Polygonal Numbers, Primes and Ternary Quadratic Forms

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Modern number theory has grown out of the early work of Fermat, Euler and Goldbach.

However, we still can get new inspirations from some classical observations of Fermat, Euler and Goldbach, and raise several difficult problems which challenge the modern number theory and appeal to new powerful methods.

Classical Results on Sums of Squares

Fermat-Euler Theorem. A prime p can be written in the form $x^2 + y^2$ with $x, y \in \mathbb{Z}$ if and only if $p \not\equiv 3 \pmod{4}$.

Lagrange's Theorem. Every $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as the sum of 4 squares.

Method. Fermat's infinite decent.

Gauss-Legendre Theorem. $n \in \mathbb{N}$ has the form $x^2 + y^2 + z^2$ if and only if n is *not* of the form $4^k(8\ell + 7)$ with $k, \ell \in \mathbb{N}$.

Tool. The theory of quadratic forms.

Sums of Triangular Numbers

A triangular number has the form $\sum_{k=0}^{n} k = n(n+1)/2$. Here are the initial 10 triangular numbers:

$$0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45.$$

Let $T_x = x(x+1)/2$. Clearly

$$T_{-x-1} = T_x, \quad \text{and} \quad 8T_x + 1 = (2x+1)^2.$$

Fermat's Assertion. Any $n \in \mathbb{N}$ is the sum of three triangular numbers.

Fermat's assertion follows from the Gauss-Legendre theorem, for,

$$n = T_x + T_y + T_z \iff 8n + 3 = (2x+1)^2 + (2y+1)^2 + (2z+1)^2.$$

Sums of Triangular Numbers

Liouville's Theorem (Liouville, 1862). Let $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b \leqslant c$. Then any $n \in \mathbb{N}$ can be written in the form $aT_x + bT_y + cT_z$ if and only if (a, b, c) is among the following triples:

$$(1,1,1),\ (1,1,2),\ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4).$$

Mixed Sums of Squares and Triangular Numbers

Euler's Observation:

$$8n+1=(2x)^2+(2y)^2+(2z+1)^2$$
$$\Longrightarrow n=\frac{x^2+y^2}{2}+T_z=\left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2}\right)^2+T_z.$$

Lionnet's Assertion (proved by Lebesgue & Réalis in 1872). Any $n\in\mathbb{N}$ is the sum of two triangular numbers and a square.

B. W. Jone and G. Pall [Acta Math. 1939]. Every $n\in\mathbb{N}$ is the sum of a square, an *even* square and a triangular number.

Theorem (i) [Z. W. Sun, Acta Arith. 2007] Any $n\in\mathbb{N}$ is the sum of an *even* square and two triangular numbers.
(ii) (Conjectured by Z. W. Sun and proved by B. K. Oh and Sun [JNT, 2009]). Any positive integer n can be written as the sum of a square, an *odd* square and a triangular number.

A Characterization of Primes Congruent to 3 mod 4

In number theory there are very few simple characterizations of primes such as Wilson's theorem. The following result provides a surprising new criterion for primes congruent to 3 mod 4.

Theorem (B. K. Oh and Sun [JNT, 2009]). $p = 2m + 1 > 1$ is a prime congruent to 3 modulo 4 if and only if $T_m = m(m+1)/2$ cannot be expressed as a sum of two odd squares and a triangular number, i.e., $p^2 = x^2 + 8(y^2 + z^2)$ for no odd integers x, y, z.

Mixed Sums of Squares and Triangular Numbers

In 2005 Z. W. Sun [Acta Arith. 2007] investigated what kind of mixed sums $ax^2+by^2+cT_z$ or $ax^2+bT_y+cT_z$ (with $a,b,c\in\mathbb{Z}^+$) are universal (i.e., all natural numbers can be so represented). This project was completed via three papers: Z. W. Sun [Acta Arith. 2007], S. Guo, H. Pan & Z. W. Sun [Integers, 2007], and B. K. Oh & Sun [JNT, 2009].

List of all universal $ax^2+by^2+cT_z$ **or** $ax^2+bT_y+cT_z$:

$T_x+T_y+z^2,\ T_x+T_y+2z^2,\ T_x+T_y+4z^2,\ T_x+2T_y+z^2,$
$T_x+2T_y+2z^2,\ T_x+2T_y+3z^2,\ T_x+2T_y+4z^2,\ 2T_2T_y+z^2,$
$2T_x+4T_y+z^2,\ 2T_x+5T_y+z^2,\ T_x+3T_y+z^2,\ T_x+4T_y+z^2,$
$T_x+4T_y+2z^2,\ T_x+6T_y+z^2,\ T_x+8T_y+z^2,\ T_x+y^2+z^2,$
$T_x+y^2+2z^2,\ T_x+y^2+3z^2,\ T_x+y^2+4z^2,\ T_x+y^2+8z^2,$
$T_x+2y^2=2z^2,\ T_x=2y^2+4z^2,\ 2T_x+y^2+z^2,\ 2T_x+y^2+2y^2,$
$4T_x+y^2+2z^2.$

Mixed Sums of Squares and Triangular Numbers

Ken Ono and K. Soundararajan [Invent. Math. 130(1997)]: Under GRH (the generalized Riemann hypothesis), we have Ramanujan's assertion that any positive odd integer greater than 2719 can be represented by the form $x^2 + y^2 + 10z^2$.

$$2n + 1 = (2x)^2 + (2y + 1)^2 + 10z^2 \iff n = 2x^2 + 4T_y + 5z^2.$$

Another Version of the Ono-Soundararajan Theorem. Under GRH, $2x^2 + 5y^2 + 4T_z$ represents all integers greater than 1359.

B. Kane and Z. W. Sun [arXiv:0808.2761] determined completely when the general form $ax^2 + by^2 + cT_z$ $(a, b, c \in \mathbb{Z}^+)$ represents sufficiently large integers and established similar results for the forms $ax^2 + bT_y + cT_z$ and $aT_x + bT_y + cT_z$. In particular, *the form $ax^2 + y^2 + T_z$ represents sufficiently large integers if and only if each odd prime divisor of a is congruent to 1 or 3 modulo 8.*

Mixed Sums of Primes and Triangular Numbers

Z. W. Sun [Acta Arith., 2007]. A positive integer n can be written as the sum of an odd square, an even square and a triangular number unless it is of the form T_m ($m > 0$) with all the prime divisors of $2m+1$ congruent to 1 mod 4.

Motivated by this result and the fact that any prime $p \equiv 1 \pmod 4$ is the sum of an odd square and an even square, Sun formulated the following conjecture.

Conjecture on Sums of Primes and Triangular Numbers (Z. W. Sun, 2008). (i) Any $n \in \mathbb{N}$ with $n \neq 216$ can be written as $p + T_x$ with $x \in \mathbb{Z}$, where p is zero or a prime.

(ii) Any odd integer $n > 3$ can be written in the form $p + 2T_x = p + x(x+1)$ with p an odd prime and $x \in \mathbb{Z}^+$.

Remark. Parts (i) and (ii) were verified for n up to 10^{12} by T. D. Noe and D. S. McNeil respectively. Note that triangular numbers are more sparse than prime numbers since the n-th prime is about $n \log n$ while $T_n \sim n^2/2$. \$1000 prize for a proof of the conjecture.

General Conjecture on Sums of Primes and Triangular Numbers (Z. W. Sun, 2008). Let $a, b \in \mathbb{N}$ and $r \in \{1, 3, 5, \ldots\}$. Then all sufficiently large integers can be written in the form $2^a p + T_x$ with $x \in \mathbb{Z}$, where p is either zero or a prime congruent to r mod 2^b. Also, all sufficiently large *odd* numbers can be written in the form $p + x(x+1)$ with $x \in \mathbb{Z}$, where p is a prime congruent to r mod 2^b.

Examples (suggested by Sun in 2008 and verified for $n \leqslant 10^{11}$):

(1) Any integer $n > 88956$ can be written in the form $p + T_x$ with $x \in \mathbb{Z}^+$, where p is either zero or a prime congruent to 1 mod 4.
(2) Each integer $n > 90441$ can be written in the form $p + T_x$ with $x \in \mathbb{Z}^+$, where p is either zero or a prime congruent to 3 mod 4.
(3) Except for 30 multiples of three (the largest of which is 49755), odd integers larger than one can be written in the form $p + x(x+1)$ with $x \in \mathbb{Z}$, where p is a prime congruent to 1 mod 4.
(4) Except for 15 multiples of three (the largest of which is 5397), odd numbers greater than one can be written in the form $p + x(x+1)$ with $x \in \mathbb{Z}$, where p is a prime congruent to 3 mod 4.

Dense Subsets of the Ring of p-adic Integers

A Special Property of Triangular Numbers. For any $a \in \mathbb{Z}^+$, $\{T_n = \binom{n+1}{2} : n \in \mathbb{N}\}$ contains a complete system of residues mod 2^a.

Z. W. Sun and W. Zhang [arXiv:0812.3089]. Let p be a prime and let $p^a \leqslant k < 2p^a$ where $a \in \mathbb{Z}^+$. Then

$$\left\{ \binom{n}{k} : n = 0, 1, 2, \ldots \right\}$$

is *dense* in the ring $\mathbb{Z}_p$ of p-adic integers, i.e., it contains a complete system of residues modulo any power of p.

Polygonal Numbers

Polygonal numbers are nonnegative integers constructed geometrically from the regular polygons. For $m = 3, 4, 5, \ldots$, the m-gonal numbers are given by

$$p_m(n) = (m-2)\binom{n}{2} + n \quad (n = 0, 1, 2, \ldots).$$

$$p_m(0) = 0,\ p_m(1) = 1,\ p_m(2) = m,\ p_m(3) = 3m{-}3,\ p_m(4) = 6m{-}8$$

$$p_3(n) = T_n,\ p_4(n) = n^2,\ p_5(n) = \frac{3n^2 - n}{2},\ p_6(n) = 2n^2{-}n = T_{2n-1}.$$

The larger m is, the more sparse m-gonal numbers are.

Euler's Discovery:

$$\frac{1}{\sum_{n=0}^{\infty} p(n)q^n} = \prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{p_5(k)} \quad (|q| < 1),$$

where $p(n)$ $(n = 1, 2, 3, \ldots)$ is the number of ways to write n as the sum of positive integers (repetition allowed) and $p(0) := 1$.

Fermat's Assertion

Fermat's Assertion (1638). Any natural number n can be written as the sum of m m-gonal numbers.

Remark. Recall that

$$p_m(\mathbb{N}) = \{0,\ 1,\ m,\ 3m-3, \ldots\}.$$

For $k < m-1$ we cannot express $m-1$ as the sum of k m-gonal numbers. Also, $2m-1$ cannot be the sum of $m-1$ m-gonal numbers. For $m = 5, 6, \ldots$, we have $p_m(\mathbb{N}) \neq p_m(\mathbb{Z})$ since $p_m(-1) = m-3 \notin p_m(\mathbb{N})$.

Those $p_5(n) = n(3n-1)/2$ ($n \in \mathbb{N}$) are called *pentagonal numbers*. Those $p_6(n) = n(2n-1)$ ($n \in \mathbb{N}$) are called *hexagonal numbers*.

In 1813 Cauchy proved Fermat's Assertion for $m \geqslant 5$.

Cauchy's Lemma. For odd integers $a, b \in \mathbb{Z}^+$ with $b^2 < 4a$ and $3a < b^2 + 2b + 4$, there exist $s, t, u, v \in \mathbb{N}$ such that $a = s^2 + t^2 + u^2 + v^2$ and $b = s + t + u + v$.

Diagonal Representations by Polygonal Numbers

$$n = \underline{p_3(x_1) + p_3(x_2) + p_3(x_3)}$$
$$n = \underline{p_4(x_1)} + p_4(x_2) + p_4(x_3) + p_4(x_4)$$
$$n = p_5(x_1) + \underline{p_5(x_2)} + p_5(x_3) + p_5(x_4) + p_5(x_5)$$
$$n = p_6(x_1) + p_6(x_2) + \underline{p_6(x_3)} + p_6(x_4) + p_6(x_5) + p_6(x_6)$$

Diagonal Representation:

$$n = \underline{p_4(x_1)} + \underline{p_5(x_2)} + \underline{p_6(x_3)}$$

Conjecture [Z. W. Sun, arxiv: 0905.0635, 2009]. Any $n \in \mathbb{N}$ can be written as the sum of a square, a pentagonal number and a hexagonal number. Also, we can write each $n \in \mathbb{N}$ as the sum of two squares and a pentagonal number, and as the sum of a triangular number, an even square and a pentagonal number.

Diagonal Representations by Polygonal Numbers

$$\begin{aligned}
n =& \underline{p_{m+1}(x_1)} + p_{m+1}(x_2) + p_{m+1}(x_3) + \cdots + p_{m+1}(x_{m+1}) \\
n =& p_{m+2}(x_1) + \underline{p_{m+2}(x_2)} + p_{m+2}(x_3) + \cdots\cdots + p_{m+2}(x_{m+2}) \\
n =& p_{m+3}(x_1) + p_{m+3}(x_2) + \underline{p_{m+3}(x_3)} + \cdots\cdots\cdots + p_{m+3}(x_{m+3}) \\
& \cdots\cdots\cdots\cdots \\
n =& p_{2m}(x_1) + p_{2m}(x_2) + p_{2m}(x_3) + \cdots\cdots + \underline{p_{2m}(x_m)} + \cdots + p_{2m}(x_{2m})
\end{aligned}$$

Diagonal Representation:

$$n = \underline{p_{m+1}(x_1)} + \underline{p_{m+2}(x_2)} + \underline{p_{m+3}(x_3)} + \cdots + \underline{p_{2m}(x_m)}$$

Conjecture on Diagonal Representations [Z. W. Sun, August 12, 2009]. Let $m \geqslant 3$ be an integer. Then any $n \in \mathbb{N}$ can be written in the form

$$p_{m+1}(x_1) + \cdots + p_{2m}(x_m) \quad \text{with } x_1, \ldots, x_m \in \mathbb{N}.$$

Strong Version of the Conjecture

Conjecture [Z. W. Sun, August 21, 2009]. Let $m \geqslant 3$ be an integer. Then any $n \in \mathbb{N}$ can be written in the form

$$p_{m+1}(x_1) + p_{m+2}(x_2) + p_{m+3}(x_3) + r$$

with $x_1, x_2, x_3 \in \mathbb{N}$ and $r \in \{0, \ldots, m-3\}$.

Remark. Clearly any $r \in \{0, \ldots, m-3\}$ can be written as $p_{m+4}(x_4) + \cdots + p_{2m}(x_m)$ with $x_4, \ldots, x_m \in \{0, 1\}$.

Verification of the Conjecture. $m = 3$ and $n \leqslant 10^6$; $4 \leqslant m \leqslant 10$ and $n \leqslant 5 \times 10^5$; $10 < m \leqslant 40$ and $n \leqslant 10^5$.

Prize: \$500 for the first rigorous proof of the full conjecture.

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Remark. Clearly any $r \in \{0, \ldots, m-3\}$ can be written as $p_{m+4}(x_4) + \cdots + p_{2m}(x_m)$ with $x_4, \ldots, x_m \in \{0, 1\}$.

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A Related Conjecture [Z. W. Sun, August 14, 2009]. For each $m = 3, 4, \ldots$ all sufficiently large integers have the form

$$p_{m+1}(x_1) + p_{m+2}(x_2) + p_{m+3}(x_3) \quad (x_1, x_2, x_3 \in \mathbb{N}).$$

Mixed Sums of Three Polygonal Numbers

Conjecture [Z. W. Sun, arxiv: 0905.0635, 2009]. Let $3 \leqslant i \leqslant j \leqslant k$ and $k \geqslant 5$. Then each $n \in \mathbb{N}$ can be written as the sum of an i-gonal number, a j-gonal number and a k-gonal number, if and only if (i, j, k) is among the following 31 triples:

$$(3,3,5),\ (3,3,6),\ (3,3,7),\ (3,3,8),\ (3,3,10),\ (3,3,12),\ (3,3,17),$$
$$(3,4,5),\ (3,4,6),\ (3,4,7),\ (3,4,8),\ (3,4,9),\ (3,4,10),\ (3,4,11),$$
$$(3,4,12),\ (3,4,13),\ (3,4,15),\ (3,4,17),\ (3,4,18),\ \underline{(3,4,27)},$$
$$(3,5,5),\ (3,5,6),\ \underline{(3,5,7)},\ (3,5,8),\ (3,5,9),\ (3,5,11),\ (3,5,13),$$
$$(3,7,8),\ (3,7,10),\ (4,4,5),\ \underline{(4,5,6)}.$$

Remark. Sun proved the 'only if' part. The 'if' part is difficult!

Universal $ap_i + bp_j + cp_k$

Let $a, b, c \in \mathbb{Z}^+$ and $i, j, k \in \{3, 4, 5, \ldots\}$. We call (ap_i, bp_j, cp_k) (or the sum $ap_i + bp_j + cp_k$) *universal* if any $n \in \mathbb{N}$ can be written as $ap_i(x) + bp_j(y) + cp_k(z)$ with $x, y, z \in \mathbb{N}$.

Theorem [Z. W. Sun, arxiv: 0905.0635, 2009]. Let $a, b, c \in \mathbb{Z}^+$ with $\max\{a, b, c\} > 1$, and let $i, j, k \in \{3, 4, \ldots\}$ with $i \leqslant j \leqslant k$ and $\max\{i, j, k\} \geqslant 5$. Suppose that (ap_i, bp_j, cp_k) is universal (over $\mathbb{N}$) with $a \leqslant b$ if $i = j$, and $b \leqslant c$ if $j = k$. Then (ap_i, bp_j, cp_k) is on the following list of 64 triples:

$(p_3, p_3, 2p_5),\ (p_3, p_3, 4p_5),\ (p_3, 2p_3, p_5),\ (p_3, 2p_3, 4p_5),\ (p_3, 3p_3, p_5),$
$(p_3, 4p_3, p_5),\ (p_3, 4p_3, 2p_5),\ (p_3, 6p_3, p_5),\ (p_3, 9p_3, p_5),\ (2p_3, 3p_3, p_5),$
$(p_3, 2p_3, p_6),\ (p_3, 2p_3, 2p_6),\ (p_3, 2p_3, p_7),\ (p_3, 2p_3, 2p_7),\ (p_3, 2p_3, p_8),$
$(p_3, 2p_3, 2p_8),\ (p_3, 2p_3, p_9),\ (p_3, 2p_3, 2p_9),\ (p_3, 2p_3, p_{10},\ (p_3, 2p_3, p_{12}),$
$(p_3, 2p_3, 2p_{12}),\ (p_3, 2p_3, p_{15}),\ (p_3, 2p_3, p_{16}),\ (p_3, 2p_3, p_{17}),\ (p_3, 2p_3, p_{23})$
$(p_3, p_4, 2p_5),\ (p_3, 2p_4, p_5),\ (p_3, 2p_4, 2p_5),\ (p_3, 2p_4, 4p_5),\ (p_3, 3p_4, p_5),$
$\underline{(p_3, 4p_4, p_5)},\ (p_3, 4p_4, 2p_5),\ (2p_3, p_4, p_5),\ (2p_3, p_4, 2p_5),\ (2p_3, p_4, 4p_5),$
$(2p_3, 3p_4, p_5),\ (3p_3, p_4, p_5),\ (p_3, 2p_4, p_6),\ (2p_3, p_4, p_6),\ (p_3, p_4, 2p_7),$
$(2p_3, p_4, p_7),\ (p_3, p_4, 2p_8),\ (p_3, 2p_4, p_8),\ (p_3, 3p_4, p_8),\ (2p_3, p_4, p_8),$
$(2p_3, 3p_4, p_8),\ (p_3, p_4, 2p_9),\ (p_3, 2p_4, p_9),\ (2p_3, p_4, p_{10}),\ (2p_3, p_4, p_{12}),$
$(p_3, 2p_4, p_{17}),\ (2p_3, p_4, p_{17}),\ (p_3, p_5, 4p_6),\ (p_3, 2p_5, p_6),\ (p_3, p_5, 2p_7),$
$(p_3, p_5, 4p_7),\ (p_3, 2p_5, p_7),\ (3p_3, p_5, p_7),\ (p_3, p_5, 2p_9),\ (2p_3, p_5, p_9),$
$(p_3, 2p_6, p_8),\ (p_3, p_7, 2p_7),\ (p_4, 2p_4, p_5),\ (2p_4, p_5, p_6).$

Conjecture [Z. W. Sun, arxiv: 0905.0635, 2009]. The above 64 triples are indeed universal.

Generalized Polygonal Numbers

For $m \in \{3, 4, 5, \ldots\}$, those $p_m(x) = (m-2)\binom{x}{2} + x$ with $x \in \mathbb{Z}$ are called *generalized m-gonal numbers*.

Generalized hexagonal numbers are identical with triangular numbers, for,

$$p_6(x) = x(2x-1) = T_{2x-1} \text{ and } T_x = p_6\left(-\frac{x}{2}\right) = p_6\left(\frac{x+1}{2}\right).$$

Let $a, b, c \in \mathbb{Z}^+$ and $i, j, k \in \{3, 4, 5, \ldots\}$. We call the sum $ap_i + bp_j + cp_k$ *universal over* $\mathbb{Z}$ if for any $n \in \mathbb{N}$ there are *integers* x, y, z such that $n = ap_i(x) + bp_j(y) + cp_k(z)$.

That $p_5 + p_5 + p_5$ is universal over $\mathbb{Z}$ (equivalently, for any $n \in \mathbb{N}$ we can write $24n + 3 = x^2 + y^2 + z^2$ with x, y, z all relatively prime to 3), was first realized by R. K. Guy [Amer. Math. Monthly 1994]. To fill in the gap in Guy's proof, one needs the following Réalis identity

$$(1^2+1^2+1^2)^2(x^2+y^2+z^2) = (x-2y-2z)^2+(y-2x-2z)^2+(z-2x-2y)^2.$$

Universal $ap_k + bp_k + cp_k$ over $\mathbb{Z}$

Theorem [Z. W. Sun, arxiv: 0905.0635, 2009]. Suppose that $ap_k + bp_k + cp_k$ is universal over $\mathbb{Z}$, where $k \in \{4, 5, 7, 8, 9, \ldots\}$, $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b \leqslant c$. Then k is equal to 5 and (a, b, c) is among the following 20 triples:

$$(1, 1, k)\ (k \in [1, 10] \setminus \{7\}),$$
$$(1, 2, 2),\ (1, 2, 3),\ (1, 2, 4),\ (1, 2, 6),\ (1, 2, 8),$$
$$(1, 3, 3),\ (1, 3, 4),\ (1, 3, 6),\ (1, 3, 7),\ (1, 3, 8),\ (1, 3, 9).$$

Conjecture [Z. W. Sun, arxiv: 0905.0635, 2009]. The above 20 triples are indeed universal over $\mathbb{Z}$.

Theorem. (i) [Z. W. Sun, arxiv: 0905.0635, 2009] The sums

$$p_5 + p_5 + 2p_5,\ p_5 + p_5 + 4p_5,\ p_5 + 2p_5 + 2p_5,$$
$$p_5 + 2p_5 + 4p_5,\ p_5 + p_5 + 5p_5,\ p_5 + 3p_5 + 6p_5$$

are universal over $\mathbb{Z}$.

(ii) [G. Fan & Z. W. Sun, arxiv:0906.2450, 2009] $p_5 + 2p_5 + 6p_5$ and $p_5 + bp_5 + 3p_5\,(b = 1, 2, 3, 4, 9)$ are universal over $\mathbb{Z}$.

(iii) (B. Kane & Sun, 2009) $p_5 + 3p_5 + 7p_5$ is universal over $\mathbb{Z}$.

For $m = 3, 4, \ldots$, clearly

$$8(m-2)p_m(x) + (m-4)^2 = ((2m-4)x - (m-4))^2.$$

Thus, for given $a, b, c \in \mathbb{Z}^+$ and $i, j, k \in \{3, 4, \ldots\}$, that $n = ap_i(x) + bp_j(y) + cp_k(z)$ for some $x, y, z \in \mathbb{Z}$ implies a representation of certain $nq + r$ (with q only depending on i, j, k and r depending on a, b, c, i, j, k) by a ternary quadratic form.

Z. W. Sun [arxiv:0905.0635, 2009] found the complete list of all possible (i, j, k) such that $p_i + p_j + p_k$ is universal over $\mathbb{Z}$. The one with $i + j + k$ maximal is $(5, 12, 76)$. Observe that $n = p_5(x) + p_{12}(y) + p_{76}(z)$ *if and only if*

$$4440(n+9)+2657 = 185(6x-1)^2+888(5y-2)^2+120(37z-18)^2.$$

It seems difficult to prove that for any $n = 9, 10, \ldots$ the equation

$$4440n + 2657 = 185x^2 + 888y^2 + 120z^2$$

has integral solutions.

Universal $ap_i + bp_j + cp_k$ over $\mathbb{N}$

As I have conjectured, there are totally $31 + 64$ universal triples (ap_i, bp_j, cp_k) with $\max\{i, j, k\} \geqslant 5$. Since

$$\{T_x + T_y : \ x, y \in \mathbb{Z}\} = \{x^2 + 2T_y : \ x, y \in \mathbb{Z}\},$$

the 95 universal triples reduce to 75 essential triples. Though we cannot prove the universality over $\mathbb{N}$ for any of them, we have proved 42 of them (including $(p_3, 4p_4, p_5)$) are universal over $\mathbb{Z}$.

For the following 33 remaining essential triples (ap_i, bp_j, cp_k), we have not yet proved the universality of $ap_i + bp_j + cp_k$ over $\mathbb{Z}$.

$(p_3, 9p_3, p_5)$, $(p_3, 2p_4, 4p_5)$, $(p_3, 4p_4, 2p_5)$, $(p_3, 2p_3, p_7)$, $(p_3, p_4, 2p_7)$, (p_3, p_5, p_7), $(p_3, p_5, 4p_7)$, $(p_3, 2p_5, p_7)$, $(p_3, p_7, 2p_7)$, $(p_3, 2p_3, 2p_8)$, (p_3, p_7, p_8), $(p_3, 2p_3, p_9)$, $(p_3, 2p_3, 2p_9)$, (p_3, p_4, p_9), $(p_3, p_4, 2p_9)$, $(p_3, 2p_4, p_9)$, $(p_3, p_5, 2p_9)$, $(2p_3, p_5, p_9)$, (p_3, p_3, p_{12}), $(p_3, 2p_3, p_{12})$, $(p_3, 2p_3, 2p_{12})$, (p_3, p_4, p_{13}), (p_3, p_5, p_{13}), $(p_3, 2p_3, p_{15})$, (p_3, p_4, p_{15}), $(p_3, 2p_3, p_{16})$, (p_3, p_3, p_{17}), $(p_3, 2p_3, p_{17})$, (p_3, p_4, p_{17}), $(p_3, 2p_4, p_{17})$, (p_3, p_4, p_{18}), $(p_3, 2p_3, p_{23})$, (p_3, p_4, p_{27}).

An Illustration of Difficulty

$p_3 + p_4 + p_{17}$ is conjectured to be universal (over $\mathbb{N}$).

$$n = p_3(x) + p_4(y) + p_{17}(z)$$
$$\iff 120n + 184 = 15(2x+1)^2 + 120y^2 + (30z-13)^2$$

So

$$\exists x, y, z \in \mathbb{N}(n = p_3(x) + p_4(y) + p_{17}(z))$$
$$\implies \exists x, y, z \in \mathbb{Z}(n = p_3(x) + p_4(y) + p_{17}(z)) \quad (\Leftarrow ?)$$
$$\iff \exists x, y, z \in \mathbb{Z}[120n + 184 = 15x^2 + 120y^2 + z^2$$
$$\& \ 2 \nmid x \ \& \ z \equiv \pm 13 \pmod{30}]$$
$$\implies \exists x, y, z \in \mathbb{Z}(120n + 184 = 15x^2 + 120y^2 + z^2 \ \& \ 2 \nmid z) \quad (\Leftarrow ?)$$
$$\iff \exists x, y, z \in \mathbb{Z}(120n + 184 = 15x^2 + 30y^2 + z^2 \ \& \ 2 \nmid z)$$

(How to prove this representation?)

Note that

$$z^2 \equiv 184 - 15 = 13^2 \pmod{120} \not\Longrightarrow z \equiv \pm 13 \pmod{30}$$

since $23^2 \equiv 13^2 \pmod{120}$.

On the Forms $p_3+p_4+p_{18}$ and $p_3+p_4+p_{27}$

Z. W. Sun [arXiv:0905.0635]: Under GRH, $p_3+p_4+p_{18}$ and $p_3+p_4+p_{27}$ are universal over $\mathbb{Z}$.

$$
\begin{aligned}
& n = p_3(x)+p_4(y)+p_{18}(z) \text{ for some } x,y,z\in\mathbb{N}\\
\Longrightarrow & n = p_3(x)+p_4(y)+p_{18}(z) \text{ for some } x,y,z\in\mathbb{Z}\\
\Longleftrightarrow & \ 64(n+1)+42 = x^2+y^2+(8z)^2 \text{ for some } x,y,z\in\mathbb{Z}
\end{aligned}
$$

(This holds under GRH)

and

$$
\begin{aligned}
& n = p_3(x)+p_4(y)+p_{27}(z) \text{ for some } x,y,z\in\mathbb{N}\\
\Longrightarrow & n = p_3(x)+p_4(y)+p_{27}(z) \text{ for some } x,y,z\in\mathbb{Z}\\
\Longleftrightarrow & \ 100(n+2)+77 = x^2+y^2+(10z)^2 \text{ for some } x,y,z\in\mathbb{Z}
\end{aligned}
$$

(This holds under GRH)

Positive Definite Quadratic Forms in $n > 3$ Variables

W. Tartakowsky (1929): For a given positive definite integral quadratic form Q in $n \geqslant 5$ variables, an integer m is represented by Q if m is locally represented by Q and sufficiently large.

Tools: Modular forms and Siegel's theorem.

H.D. Kloosterman [Acta Math. 49(1926)]. For a given positive definite integral quadratic form Q in 4 variables, an integer m is represented by Q if m is locally represented by Q and sufficiently large with a priori bounded divisibility at the anisotropic primes.

Tools: The circle method and Kloosterman sums.

Reference: J. Hanke, *Some recent results about (ternary) quadratic forms, preprint.*

Regular Ternary Quadratic Forms

A positive (definite) ternary quadratic form

$$Q(x, y, z) = ax^2 + by^2 + cz^2 + dyz + exz + fxy$$

with $a, b, c, d, e, f \in \mathbb{Z}$ is said to be *regular* if it represents an integer n (i.e., $Q(x, y, z) = n$ for some $x, y, z \in \mathbb{Z}$) if and only if it locally represents n (i.e., for any prime p the equation $Q(x, y, z) = n$ has integral solutions in the p-adic field $\mathbb{Q}_p$; in other words, for any $m \in \mathbb{Z}^+$ the congruence $Q(x, y, z) \equiv n \pmod{m}$ is solvable over $\mathbb{Z}$).

A full list of positive regular ternary quadratic forms was given by W. C. Jagy, I. Kaplansky and A. Schiemann [*There are 913 regular ternary forms*, Mathematika 44(1997), 332–341]. There are totally 102 regular forms $ax^2 + by^2 + cz^2$ with $1 \leqslant a \leqslant b \leqslant c$ and $\gcd(a, b, c) = 1$; for each of them those positive integers not represented by the form were described explicitly in Dickson's book published in 1939. However, we often meet irregular positive ternary quadratic forms when we investigate the universality of $ap_i + bp_j + cp_k$ over $\mathbb{Z}$.

Connection to Modular Forms

For a positive definite integral quadratic form $Q(x, y, z)$, we define

$$r_Q(n) := |\{(x, y, z) \in \mathbb{Z}^3 : \; Q(x, y, z) = n\}|.$$

The theta series

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) e^{2\pi i n z}$$

is a holomorphic function in the complex upper half-plane

$$\mathcal{H} = \{z \in \mathbb{C} : \; \mathrm{Im}(z) > 0\}.$$

Furthermore, there is a congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \; c \equiv 0 \pmod{N} \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$ and a Dirichlet character χ_Q mod N such that

$$\theta_Q\left(\frac{az+b}{cz+d}\right) = \chi_Q(d)(cz+d)^{3/2}\theta_Q(z)$$

$$\text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ and } z \in \mathcal{H}.$$

Some Results on Ternary Quadratic Forms

Theorem. [Z. W. Sun, arxiv:0905.0635, 2009] Let $n \in \mathbb{N}$. Then

(i)

$$6n + 1 = x^2 + 3y^2 + 24z^2 \quad \text{for some } x, y, z \in \mathbb{Z}$$

and consequently

$$n = T_x + (2y)^2 + p_5(z) \quad \text{for some } x, y, z \in \mathbb{Z}.$$

[*The proof needs several technique lemmas.*]

(ii)

$$12n + 4 = x^2 + 3y^2 + 3z^2 \quad \text{for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid x,$$

$$12n + 4, 12n + 8 = 3x^2 + y^2 + z^2 \quad \text{for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid x.$$

[*The proof has the same flavor with Fermat's infinite descent.*]

$$12n + 5 = x^2 + y^2 + (6z)^2 \quad \text{for some } x, y, z \in \mathbb{Z}.$$

Consequently, $n = T_x + y^2 + p_{11}(z) \quad \text{for some } x, y, z \in \mathbb{Z}.$

(iii) Let $r \in \{1, 9\}$.

$$\begin{aligned} & 20n + r \text{ is not a square} \\ \Longrightarrow\ & 20n + r = 5x^2 + 5y^2 + 4z^2 \quad \text{for some } x, y, z \in \mathbb{Z}. \end{aligned}$$

Consequently, if $20(n+1)+1$ is not a square then there are $x, y, z \in \mathbb{Z}$ such that $n = p_3(x) + p_3(y) + p_{12}(z)$.

(iv)

$$\begin{aligned} & 7n + 4 \text{ is not squarefree} \\ \Longrightarrow\ & n = T_x + y^2 + p_9(z) \quad \text{for some } x, y, z \in \mathbb{Z}. \end{aligned}$$

(iii) Let $r \in \{1, 9\}$.

$$\begin{aligned} & 20n + r \text{ is not a square} \\ \Longrightarrow\ & 20n + r = 5x^2 + 5y^2 + 4z^2 \quad \text{for some } x, y, z \in \mathbb{Z}. \end{aligned}$$

Consequently, if $20(n+1)+1$ is not a square then there are $x, y, z \in \mathbb{Z}$ such that $n = p_3(x) + p_3(y) + p_{12}(z)$.

(iv)

$$\begin{aligned} & 7n + 4 \text{ is not squarefree} \\ \Longrightarrow\ & n = T_x + y^2 + p_9(z) \quad \text{for some } x, y, z \in \mathbb{Z}. \end{aligned}$$

Lemma. (i) (Sun) For any $n \in \mathbb{N}$ we have

$$\begin{aligned} & |\{(x, y) \in \mathbb{Z}^2 :\ x^2 + 3y^2 = 8n + 4 \text{ and } 2 \nmid x\}| \\ & = \frac{2}{3} |\{(x, y) \in \mathbb{Z}^2 :\ x^2 + 3y^2 = 8n + 4\}|. \end{aligned}$$

(ii) (Sun) A positive integer $w = x^2 + 7y^2 \equiv 0 \pmod 8$ can be written as $u^2 + 7v^2$ with u, v *odd*.

(iii) (X. Wang & D. Pei, 2001) If $7n + 4$ is not squarefree, then $7n + 4 = x^2 + 7y^2 + 7z^2$ for some $x, y, z \in \mathbb{Z}$ (this is related to a conjecture of Kaplansky).

A Conjecture on $p + ax^2$

Goldbach (1752): Whether odd $n > 1$ has the form $p + 2x^2$?
Counterexamples: 5777 and 5993 (Stern and his students, 1856).

Conjecture [Z. W. Sun, arxiv:0905.0635, 2009]. If $a \in \mathbb{Z}^+$ is not a square, then sufficiently large integers relatively prime to a has the form $p + ax^2$ with p a prime and $x \in \mathbb{Z}$, i.e., the set $S(a)$ given by

$$\{n > 1 : \gcd(a, n) = 1, \text{ and } n \neq p + ax^2 \text{ for any prime } p \text{ and } x \in \mathbb{Z}\}$$

is finite. In particular,

$$S(6) = \emptyset, \quad S(12) = \{133\}, \quad S(30) = \{121\},$$
$$S(3) = \{4, 28, 52, 133, 292, 892, 1588\},$$
$$S(18) = \{187, 1003, 5777, 5993\},$$
$$S(24) = \{25, 49, 145, 385, 745, 1081, 1139, 1561, 2119, 2449, 5299\}.$$

Remark. The above conjecture implies that $p_3 + 2p_4 + p_9$ and $p_3 + p_4 + p_{13}$ are universal over $\mathbb{Z}$.

A Conjecture on $p + 2p_m(x)$

Conjecture [Z. W. Sun, arxiv:0905.0635, 2009]

(i) Let $m \in \{5, 6, \ldots\}$ with $m \not\equiv 2 \pmod 8$. Then all sufficiently large odd integers have the form $p + 2p_m(x)$ with p a prime and $x \in \mathbb{N}$.

(ii) Any odd number $n > 1$ other than 135, 345, 539 can be written in the form $p + 2p_5(x) = p + 3x^2 - x$ with p a prime and $x \in \mathbb{N}$. Moreover, we can require that

$$\begin{aligned} p &\equiv 1 \pmod 4 \quad \text{if } n > 16859, \\ p &\equiv 3 \pmod 4 \quad \text{if } n > 27695, \\ p &\equiv 1 \pmod 6 \quad \text{if } n > 12845, \\ p &\equiv 5 \pmod 6 \quad \text{if } n > 15865. \end{aligned}$$

A Conjecture on Sums of Three Squares

$$4n + 1 = (2x + 1)^2 + (2y)^2 + (2z)^2$$
$$4n + 2 = (2x)^2 + (2y + 1)^2 + (2z + 1)^2$$

Conjecture (Z. W. Sun, 2009). Let

$$N_1 = 14617,\ N_2 = 15618,\ N_3 = 25582.$$

For any $i \in \{1, 2, 3\}$, each $n > N_i$ with $n \equiv 1, 2 \pmod 4$ can be written in the form $x_1^2 + x_2^2 + x_3^2$ with $0 \leqslant x_1 \leqslant x_2 \leqslant x_3$ and $x_i \equiv n \pmod 2$.

Difficulty. All known proofs of the Gauss-Legendre theorem deal with quadratic forms with integer variables; it seems that none of them can be adapted to yield a proof of the above conjecture. New ideas are needed!

An Application

Let $n \in \mathbb{Z}^+$. By the above conjecture, there is an $n_0 \in \mathbb{N}$ with $n_0 < n$ such that $p_6(n) - p_6(n_0) =$ odd square $+$ even square.

$$8n - 2 = (2a+1)^2 + (2b+1)^2 + (2c)^2$$
$$(2a+1 \geqslant 2b+1 \geqslant 2c \geqslant 0)$$
$$\Longrightarrow 4n - 1 = (a+b+1)^2 + (a-b)^2 + 2c^2 = x^2 + y^2 + 2c^2$$
$$[x = a+b+1,\ y = a-b,\ 2 \nmid c,\ x^2 + y^2 \geqslant (2c)^2]$$
$$\Longrightarrow (4n-1)^2 = (x^2 + y^2 + 2c^2)^2 = (x^2 + y^2 - 2c^2)^2 + 8c^2(x^2 + y^2)$$
$$\Longrightarrow 8p_6(n) + 1 = (4n-1)^2 = (4n_0 - 1)^2 + 8((cx)^2 + (cy)^2)$$
$$[4n_0 - 1 = x^2 + y^2 - 2c^2 \geqslant 0]$$
$$\Longrightarrow p_6(n) - p_6(n_0) = (cx)^2 + (cy)^2 = \text{odd square} + \text{even square}.$$

Conjecture [Z. W. Sun, 2009]. Let $m > 2$ be an integer. Then $m \equiv 1, 2 \pmod 4$ if and only if each *sufficiently large m*-gonal number can be written as the sum of a smaller m-gonal number, an odd square and an even square.

A New Kind of Numbers

An integer $m \geqslant 3$ is said to be *good* if for *any* $n \in \mathbb{Z}^+$ there is $n_0 \in \mathbb{N}$ such that $p_m(n) - p_m(n_0)$ can be written as the sum of an odd square and an even square.

$$m \text{ is good} \implies m \equiv 2 \pmod 4 \quad \text{(Easy!)}.$$

Is 6 a good number ? By Sun [Acta Arith. 2007], if $n \in \mathbb{Z}^+$ then $p_6(n) = T_{2n-1}$ can be written in the form $(2x+1)^2 + (2y)^2 + T_z$.

Conjecture [Z. W. Sun, 2009]. (i) There are infinitely many good numbers.
(ii) Let a_n be the nth good number. Then

$$\lim_{n\to\infty} \frac{a_n}{n^2/\log n} = c \approx 2.5.$$

(iii) Good numbers below 300 are as follows:

6, 10, 14, 26, 38, 42, 50, 62, 98,
114, 122, 146, 170, 206, 230, 290, 294.

Thank you!