Conjectures and Results on Super Congruences and Series related to π and Other Constants

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

May 2, 2010

Abstract

If a p-adic congruence happens to hold modulo a higher power of p then it is called a super congruence. The topic of super congruences is related to the p-adic Gamma function, Gauss and Jacobi sums, hypergeometric series, modular forms, Calabi-Yau manifolds, representations of p by certain quadratic forms, and some sophisticated combinatorial identities involving harmonic numbers. Recently the speaker formulated many conjectures on super congruences and revealed that super congruences are related to Euler numbers and series with summation related to π and other constants. In this talk we will analyze few typical conjectures of the speaker and introduce related progress.

Part A. Previous Work by Others

Classical congruences for central binomial coefficients

A central binomial coefficient has the form

$$\binom{2k}{k}\ (k\in\mathbb{N}=\{0,1,2,\ldots\}).$$

If $p=2n+1$ is an odd prime, then

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv 0 \pmod p\ \text{ for every } k=\frac{p+1}{2},\ldots,p-1.$$

Wolstenholme's Congruence. For any prime $p>3$ we have

$$H_{p-1}=\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2}$$

and

$$\binom{2p-1}{p-1}=\frac{1}{2}\binom{2p}{p}\equiv 1 \pmod{p^3}.$$

Remark. In 1900 Glaiser proved that for any prime $p>3$ we have

$$\binom{2p-1}{p-1}\equiv 1-\frac{2}{3}p^3B_{p-3} \pmod{p^4}.$$

Classical congruences for central binomial coefficients

Morley's Congruence. For any prime $p > 3$ we have

$$\binom{p-1}{(p-1)/2} \equiv \left(\frac{-1}{p}\right) 4^{p-1} \pmod{p^3}.$$

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

Beukers' Conjecture for Apéry Numbers

In 1978 Apéry proved that $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is irrational! During his proof he used the sequence $\{B(n)/A(n)\}_{n=1}^{\infty}$ of rational numbers to approximate $\zeta(3)$, where

$$A(0) = 1,\ A(1) = 5,\ B(0) = 0,\ B(1) = 6,$$

and both $\{A(n)\}_{n\geqslant 0}$ and $\{B(n)\}_{n\geqslant 0}$ satisfy the recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}\ (n = 1, 2, \ldots).$$

In fact,

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

and these numbers are called *Apéry numbers*.

Dedekind eta function in the theory of modular forms:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau}$$

Note that $q| < 1$ if $\tau \in \mathbb{H} = \{z \in \mathbb{C} :\ \mathrm{Im}(z) > 0\}$.

Beukers' Conjecture (1985). For any prime $p > 3$ we have

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2},$$

where $a(n)$ $(n = 1, 2, 3, \ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = \sum_{n=1}^{\infty} a(n)q^n.$$

A Simple Observation. Let $p = 2n+1$ be an odd prime. Then

$$\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{n}{k}\binom{-n-1}{k}$$
$$=\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k} \equiv \binom{-1/2}{k}^2$$
$$=\left(\binom{2k}{k}/(-4)^k\right)^2 = \binom{2k}{k}^2/16^k \pmod{p^2}.$$

Thus Beukers' conjecture has the following equivalent form:

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^4}{256^k} \equiv a(p) \pmod{p^2}.$$

Ahlgren and Ono's Proof of the Beukers' conjecture

S. Ahlgren and Ken Ono [J. Reine Angew. Math. 518(2000)]: The Beukers conjecture is true!

Outline of their proof. First show that $a(p)$ can be expressed as a special value of the Gauss hypergeometric function ${}_4F_3(\lambda)$ defined in terms of Jacobi sums. Then rewrite Jacobi sums in terms of Gauss' sums and apply the Gross-Koblitz formula to express Gauss sums in terms of the p-adic Gamma function $\Gamma_p(x)$. Finally use combinatorial properties of $\Gamma_p(x)$ and some sophisticated combinatorial identities involving harmonic numbers $H_n = \sum_{0<k\leqslant n} 1/k$.

Ahlgren and Ono's Proof of the Beukers' conjecture

Key steps in Ahlgren and Ono's proof.

(i) For an odd prime p let $N(p)$ denote the number of $\mathbb{F}_p$-points of the following Calabi-Yau threefold

$$x + \frac{1}{x} + y + \frac{1}{y} + z + \frac{1}{z} + w + \frac{1}{w} = 0.$$

Then

$$a(p) = p^3 - 2p^2 - 7 - N(p).$$

(ii) For any positive integer n we have

$$\sum_{k=1}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 (1 + 2kH_{n+k} + 2kH_{n-k} - 4kH_k) = 0.$$

T. Kilbourn [Acta Arith. 123(2006)]: For any odd prime p we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^4}{256^k} \equiv a(p) \pmod{p^3}.$$

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$ and $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$.

Greene's character sum analogue of hypergeometric series

Let p be an odd prime. For two Dirichlet characters A and B mod p, define the *normalized Jacobi sum*

$$\binom{A}{B} := \frac{B(-1)}{p} J(A, \bar{B}) = \frac{B(-1)}{p} \sum_{x=0}^{p-1} A(x)\bar{B}(1-x).$$

Given Dirichlet characters $A_0, A_1, \ldots, A_r$ and $B_1, \ldots, B_r$ mod p, Greene [Trans. AMS, 1987] defined

$$\begin{aligned}&{}_{r+1}F_r(A_0, A_1, \ldots, A_r; B_1, \ldots, B_r \mid x)_p \\ =&\frac{p}{p-1} \sum_{\chi} \binom{A_0\chi}{\chi} \prod_{i=1}^{r} \binom{A_i\chi}{B_i\chi} \chi(x).\end{aligned}$$

$$\begin{aligned}&{}_{r+1}F_r(A_0, A_1, \ldots, A_r; B_1, \ldots, B_r \mid x)_p \\ =&\frac{A_r B_r(-1)}{p} \sum_{y=0}^{p-1} {}_rF_{r-1}(A_0, A_1, \ldots, A_{r-1}; B_1, \ldots, B_{r-1} \mid xy)_p \\ &\times A_r(y)\bar{A}_r B_r(1-y) \quad \text{(Greene, 1987).}\end{aligned}$$

Connections to elliptic curves over $\mathbb{F}_p$

Let ε_p be the trivial character with $\varepsilon_p(x) = 1$ for all $x \not\equiv 0 \pmod{p}$. Let ϕ_p be the Legendre character given by $\phi_p(x) = (\frac{x}{p})$. Set

$$_{r+1}F_r(x)_p := {}_{r+1}F_r(\phi_p, \phi_p, \ldots, \phi_p; \varepsilon_p, \ldots, \varepsilon_p \mid x)_p.$$

Consider the curve over $\mathbb{Q}$ defined by

$$_2E_1(\lambda) : \; y^2 = x(x-1)(x-\lambda).$$

If $\lambda \in \mathbb{Q} \setminus \{0, 1\}$, then $_2E_1(\lambda)$ is an elliptic curve with

$$\Delta({}_2E_1(\lambda)) = 16\lambda^2(\lambda-1)^2 \;\; \text{(discriminant)}$$

$$j({}_2E_1(\lambda)) = \frac{256(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} \;\; (j-invariant).$$

Connections to elliptic curves over $\mathbb{F}_p$

Suppose $\lambda(\lambda-1) \not\equiv 0 \pmod{p}$. Then p is a prime of good reduction for ${}_2E_1(\lambda)$, and we define

$$ {}_2a_1(p;\lambda) := p + 1 - |{}_2E_1(\lambda)_p| $$

where $|{}_2E_1(\lambda)_p|$ denotes the number of $\mathbb{F}_p$-points of ${}_2E_1(\lambda)_p$ including the point at infinity. It is known that

$$ {}_2a_1(p;\lambda) = -\sum_{x=0}^{p-1} \phi_p(x(x-1)(x-\lambda)) $$

and

$$ {}_2F_1(\lambda)_p = -\frac{\phi_p(-1)}{p} {}_2a_1(p;\lambda) = \left(\frac{-1}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x(x-1)(x-\lambda)}{p}\right). $$

Conjectures of Rodriguez-Villegas

In 2001 Rodriguez-Villegas conjectured 22 congruences which relate truncated hypergeometric series to the number of $\mathbb{F}_p$-points of some family of Calabi-Yau manifolds. Here we list some of them.

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) = \left(\frac{-3}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6 \pmod{p^2}.$$

Where the denominators and $\left(\frac{\cdot}{p}\right)$ come from?

By Stirling's formula,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to +\infty$$

It follows that

$$\begin{aligned}
\binom{2k}{k}^2 &\sim \frac{16^k}{k\pi}, \\
\binom{2k}{k}^3 &\sim \frac{64^k}{(k\pi)^{3/2}}, \\
\binom{2k}{k}\binom{3k}{k} &\sim \frac{\sqrt{3}\ 27^k}{2k\pi}, \\
\binom{2k}{k}\binom{4k}{2k} &\sim \frac{64^k}{\sqrt{2}k\pi}, \\
\binom{3k}{k}\binom{6k}{3k} &\sim \frac{432^k}{2k\pi}.
\end{aligned}$$

A theorem of Stienstra and Beukers

J. Stienstra and F. Beukers [Math. Ann. 27(1985)]:

$$[q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6$$

$$=\begin{cases}4x^2-2p & \text{if } p=1 \pmod 4 \ \&\ p=x^2+y^2 \text{ with } 2\nmid x \ \&\ 2\mid y,\\ 0 & \text{if } p\equiv 3 \pmod 4;\end{cases}$$

$$[q^p]q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2$$

$$=\begin{cases}4x^2-2p & \text{if } p=1,3 \pmod 8 \ \&\ p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 5,7 \pmod 8;\end{cases}$$

$$[q^p]q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3$$

$$=\begin{cases}4x^2-2p & \text{if } p=1 \pmod 3 \ \&\ p=x^2+3y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Progress on Rodriguez-Villegas conjectures

The congruences we list have been confirmed, see,

E. Motenson, J. Number Theory 99(2003); Trans. AMS 355(2003); Proc. AMS 133(2005).

Many of the 22 conjectures remain open.

Ramanujan's series for $1/\pi$

Here are 5 of the 17 Ramanujan series recorded by him in 1914:

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{6k+1}{4^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}=\frac{4}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{6k+1}{(-8)^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{42k+5}{64^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{20k+3}{(-4)^k}\cdot\frac{(1/2)_k(1/4)_k(3/4)_k}{(1)_k^3}=\sum_{k=0}^{\infty}(20k+3)\frac{\binom{4k}{k,k,k,k}}{(-1024)^k}=\frac{8}{\pi}.$$

Remark. The first one was actually proved by G. Bauer in 1859.

Hamme's Conjectures

L. Van Hamme [1997] conjectured the p-adic analogues of the above first 4 identities and W. Zudilin [JNT, 2009] obtained the p-adic analogue of the last identity.

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{-1}{p}\right)p \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{256^k} \equiv \left(\frac{-1}{p}\right)p \pmod{p^4},$$

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k} \equiv \left(\frac{-2}{p}\right)p \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv \left(\frac{-1}{p}\right)5p \pmod{p^4},$$

$$\sum_{k=0}^{p-1}(20k+3)\frac{\binom{4k}{k,k,k,k}}{(-1024)^k} \equiv \left(\frac{-1}{p}\right)3p \pmod{p^3}.$$

Progress on Hamme's conjectures

The first of the above congruence was proved by E. Mortenson [Proc. AMS 136(2008)] and the second one was recently shown by Ling Long, while the last was confirmed by Zudilin via the WZ method. The third and the fourth remain open.

The p-adic Gamma function plays an important role in Hamme's formulation of those conjectures. It is defined in the following way:

$$\Gamma_p(n) := (-1)^n \prod_{\substack{1<k<n \\ p\nmid n}} k \quad (n = 1, 2, 3, \ldots)$$

and

$$\Gamma_p(x) = \lim_{n \to x} \Gamma_p(n) \ \text{ for any } p-\text{adic integer } x.$$

Teichmüller character and Gauss sums

Let p be an odd prime and let $\mathbb{Z}_p$ be the ring of p-adic integers. The *Teichmüller character* $\omega : \mathbb{F}_p :\to \mathbb{Z}_p$ is given by

$$\omega(a) = \lim_{n\to\infty} a^{p^n} \quad (p\text{-adic limit}).$$

Note that $\omega(a) \equiv a \pmod{p}$ and ω is a primitive (multiplicative) character of $\mathbb{F}_p$.

Let $\mathbb{C}_p$ be the completion of the algebraic closure of the p-adic field $\mathbb{Q}_p$ and let $\pi \in \mathbb{C}_p$ be a solution of the equation $x^{p-1} = -p$. Then there is a unique pth root $\zeta_p \in \mathbb{C}_p$ of unity satisfying $\zeta_p \equiv \pi + 1 \pmod{\pi^2}$. For a character $\chi : \mathbb{F}_p \to \mathbb{C}_p$ define the *Gauss sum*

$$g(\chi) = \sum_{a=0}^{p-1} \chi(a)\zeta_p^a.$$

The Gross-Koblitz formula

The following result is very famous and quite useful.

The Gross-Koblitz formula. For any odd prime p we have

$$g(\bar{\omega}^k) = -\pi^k \Gamma_p\left(\frac{k}{p-1}\right) \ (k = 0, 1, \ldots, p-2).$$

A Connection of Γ_p to Binomial Coefficients
If $p = 2n+1$ is an odd prime, then for $k = 0, \ldots, n$ we have

$$(-1)^{(p+1)/2}\frac{\Gamma_p(k+1/2)^2}{\Gamma_p(k+1)^2} \equiv \binom{n}{k}\binom{n+k}{k}(-1)^k \equiv \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}.$$

Some Series for π

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty} \frac{545140134k + 13591409}{(-640320)^{3k}} \binom{6k}{3k} \binom{3k}{k, k, k} = \frac{3 \times 53360^2}{2\pi\sqrt{10005}}.$$

This yielded the record for the calculation of π during 1989-1994.

D. Zeilberger (1993):

$$\sum_{k=1}^{\infty} \frac{21k - 8}{k^3 \binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}.$$

T. Amdeberhan and D. Zeilberger (1997):

$$\sum_{k=1}^{\infty} \frac{(-1)^k (205k^2 - 160k + 32)}{k^5 \binom{2k}{k}^5} = -2\zeta(3).$$

A Conjecture of J. Guillera (2003):

$$\sum_{k=1}^{\infty} \frac{(21k^3 - 22k^2 + 8k - 1)256^k}{k^7 \binom{2k}{k}^7} = \frac{\pi^4}{8}.$$

Part B. My Results and Conjectures

Some Joint Work

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\ (\mathrm{mod}\ p)\quad (d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\ (\mathrm{mod}\ p)\quad \text{for } p>3.$$

Sun & R. Tauraso [arXiv:0709.1665, Adv. in Appl. Math.].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\ (\mathrm{mod}\ p^2),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\ (\mathrm{mod}\ p^3)\quad \text{for } p>3,$$

L. L. Zhao, H. Pan and Z. W. Sun [Proc. AMS, 2010]

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv 0\ (\mathrm{mod}\ p).$$

My own results

Recall that if $p/2 < k < p$ then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p}.$$

Thus

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \pmod{p},$$

where m is an integer with $p \nmid m$.

In 2009 I [arXiv:0909.5648, arXiv:0911.3060, 0909.3808] determined

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \bmod p^2, \quad \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \bmod p^2, \quad \sum_{k=0}^{p-1} \frac{\binom{3k}{k}}{m^k} \bmod p$$

in terms of linear recurrences.

Some particular congruences due to me

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{8^k}\equiv\frac{3}{4}\left(\left(\frac{p}{5}\right)-1\right)\pmod p,$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{7^k}\equiv\begin{cases}-2\pmod p & \text{if } p\equiv\pm2\pmod 7,\\ 1\pmod p & \text{otherwise.}\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k}}{5^k}\equiv\begin{cases}1\pmod p & \text{if } p\equiv1\pmod 5\ \&\ p\neq11,\\ -1/11\pmod p & \text{if } p\equiv2,3\pmod 5,\\ -9/11\pmod p & \text{if } p\equiv4\pmod 5.\end{cases}$$

If $p\equiv1\pmod 3$ then

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{6^k}\equiv2^{(p-1)/3}\pmod p.$$

Connection between super congruences and Euler numbers

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1,\ \sum_{2|k} \binom{n}{k} E_{n-k} = 0\ (n = 1, 2, 3, \ldots).$$

It is known that $E_1 = E_3 = E_5 = \cdots = 0$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}\ \left(|x| < \frac{\pi}{2}\right).$$

Z. W. Sun [arXiv:1001.4453].

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{8^k} \equiv \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right) \frac{p^2}{4} E_{p-3} \pmod{p^3}.$$

.

Connection between super congruences and Euler numbers

Theorem (Sun, 2010). For any prime $p > 3$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2\binom{2k}{k}} \equiv (-1)^{(p-1)/2} \frac{4}{3} E_{p-3} \pmod{p},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3}.$$

Remark. Note that

$$\lim_{k\to+\infty} \frac{k\binom{2k}{k}^2}{16^k} = \frac{1}{\pi} \text{ and } \sum_{k=1}^{\infty} \frac{1}{k^2\binom{2k}{k}} = \frac{\pi^2}{18}.$$

Some auxiliary results needed for the proof

A Lemma (Sun, 2010). (i) If $p = 2n+1$ is an odd prime, then

$$\binom{n}{k}\binom{n+k}{k}(-1)^k\left(1-\frac{p}{4}(H_{n+k}-H_{n-k})\right)\equiv\frac{\binom{2k}{k}^2}{16^k}\pmod{p^4}.$$

(ii) We have

$$(-1)^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k(H_{n+k}-H_{n-k})=\frac{3}{2}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}.$$

Some auxiliary identities:

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}=\frac{n+1}{3}\binom{2n+1}{n}\sum_{k=1}^{n}\frac{1}{k^2\binom{n}{k}^2}\ \text{(Staver)},$$

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^2\binom{n}{k}\binom{n+k}{k}}=(-1)^{n-1}\left(3\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}}+2\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right)\ \text{(Apéry)}$$

$$\sum_{k=1}^{n}\frac{1}{k^2\binom{n+k}{k}}=3\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}}-\sum_{k=1}^{n}\frac{1}{k^2}\ \text{(Sun)}.$$

Six conjectured series for π^2 and other constants

Conjecture (Z. W. Sun, 2010): We have

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},$$

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=-27\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

Conjecture involving $x^2 + 7y^2$

Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1 \ \& \ p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1. \end{cases}$$

Moreover,

$$\sum_{k=0}^{(p-1)/2} (21k + 8)\binom{2k}{k}^3 \equiv 8p + \left(\frac{-1}{p}\right) 32p^3 E_{p-3} \pmod{p^4}.$$

Remark. M. Jameson and K. Ono are working on the first part of this conjecture but they have not yet got a proof.

Conjecture involving $x^2 + 11y^2$

Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}$$

$$\equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{11}) = 1 \ \&\ 4p = x^2 + 11y^2\ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases}$$

Furthermore,

$$\sum_{k=0}^{p-1}(11k+3)\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k} \equiv 3p + \frac{7}{2}p^4 B_{p-3} \pmod{p^5},$$

$$p\sum_{k=1}^{(p-1)/2} \frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} \equiv 32\frac{2^{p-1}-1}{p} - \frac{64}{3}p^2 B_{p-3} \pmod{p^3}.$$

Remark. It is well-known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $(\frac{p}{11}) = 1$ we can write $4p = x^2 + 11y^2$ with $x, y \in \mathbb{Z}$.

Conjecture involving $x^2 + 163y^2$

Let $p > 5$ be a prime with $p \neq 23, 29$.

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{(-640320)^{3k}} \equiv \begin{cases} (\frac{-10005}{p})(x^2 - 2p) \pmod{p^2} & \text{if } (\frac{p}{163}) = 1 \ \&\ 4p = x^2 + 163y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{163}) = -1. \end{cases}$$

Remark. It is well known that the only imaginary quadratic fields with class number one are those $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For each of the 9 values of d we have corresponding conjectures similar to the above one.

Conjecture for $\mathbb{Q}(\sqrt{-d})$ with class number two

Let $d > 0$ be a squarefree integer. It is known that $\mathbb{Q}(\sqrt{-d})$ has class number two if and only if d is among

$5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427.$

Except for $d = 35, 91, 115, 187, 235, 403, 427$ we have found explicit conjectures involving $x^2 + dy^2$.

Conjecture for $\mathbb{Q}(\sqrt{-15})$ (Sun). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \ \&\ p = x^2 + 15y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \ \&\ p = 5x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1. \end{cases}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{15k+4}{(-27)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 4 \left(\frac{p^a}{3}\right) \pmod{p^2}.$$

One more conjecture

Conjecture (Sun, 2010). Let $p > 7$ be a prime and let $H_{p-1} = \sum_{k=1}^{p-1} 1/k \equiv -p^2 B_{p-3}/3 \pmod{p^3}$. Then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv -\frac{2}{p^2} H_{p-1} \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} \frac{1}{k^4 \binom{2k}{k}} - \frac{H_{p-1}}{p^3} \equiv -\frac{7}{45} p B_{p-5} \pmod{p^2}.$$

Also,

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3 \binom{2k}{k}} \equiv -2 B_{p-3} \pmod{p}.$$

Motivation.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5}\zeta(3) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} = \frac{17}{36}\zeta(4).$$

More Conjectures on Congruences

For more conjectures of mine on congruences, see

Z. W. Sun, *Open Conjectures on Congruences*,

`arXiv:0911.5665`.

You are welcome to solve my conjectures!

Thank you!