k} \pmod{p}.$$

This is easy. In fact,

$$\begin{aligned}\binom{2k}{k} = \binom{-1/2}{k}(-4)^k \equiv & \binom{n}{k}(-4)^k = \binom{n}{n-k}(-4)^k \\ \equiv & \binom{-1/2}{n-k}(-4)^k = \frac{\binom{2(n-k)}{n-k}}{(-4)^{n-k}}(-4)^k \\ \equiv & (-1)^n\binom{2(n-k)}{n-k}16^k \pmod{p}.\end{aligned}$$

An Example

Let $p = 2n + 1$ be an odd prime. Then

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv \sum_{k=0}^{n}(21(n-k)+8)\left(\frac{-1}{p}\right)\left(\frac{\binom{2k}{k}}{16^k}\right)^3$$

$$\equiv \frac{(-1)^{(p+1)/2}}{2}\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \pmod{p}.$$

This relates the Zeilberger series

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{6}$$

to the Ramanujan series

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} = \frac{16}{\pi}.$$

A transformation via dual sequences

For a sequence $a_0, a_1, a_2, \ldots$ of complex numbers, define

$$a_n^* = \sum_{k=0}^{n} \binom{n}{k} (-1)^k a_k \quad \text{for all } n \in \mathbb{N} = \{0, 1, 2, \ldots\}$$

and call $(a_n^*)_{n\in\mathbb{N}}$ the *dual sequence* of $(a_n)_{n\in\mathbb{N}}$. It is well known that $a_n^{**} = a_n$ for all $n \in \mathbb{N}$.

For example,

$$w_n := \sum_{k=0}^{\lfloor n/3 \rfloor} (-1)^k 3^{n-3k} \binom{n}{3k} \binom{3k}{k} \binom{2k}{k} = 3^n a_n^*$$

where

$$a_n = \begin{cases} \binom{3k}{k}\binom{2k}{k}/27^k & \text{if } n = 3k, \\ 0 & \text{if } 3 \nmid n. \end{cases}$$

On March 10, 2011, I realized that if $|m-4| > 4$ then

$$\sum_{n=0}^{\infty} (bmn+2b+(m-4)c) \frac{\binom{2n}{n} a_n^*}{(4-m)^n} = (m-4)\sqrt{\frac{m-4}{m}} \sum_{k=0}^{\infty} (bk+c) \frac{\binom{2k}{k} a_k}{m^k}.$$

Congruences for dual sequences

Z.-W. Sun [Nanjing Univ. J. Math. Biquarterly 32(2015)]: Let p be an odd prime and let m be an integer with $p \nmid m(m-4)$. Let α be a positive integer, and let $a_0, a_1, \ldots, a_{p^\alpha-1}$ be p-adic integers. Then we have the congruences

$$\sum_{k=0}^{p^\alpha-1} \frac{\binom{2k}{k}}{(4-m)^k} a_k^* \equiv \left(\frac{m(m-4)}{p^\alpha}\right) \sum_{k=0}^{p^\alpha-1} \frac{\binom{2k}{k}}{m^k} a_k \pmod{p}$$

and

$$m \sum_{k=0}^{p^\alpha-1} \frac{k\binom{2k}{k}}{(4-m)^k} a_k^* \equiv \left(\frac{m(m-4)}{p^\alpha}\right) \sum_{k=0}^{p^\alpha-1} ((m-4)k-2) \frac{\binom{2k}{k}}{m^k} a_k \pmod{p},$$

where $(\frac{\cdot}{p^\alpha})$ denotes the Jacobi symbol.

If $(-1)^k a_k = f_k := \sum_{j=0}^k \binom{k}{j}^3$ $(k = 0, 1, \ldots)$, then $a_n^* = \sum_{k=0}^n \binom{n}{k} f_k = g_n$, where $g_n = \sum_{k=0}^n \binom{n}{k}^2 \binom{2k}{k}$.

Main References:

1. Z.-W. Sun, *List of conjectural series for powers of π and other constants*, preprint, `arXiv:1102.5649`, 2011-2014.

2. Z.-W. Sun, *Conjectures and results on x^2 mod p^2 with $4p = x^2 + dy^2$*, in: Number Theory and Related Area (eds., Y. Ouyang, C. Xing, F. Xu and P. Zhang), Adv. Lect. Math. 27, Higher Education Press and Internat. Press, Beijing-Boston, 2013, pp. 149–197.

Thank you!