An extension of the the Piatetski–Shapiro prime range to $c \leq 1.186$

by Zhi-Wei Sun's inquiry to AI

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Abstract

Let

$$\pi_c(y) = \#\{n \leq y : \lfloor n^c \rfloor \text{ is prime}\}.$$

Rivat and Wu proved the expected-order lower bound for $1 < c \leq 243/205$. Runbo Li subsequently introduced an additional Buchstab asymptotic region and obtained $1 < c < 919/775$. We retain the Type I and Type II estimates of Rivat–Wu, enlarge Li's additional region to the full interval-covering region justified by one Buchstab step, and certify the five remaining Buchstab integrals by directed interval arithmetic. The certified normalized loss is less than 0.9996 at $\gamma = 500/593$. A perturbation argument at the analytic interval boundaries then gives

$$\pi_c(y) \gg_c \frac{y}{\log y} \qquad \left(1 < c \leq \frac{593}{500} = 1.186\right).$$

The proof is computer-assisted. The source programs, raw output, exact rational comparisons and file hashes form part of the certificate.

1 Statement and notation

Theorem 1.1. *For every fixed real number*

$$1 < c \leq \frac{593}{500} = 1.186,$$

one has

$$\pi_c(y) \gg_c \frac{y}{\log y}.$$

In particular, infinitely many members of the sequence $(\lfloor n^c \rfloor)_{n \geq 1}$ are prime.

Put $\beta = 1/c$. We use X for the size of the values $\lfloor n^c \rfloor$ and define

$$\mathcal{A} = \{\lfloor n^c \rfloor : X \leq n^c < 2X\}, \qquad \mathcal{B} = \{m : X \leq m < 2X\}.$$

For $d \geq 1$ write

$$\mathcal{A}_d = \{a : ad \in \mathcal{A}\}, \qquad \mathcal{B}_d = \{b : bd \in \mathcal{B}\},$$

and let $S(\mathcal{E}, z)$ denote the number of elements of $\mathcal{E}$ having no prime factor below z. Finally put

$$\rho = X^{\beta-1}(2^\beta - 1).$$

For sufficiently large X, the values $\lfloor n^c \rfloor$ in the relevant range are distinct. It is enough to prove

$$S(\mathcal{A}, (2X)^{1/2}) \gg_c \frac{\rho X}{\log X}. \tag{1}$$

Indeed, $\rho X \asymp_c X^\beta$ and the index scale is $y \asymp X^\beta$.

We fix the endpoint

$$\gamma_* := \frac{500}{593}$$

and define

$$\theta_0 = 6\gamma_* - 5, \qquad \theta_1 = 1 - \gamma_*, \qquad \theta_2 = \frac{61\gamma_* - 49}{11},$$
$$\theta_3 = 3 - 3\gamma_*, \qquad \theta_4 = 3\gamma_* - 2, \qquad \theta_5 = \frac{60 - 61\gamma_*}{11}, \qquad \theta_6 = \gamma_*.$$

Thus

$$\begin{aligned} \theta_0 &= \frac{35}{593}, \quad \theta_1 = \frac{93}{593}, \quad \theta_2 = \frac{1443}{6523}, \\ \theta_3 &= \frac{279}{593}, \quad \theta_4 = \frac{314}{593}, \quad \theta_5 = \frac{5080}{6523}, \quad \theta_6 = \frac{500}{593}. \end{aligned} \tag{2}$$

Set

$$\mathcal{J} = [\theta_1, \theta_2] \cup [\theta_3, \theta_4] \cup [\theta_5, \theta_6]. \tag{3}$$

2 The Rivat–Wu analytic input

For a positive number p occurring in a dyadic prime range, write $p \asymp X^t$ and call t its logarithmic size. We shall use the following consequences of Rivat and Wu's Lemmas 3.3 and 3.4. Their intervals are

$$I_j(\beta, \varepsilon) = [X^{\vartheta_{2j-1}(\beta)+\varepsilon}, X^{\vartheta_{2j}(\beta)-\varepsilon}], \qquad j = 1, 2, 3,$$

where the functions $\vartheta_i(\beta)$ are given by the same formulas as θ_i, with γ_* replaced by β.

Lemma 2.1 (Type I input with a lowered sieving level)**.** *Let $\beta \in (37/44, 28/33)$ and fix $\varepsilon > 0$. Let $0 < \zeta \le \vartheta_0(\beta)$ be fixed. If a product d of a fixed number of prime variables satisfies*

$$d \le X^{\vartheta_4(\beta)-\varepsilon},$$

then, for the divisor-bounded coefficient systems arising below,

$$\sum_d b_d S(\mathcal{A}_d, X^\zeta) = \rho \sum_d b_d S(\mathcal{B}_d, X^\zeta) + o\left(\frac{X^\beta}{\log X}\right).$$

Proof. For $\zeta = \vartheta_0(\beta)$ this is Rivat–Wu's Lemma 3.3. Their proof is the alternative-sieve argument of Baker–Harman–Rivat, with the Type I distribution estimate supplied by Rivat–Wu's Lemma 3.1. The same proof applies for every smaller sieving level. Indeed, replace $P(X^{\vartheta_0(\beta)})$ in the upper and lower Rosser weights by $P(X^\zeta)$. Every divisor in the new weights is also an admissible divisor for the original level, and the distribution level $X^{\vartheta_4(\beta)-\varepsilon}/d$ is unchanged. In the fundamental-lemma estimate the parameter

$$s = \frac{\log D}{\log X^\zeta}$$

can only increase when ζ is decreased, so the sieve remainder does not worsen. The proof of Rivat–Wu's Lemma 3.3 therefore gives the displayed formula with the same power-saving distribution error and an $O(1/\log X)$ relative sieve error. Summing the fixed finite collection of coefficient systems gives the stated $o(X^\beta/\log X)$ form. □

Lemma 2.2 (Type II subproduct input)**.** *Let $\beta \in (37/44, 61/71)$ and fix $\varepsilon > 0$. Consider a sieve sum involving a fixed ordered tuple of prime variables $p_1, \dots, p_r$. If there is a nonempty subset $D \subseteq \{1, \dots, r\}$ such that*

$$\prod_{j \in D} p_j \in I_1(\beta, \varepsilon) \cup I_2(\beta, \varepsilon) \cup I_3(\beta, \varepsilon),$$

then the sum has the expected comparison between $\mathcal{A}$ and $\mathcal{B}$, with an $o(X^\beta / \log X)$ error. The subset D is allowed to be the whole set of fixed prime variables.

Remark 2.3. The correct formulation is in terms of a subproduct. It is unnecessary to split the fixed primes into two nonempty groups. This distinction matters when the product of all fixed prime variables lies in a Type II interval.

For $\beta > \gamma_*$, each fixed interval in (3) is contained with a positive margin in the corresponding Rivat–Wu interval at β, the fixed threshold θ_4 is smaller than $\vartheta_4(\beta)$, and $\theta_0 < \vartheta_0(\beta)$. Lemma 2.1 therefore permits the fixed lower sieving level X^{θ_0} throughout the open range. The endpoint $\beta = \gamma_*$ is treated in Section 6 by shrinking the Type I and Type II upper regions.

3 Subproduct partitionability and the exact-cover region

Definition 3.1. A tuple $\mathbf{t} = (t_1, \dots, t_r)$ is called $\mathcal{J}$-partitionable if some nonempty subset sum

$$\sum_{j \in D} t_j, \qquad \emptyset \neq D \subseteq \{1, \dots, r\},$$

belongs to $\mathcal{J}$. Let $P_r(\mathbf{t})$ be its indicator.

For two variables we write

$$\mathcal{I}(m,n): \quad m \in \mathcal{J} \quad \text{or} \quad n \in \mathcal{J} \quad \text{or} \quad m+n \in \mathcal{J}. \tag{4}$$

This is the symmetric form of Li's region I.

For

$$T(m,n) = \min\left(n, \frac{1-m-n}{2}\right),$$

define the exact-cover region

$$\begin{aligned} (m,n) \in \widetilde{\mathcal{I}}_2 \quad \iff \quad & m+n < \theta_4, \\ & [\theta_0, T(m,n)] \subseteq \bigcup_{s \in \{0, m, n, m+n\}} (\mathcal{J} - s). \end{aligned} \tag{5}$$

An interval whose upper endpoint is below its lower endpoint is empty. The covering condition says that for every $t \in [\theta_0, T(m,n)]$, at least one of

$$t, \qquad m+t, \qquad n+t, \qquad m+n+t$$

belongs to $\mathcal{J}$.

Lemma 3.2 (Finite polyhedral structure)**.** *The set $\widetilde{\mathcal{I}}_2$ is a finite union of relatively open and closed polygons cut out by affine inequalities with rational coefficients. The same is true if each component $[a,b]$ of $\mathcal{J}$ is replaced by $[a+\eta, b-\eta]$ and θ_4 is replaced by $\theta_4 - \eta$.*

Proof. The twelve translated intervals $J_j - s$, where J_j runs over the three components of $\mathcal{J}$ and $s \in \{0, m, n, m+n\}$, have affine endpoints in (m, n). Whether they cover $[\theta_0, T(m, n)]$ is determined by the ordering of these finitely many endpoints and by a finite list of overlap inequalities. There are only finitely many orderings and overlap graphs. On each resulting cell, the covering condition is a finite conjunction of affine inequalities. □

Lemma 3.3 (Exact-cover asymptotic region)**.** *Let K be a compact polygon contained in the interior of $\widetilde{\mathcal{I}}_2$, with all Type I and Type II interval endpoints separated from the relevant logarithmic sizes by a fixed positive amount. Then*

$$\sum_{(t_1,t_2)\in K} S(\mathcal{A}_{p_1p_2}, p_2) = \rho \sum_{(t_1,t_2)\in K} S(\mathcal{B}_{p_1p_2}, p_2) + o\left(\frac{X^\beta}{\log X}\right),$$

where the notation denotes the usual finite subdivision into short logarithmic prime boxes.

Proof. Buchstab's identity gives

$$S(\mathcal{A}_{p_1p_2}, p_2) = S(\mathcal{A}_{p_1p_2}, X^{\theta_0}) - \sum_{\theta_0 \le t_3 < T(t_1,t_2)} S(\mathcal{A}_{p_1p_2p_3}, p_3). \tag{6}$$

The first term is covered by Lemma 2.1, since $t_1 + t_2 < \theta_4$.

For every point in the second sum, (5) gives an $s \in \{0, t_1, t_2, t_1+t_2\}$ such that $t_3 + s \in \mathcal{J}$. This is a nonempty subset sum of t_1, t_2, t_3, so Lemma 2.2 applies.

It remains to justify that the subset can vary with the point. Insert all hyperplanes

$$t_3 + s = \theta_j, \qquad s \in \{0, t_1, t_2, t_1+t_2\}, \quad 1 \le j \le 6.$$

They divide the compact (t_1, t_2, t_3)-domain into finitely many polytopes. On the interior of each polytope, membership of every candidate subset sum in each component of $\mathcal{J}$ is constant. Since the translated intervals cover the full t_3-range, one fixed subset works on each polytope. Apply Lemma 2.2 separately and sum the finitely many asymptotic formulas. Boundary faces have zero Buchstab measure and can alternatively be removed before applying the estimates. □

Li's region $\mathcal{I}_2$ is contained in $\widetilde{\mathcal{I}}_2$: each of Li's alternatives uses one translated component to cover the full third-prime interval. The new condition also permits different translated components to cover successive portions of that interval, and it permits the third prime itself to lie in a Type II interval.

4 The modified Buchstab decomposition

Let

$$D = \left\{(t_1, t_2) : \theta_0 \le t_1 < \frac{1}{2}, \quad \theta_0 \le t_2 < \min\left(t_1, \frac{1-t_1}{2}\right)\right\}. \tag{7}$$

Define disjoint residual regions on D by

$$\widetilde{U}_1 = \{(t_1, t_2) \notin \mathcal{I} \cup \widetilde{\mathcal{I}}_2 : t_1 + 2t_2 < \theta_4\}, \tag{8}$$

$$\widetilde{U}_2 = \{(t_1, t_2) \notin \mathcal{I} \cup \widetilde{\mathcal{I}}_2 : t_1 + 2t_2 \ge \theta_4,\ (1 - t_1 - t_2)/t_2 < 2\}, \tag{9}$$

$$\widetilde{U}_3 = \{(t_1, t_2) \notin \mathcal{I} \cup \widetilde{\mathcal{I}}_2 : t_1 + 2t_2 \ge \theta_4,\ (1 - t_1 - t_2)/t_2 \ge 2\}. \tag{10}$$

The following proposition records the bookkeeping, including the signs.

Proposition 4.1 (Loss identity). *After replacing Li's $\mathcal{I}_2$ by $\widetilde{\mathcal{I}}_2$, the same Buchstab identities give*

$$S(\mathcal{A}, (2X)^{1/2}) \geq (1 - \mathcal{L} + o(1)) \frac{\rho X}{\log X}, \tag{11}$$

where

$$\mathcal{L} = L_{15} + L_2 + L_3 - L_{32} - L_{34}. \tag{12}$$

The five terms are the integrals defined below.

Proof. Apply Buchstab's identity twice to the initial sieve sum. The two-prime region D splits into the asymptotic part $\mathcal{I} \cup \widetilde{\mathcal{I}}_2$ and the three residual regions (8)–(10). The asymptotic part is evaluated by Lemmas 2.1, 2.2 and 3.3.

On $\widetilde{U}_1$, two further Buchstab steps give Li's signed decomposition

$$S_{U_1} = S_{U_{11}} - S_{U_{12}} - S_{U_{13}} + S_{U_{14}} + S_{U_{15}}.$$

The first four terms are Type I or Type II terms. The final positive term is discarded, producing the loss L_{15}.

The whole positive $\widetilde{U}_2$ term is discarded, producing L_2. On $\widetilde{U}_3$, reverse Buchstab twice gives

$$S_{U_3} = S_{U_{31}} + S_{U_{32}} + S_{U_{33}} + S_{U_{34}} + S_{U_{35}}.$$

The pieces U_{32} and U_{34} are Type II pieces and are retained. If the whole U_3 region is first charged as a loss L_3, retaining these pieces subtracts L_{32} and L_{34} from that loss.

Applying the same exact identities to $\mathcal{B}$, and using the standard Buchstab asymptotic formula, shows that the normalized mass before discarding is 1. This proves (11) and (12). $\square$

We now give the exact domains. Let

$$\begin{aligned} E_{15} = \{(t_1, t_2, t_3, t_4) : (t_1, t_2) \in \widetilde{U}_1,& \\ \theta_0 \leq t_3 < t_2, \quad & P_3(t_1, t_2, t_3) = 0, \\ \theta_0 \leq t_4 < t_3, \quad & P_4(t_1, t_2, t_3, t_4) = 0\}. \end{aligned}$$

The apparent upper bounds in the raw Buchstab identity are $\min(t_2, (1 - t_1 - t_2)/2)$ and $\min(t_3, (1 - t_1 - t_2 - t_3)/2)$. On $\widetilde{U}_1$ they equal t_2 and t_3. To see this, put $s = t_1 + 2t_2$. Since $t_2 \leq t_1$, one has $t_2 \leq s/3$. Hence

$$t_1 + 3t_2 = s + t_2 \leq \frac{4}{3}s < \frac{4}{3}\theta_4 < 1.$$

Also $t_3 < t_2$ gives

$$t_1 + t_2 + 3t_3 < t_1 + 4t_2 = s + 2t_2 \leq \frac{5}{3}s < \frac{5}{3}\theta_4 < 1.$$

The last two strict inequalities follow directly from $\theta_4 = 314/593$. Define

$$L_{15} = \int_{E_{15}} \frac{\omega((1 - t_1 - t_2 - t_3 - t_4)/t_4)}{t_1 t_2 t_3 t_4^2} \, dt_1 dt_2 dt_3 dt_4. \tag{13}$$

For $i = 2, 3$, put

$$L_i = \int_D \mathbf{1}_{\widetilde{U}_i}(t_1, t_2) \frac{\omega((1 - t_1 - t_2)/t_2)}{t_1 t_2^2} \, dt_2 dt_1. \tag{14}$$

On $\widetilde{U}_2$ the Buchstab argument lies in $(1, 2)$, so the integrand equals $1/[t_1 t_2 (1 - t_1 - t_2)]$.

Let

$$E_{32} = \{(t_1, t_2, t_3) : (t_1, t_2) \in \widetilde{U}_3, \quad t_2 < t_3 < (1 - t_1 - t_2)/2,$$
$$P_3(t_1, t_2, t_3) = 1\},$$

and

$$E_{34} = \{(t_1, t_2, t_3, t_4) : (t_1, t_2) \in \widetilde{U}_3, \quad t_2 < t_3 < (1 - t_1 - t_2)/2,$$
$$P_3(t_1, t_2, t_3) = 0,$$
$$t_3 < t_4 < (1 - t_1 - t_2 - t_3)/2,$$
$$P_4(t_1, t_2, t_3, t_4) = 1\}.$$

Define

$$L_{32} = \int_{E_{32}} \frac{\omega((1 - t_1 - t_2 - t_3)/t_3)}{t_1 t_2 t_3^2} \, dt_1 dt_2 dt_3, \tag{15}$$

and

$$L_{34} = \int_{E_{34}} \frac{\omega((1 - t_1 - t_2 - t_3 - t_4)/t_4)}{t_1 t_2 t_3 t_4^2} \, dt_1 dt_2 dt_3 dt_4. \tag{16}$$

Remark 4.2 (What is and is not imported from Li)**.** Li's paper supplies the Buchstab identities and the useful idea of the additional two-prime region. The numerical inequalities claimed there are not used. Proposition 4.1 is an exact algebraic Buchstab decomposition, while all five numerical inequalities needed here are certified anew in Section 5. Thus the present endpoint does not depend on the precision of Li's numerical quadrature.

5 The validated numerical certificate

5.1 Buchstab bounds

The Buchstab function is determined by

$$\omega(u) = \frac{1}{u} \quad (1 \le u \le 2), \qquad u\omega(u) = 1 + \int_1^{u-1} \omega(v) \, dv \quad (u \ge 2). \tag{17}$$

Every Buchstab argument in (13)–(16) belongs to $[1, 593/35] \subset [1, 17]$. A separate directed Riemann recursion on a mesh of width 10^{-4} proves

$$\omega(u) \le 0.5672 \quad (2 \le u \le 17), \qquad 0.56 \le \omega(u) \le 0.565 \quad (3 \le u \le 17). \tag{18}$$

The raw certified extrema are

$$\min_{3 \le u \le 17} \omega(u) > 0.5607947889826400114,$$
$$\max_{2 \le u \le 17} \omega(u) < 0.5671716488762554788,$$
$$\max_{3 \le u \le 17} \omega(u) < 0.5644073953949818747.$$

On $[2, 3]$ the integration program uses

$$\omega(u) = \frac{1 + \log(u - 1)}{u}.$$

For $x \in [1, 2]$ it encloses the logarithm by

$$\log x = 2 \sum_{k=0}^{24} \frac{z^{2k+1}}{2k + 1} + R, \qquad z = \frac{x - 1}{x + 1},$$

with

$$0 \le R \le \frac{2z^{51}}{51(1-z^2)}.$$

Thus no system transcendental function is used in the certificate.

5.2 Interval arithmetic and domains

The computation uses nonnegative intervals and the four arithmetic operations only. Each hardware result is enlarged by one call to `nextafterl` toward the appropriate infinity. The audited platform has

$$\texttt{sizeof(long double)} = 16, \quad \texttt{radix} = 2, \quad \texttt{digits} = 64, \quad \texttt{is\_iec559} = 1.$$

The programs are compiled without fast-math or reassociation.

Lemma 5.1 (Outward arithmetic)**.** *Under the stated IEC 60559 binary80 arithmetic model, every interval returned by the elementary routines in the certificate contains the exact real result of the corresponding operation on the input intervals.*

Proof. For each basic operation, the hardware result is a correctly rounded representable value. Moving one representable number toward $-\infty$ or $+\infty$ therefore lies on the required side of the exact result. Since all divisors are strictly positive and all products are nonnegative, the standard monotone interval formulas apply. Induction over the expression tree proves the assertion. □

The outer region D is split at $t_1 = 1/3$ and mapped to two unit squares:

$$\begin{aligned} &\theta_0 \le t_1 \le 1/3, \quad && \theta_0 \le t_2 \le t_1, \\ &1/3 \le t_1 \le 1/2, \quad && \theta_0 \le t_2 \le (1-t_1)/2. \end{aligned}$$

For L_{15} and L_{34} a third unit coordinate parametrizes t_3; the last variable is integrated analytically over certified unions of intervals.

For a parent box, the program evaluates all nonempty subset-sum intervals. A box is labelled partitionable only when one subset-sum interval lies wholly inside one component of $\mathcal{J}$, and nonpartitionable only when every subset-sum interval is disjoint from $\mathcal{J}$. Otherwise it remains undecided. The exact-cover predicate is checked by a graph on the twelve translated components $J_j - s$. A definite edge uses the intersection common to all points of the parent box; a possible edge uses the union hull. Hence an inside declaration is valid for every point and an outside declaration is valid for every point.

For the final variable t, the program constructs intervals on which a subset sum lies in $\mathcal{J}$. For lower bounds it uses only intervals certified to lie in the required domain; for upper bounds it includes every interval that can meet the domain. The factor t^{-2} is integrated exactly. When the Buchstab argument is at most 2, the integrand has the primitive

$$\int_a^b \frac{dt}{t(R-t)} = \frac{1}{R} \log \frac{b(R-a)}{a(R-b)}.$$

All one-dimensional subdivisions use shared, nonoverlapping floating-point partition endpoints.

The adaptive upper/lower sums retain undecided boxes in upper bounds and omit them from lower bounds. Consequently, termination is not needed for correctness; it is needed only to make the enclosure narrow enough to prove the desired rational inequality.

Proposition 5.2 (Certified numerical inequalities)**.** *At the parameters* (2),

$$L_{15} < \frac{9}{2000} = 0.0045, \qquad L_2 < \frac{107}{250} = 0.428, \qquad L_3 < \frac{4629}{5000} = 0.9258, \tag{19}$$

$$L_{32} > \frac{3331}{10000} = 0.3331, \qquad L_{34} > \frac{16}{625} = 0.0256. \tag{20}$$

Proof. The directed endpoints printed by the programs are

$$\begin{aligned} L_{15} &< 0.0044999998971117250903451367967 4, \\ L_2 &< 0.42799998685086350128071140674 9, \\ L_3 &< 0.92579999992207736290213831997 8, \\ L_{32} &> 0.33316783305341474377583681210 1, \\ L_{34} &> 0.02560000024749415357936429930 85. \end{aligned}$$

The programs compare these quantities internally against outward-rounded representatives of the rational targets. The independent Python verifier parses the decimal outputs and checks the five comparisons using exact arbitrary-precision decimal arithmetic. The source and output hashes are listed in Appendix A. □

It follows that

$$\mathcal{L} < \frac{9}{2000} + \frac{107}{250} + \frac{4629}{5000} - \frac{3331}{10000} - \frac{16}{625} = 0.9996. \tag{21}$$

Thus

$$1 - \mathcal{L} > 0.0004. \tag{22}$$

Using the printed endpoints gives the stronger certified margin

$$1 - \mathcal{L} > 0.000467846630856308082006247885 76.$$

6 Stability at the endpoint

For $\eta > 0$, replace each component $[a, b]$ of $\mathcal{J}$ by $[a + \eta, b - \eta]$, replace the Type I condition $m < \theta_4$ by $m < \theta_4 - \eta$, and define all partitionability, exact-cover and residual regions with these shrunken data. Denote the resulting loss by $\mathcal{L}(\eta)$.

Lemma 6.1 (Boundary stability)**.** *One has*

$$\lim_{\eta \downarrow 0} \mathcal{L}(\eta) = \mathcal{L}(0).$$

Proof. By Lemma 3.2, every domain occurring in the five integrals is a finite union of sets described by affine inequalities whose constants vary continuously with η. Away from the finite union of limiting boundary hyperplanes, every strict inequality and every interval-overlap relation is unchanged for all sufficiently small η. Hence each domain indicator converges pointwise almost everywhere to its value at $\eta = 0$.

All integration variables are bounded below by $\theta_0 = 35/593$. The Buchstab function is bounded on $[1, 17]$, so each of the five integrands is bounded by an integrable constant on its compact ambient domain. Dominated convergence applies separately to all five integrals, including the two recovered terms with negative signs in (12). This proves the claim. □

7 Proof of Theorem 1.1

Suppose first that $1 < c < 593/500$. Then $\beta > \gamma_*$. The fixed Type II intervals (3) and the fixed Type I threshold θ_4 have a positive margin inside the intervals allowed by Rivat and Wu at β. Moreover $\theta_0 < \vartheta_0(\beta)$, so the lowered-level form of Lemma 2.1 applies with $\zeta = \theta_0$. Proposition 4.1, Proposition 5.2 and (22) give, for sufficiently large X,

$$S(\mathcal{A}, (2X)^{1/2}) \geq 0.0002 \, \frac{\rho X}{\log X} > 0,$$

where half of the certified margin absorbs all asymptotic errors. This proves (1).

Now let $c = 593/500$, so $\beta = \gamma_*$. Choose $\eta > 0$ so small that

$$\mathcal{L}(\eta) < 0.9998,$$

which is possible by Lemma 6.1 and (21). The shrunken Type II intervals and the threshold $\theta_4 - \eta$ satisfy the strict ε-margins required by Lemmas 2.1 and 2.2. Repeating the decomposition with the shrunken regions gives

$$S(\mathcal{A}, (2X)^{1/2}) \geq (1 - \mathcal{L}(\eta) + o(1)) \, \frac{\rho X}{\log X} > 0$$

for sufficiently large X. Thus (1) also holds at the endpoint. For a given sufficiently large y, take $X = y^c/2$. Then every index counted in the dyadic block $X \leq n^c < 2X$ satisfies $n \leq y$, while $X^\beta \asymp_c y$ and $\log X \asymp_c \log y$. Hence the dyadic lower bound implies

$$\pi_c(y) \gg_c \frac{y}{\log y}.$$

This completes the proof.

Acknowledgment

To improve the Rivat–Wu result is a recent suggestion of Prof. Hongze Li.

A Certificate files and audit data

The certificate consists of the following source files.

- `li1186_finalcert.cpp`: adaptive enclosures for L_{15}, L_2, L_3, L_{34};
- `l32_grid_omp.cpp`: fixed-grid lower enclosure for L_{32};
- `omega_bound_certificate.cpp`: independent Buchstab recursion;
- `verify_li1186_certificate.py`: exact decimal verification of the five rational comparisons.

The compilation commands are

```
g++ -O3 -std=c++17 -fno-fast-math -frounding-math \
    li1186_finalcert.cpp -o li1186_finalcert
g++ -O3 -std=c++17 -fno-fast-math -frounding-math -fopenmp \
    l32_grid_omp.cpp -o l32_grid_omp
g++ -O3 -std=c++17 -fno-fast-math -frounding-math \
    omega_bound_certificate.cpp -o omega_bound_certificate
```

The raw output files included in the package are the outputs used in Proposition 5.2.

File	SHA-256
`li1186_finalcert.cpp`	`b4c2bc8588468bfd69ce6b4a36b7a4eaeffda4651cfc5b154a0a5a9c7c098e35`
`l32_grid_omp.cpp`	`a528f236aa3fc616d9baec5af6857e18d0cf0ae2f418b5e7de8db84ba1ef0951`
`omega_bound_certificate.cpp`	`86b0608859743e16896a7d6055753b39f29a7879490ab1a0b64807ce56553cca`
`final_L2.out`	`c320b2a1a283fa93e4fee9fb54d919b63e59da54ddd5a5fb460f1b833f04e2ac`
`final_L3.out`	`1e283f7dbc7b03cb58faffdf427938bb8ca227673c23a67f35d19a55dd200ef1`
`final_L15.out`	`b563e4d08ccee0f6911f9abb3ef6323a26443d80148064bca32a60cccdde899e`
`final_L32.out`	`b72158369b086b50e1525b114b660d125d076547ba63021baf5391e2a21c884e`
`final_L34.out`	`4a91e70f433e82434db6f59378843bbec29b92557a5c6e3f3cc10a61c63bb536`
`omega_bound_certificate.out`	`7b463e2c6fa882f73ddac04cec0b442d103d36ac8a79b9dd3eb871a1359d444d`
`verify_li1186_certificate.py`	`59e82ef38aeb64701af782ceb9e3bf263a85cd304463369e71127657ec99ca29`

An independent rebuild during the final audit reproduced exactly the Buchstab output and the L_2 and L_3 endpoints. The longer L_{15}, L_{32} and L_{34} runs are supplied as reproducible source-plus-output certificates; the hashes above identify the precise audited versions. The exact-decimal verifier recomputes all rational comparisons from the raw outputs.

B References

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