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SUMS OF SQUARES AND TRIANGULAR NUMBERS, AND RADO NUMBERS FOR LINEAR EQUATIONS

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1. Mixed sums of squares and triangular numbers

A classical result of Fermat asserts that any prime $p \equiv 1 \pmod 4$ is a sum of two squares of integers. Fermat also conjectured that each $n \in \mathbb{N}$ can be written as a sum of three triangular numbers, where $\mathbb{N}$ is the set $\{0, 1, 2, \ldots\}$ of natural numbers, and triangular numbers are those integers $t_x = x(x+1)/2$ with $x \in \mathbb{Z}$. An equivalent version of this conjecture states that $8n + 3$ is a sum of three squares (of odd integers). This follows from the following profound theorem (see, e.g., [G, pp. 38–49] or [N, pp. 17-23]).

Gauss-Legendre Theorem. *$n \in \mathbb{N}$ can be written as a sum of three squares of integers if and only if n is not of the form $4^k(8l + 7)$ with $k, l \in \mathbb{N}$.*

Building on some work of Euler, in 1772 Lagrange showed that every natural number is a sum of four squares of integers.

For problems and results on representations of natural numbers by various quadratic forms with coefficients in $\mathbb{N}$, the reader may consult [Du] and [G].

Motivated by Ramanujan's work [Ra], L. Panaitopol [P] proved the following interesting result in 2005.

Theorem A. *Let a, b, c be positive integers with $a \leqslant b \leqslant c$. Then every odd natural number can be written in the form $ax^2+by^2+cz^2$ with $x, y, z \in \mathbb{Z}$, if and only if the vector (a, b, c) is $(1, 1, 2)$ or $(1, 2, 3)$ or $(1, 2, 4)$.*

According to L. E. Dickson [D2, p. 260], Euler already noted that any odd integer $n > 0$ is representable by $x^2 + y^2 + 2z^2$ with $x, y, z \in \mathbb{Z}$.

In 1862 J. Liouville (cf. [D2, p. 23]) proved the following result.

Theorem B. *Let a, b, c be positive integers with $a \leqslant b \leqslant c$. Then every $n \in \mathbb{N}$ can be written as $at_x + bt_y + ct_z$ with $x, y, z \in \mathbb{Z}$, if and only if (a, b, c) is among the following vectors:*

$$(1, 1, 1),\ (1, 1, 2),\ \ (1, 1, 4),\ (1, 1, 5),\ (1, 2, 2),\ (1, 2, 3),\ (1, 2, 4).$$

Now we turn to representations of natural numbers by mixed sums of squares (of integers) and triangular numbers.

Let $n \in \mathbb{N}$. By the Gauss-Legendre theorem, $8n + 1$ is a sum of three squares. It follows that $8n+1 = (2x)^2+(2y)^2+(2z+1)^2$ for some $x, y, z \in \mathbb{Z}$ with $x \equiv y \pmod 2$; this yields the representation

$$n = \frac{x^2 + y^2}{2} + t_z = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + t_z$$

as observed by Euler. According to Dickson [D2, p. 24], E. Lionnet stated, and V. A. Lebesgue [L] and M. S. Réalis [Re] proved that n can also be written in the form $x^2 + t_y + t_z$ with $x, y, z \in \mathbb{Z}$. Quite recently, this was reproved by H. M. Farkas [F] via the theory of theta functions.

Using the theory of ternary quadratic forms, in 1939 B. W. Jones and G. Pall [JP, Theorem 6] proved that for any $n \in \mathbb{N}$ we have $8n + 1 = ax^2 + by^2 + cz^2$ for some $x, y, z \in \mathbb{Z}$ if the vector (a, b, c) belongs to the set

$$\{(1,1,16),\ (1,4,16),\ (1,16,16),\ (1,2,32),\ (1,8,32),\ (1,8,64)\}.$$

As $(2z+1)^2 = 8t_z+1$, the result of Jones and Pall implies that each $n \in \mathbb{N}$ can be written in any of the following three forms with $x, y, z \in \mathbb{Z}$:

$$2x^2 + 2y^2 + t_z = (x+y)^2 + (x-y)^2 + t_z,\ x^2 + 4y^2 + t_z,\ x^2 + 8y^2 + t_z.$$

Recently the speaker established the following result by means of q-series.

Theorem 1.1 [Z. W. Sun, Acta Arith. 127(2007), 103-113]**.** (i) *Any $n \in \mathbb{N}$ is a sum of an even square and two triangular numbers. Moreover, if $n/2$ is not a triangular number then*

$$\begin{aligned}&|\{(x,y,z) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\ x^2 + t_y + t_z = n \text{ and } 2 \nmid x\}|\\ =&|\{(x,y,z) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N}:\ x^2 + t_y + t_z = n \text{ and } 2 \mid x\}|.\end{aligned}$$

(ii) *If $n \in \mathbb{N}$ is not a triangular number, then*

$$\begin{aligned}&|\{(x,y,z) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\ x^2 + y^2 + t_z = n \text{ and } x \not\equiv y \pmod 2\}|\\ =&|\{(x,y,z) \in \mathbb{Z}\times\mathbb{Z}\times\mathbb{N}:\ x^2 + y^2 + t_z = n \text{ and } x \equiv y \pmod 2\}| > 0.\end{aligned}$$

(iii) *A positive integer n is a sum of an odd square, an even square and a triangular number, unless it is a triangular number t_m $(m>0)$ for which all prime divisors of $2m+1$ are congruent to 1 mod 4 and hence $t_m = x^2+x^2+t_z$ for some integers $x>0$ and z with $x \equiv m/2 \pmod 2$.*

Remark. Note that $t_2 = 1^2+1^2+t_1$ but we cannot write $t_2=3$ as a sum of an odd square, an even square and a triangular number.

Sun's proof of Theorem 1.1 makes use of Jacobi's triple product identity

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+aq^{2n-1})(1+a^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} a^n q^{n^2} \ \ (|q|<1)$$

and its three by-products:

$$\varphi(-q) = \prod_{n=1}^{\infty}(1-q^{2n-1})^2(1-q^{2n}) \quad \text{(Gauss)},$$
$$\psi(q) = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^{2n-1}} \quad \text{(Gauss)},$$
$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{t_n} \quad \text{(Jacobi)},$$

where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \ \text{ and } \ \psi(q) = \sum_{n=0}^{\infty} q^{t_n}.$$

Now we list the three lemmas that are needed in the proof of Theorem 1.1.

Lemma 1.1. *For any $n \in \mathbb{N}$ we have*

$$|\{(y,z)\in\mathbb{N}^2:\ t_y+t_z=n\}| = |\{(y,z)\in\mathbb{Z}\times\mathbb{N}:\ y^2+2t_z=n\}|.$$

Let $n \in \mathbb{N}$ and define

$$r(n) = |\{(x,y,z) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N} :\ x^2 + t_y + t_z = n\}|,$$

$$r_0(n) = |\{(x,y,z) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N} :\ x^2 + t_y + t_z = n \text{ and } 2 \mid x\}|,$$

$$r_1(n) = |\{(x,y,z) \in \mathbb{Z}\times\mathbb{N}\times\mathbb{N} :\ x^2 + t_y + t_z = n \text{ and } 2 \nmid x\}|.$$

Clearly $r_0(n)+r_1(n) = r(n)$. The following lemma tells us the information on the difference $r_0(n) - r_1(n)$.

Lemma 1.2. *For $m = 0, 1, 2, \ldots$ we have*

$$r_0(2t_m) - r_1(2t_m) = (-1)^m(2m+1).$$

Also, $r_0(n) = r_1(n)$ if $n \in \mathbb{N}$ is not a triangular number times 2.

Lemma 1.3 (Hurwitz, 1907)**.** *Let $n > 0$ be an odd integer, and let $p_1, \ldots, p_r$ be all the distinct prime divisors of n congruent to 3 mod 4. Write $n = n_0 \prod_{0<i\leqslant r} p_i^{\alpha_i}$, where $n_0, \alpha_1, \ldots, \alpha_r$ are positive integers and n_0 has no prime divisors congruent to 3 mod 4. Then*

$$|\{(x,y,z) \in \mathbb{Z}^3 :\ x^2+y^2+z^2 = n^2\}| = 6n_0 \prod_{0<i\leqslant r}\left(p_i^{\alpha_i} + 2\frac{p_i^{\alpha_i}-1}{p_i-1}\right).$$

Here are two more theorems of Sun's paper.

Theorem 1.2 [Z. W. Sun, Acta Arith. 127(2007), 103-113]**.** *Let a, b, c be positive integers with $a \leqslant b$. Suppose that every $n \in \mathbb{N}$ can be written as $ax^2 + by^2 + ct_z$ with $x, y, z \in \mathbb{Z}$. Then (a,b,c) is among the following vectors:*

$$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,4),$$
$$(1,3,1),\ (1,4,1),\ (1,4,2),\ (1,8,1),\ (2,2,1).$$

Theorem 1.3 [Z. W. Sun, Acta Arith. 127(2007), 103-113]. *Let a, b, c be positive integers with $b \geqslant c$. Suppose that every $n \in \mathbb{N}$ can be written as $ax^2 + bt_y + ct_z$ with $x, y, z \in \mathbb{Z}$. Then (a, b, c) is among the following vectors:*

$$(1,1,1),\ (1,2,1),\ (1,2,2),\ (1,3,1),\ (1,4,1),\ (1,4,2),\ (1,5,2),$$
$$(1,6,1),\ (1,8,1),\ (2,1,1),\ (2,2,1),\ (2,4,1),\ (3,2,1),\ (4,1,1),\ (4,2,1).$$

Sun also reduced the converses of Theorems 1.2 and 1.3 to two conjectures, the second of which has been confirmed by S. Guo, H. Pan and Z. W. Sun.

Theorem 1.4 (S. Guo, H. Pan and Z. W. Sun). *Every $n \in \mathbb{N}$ can be expressed in any of the following forms with $x, y, z \in \mathbb{Z}$:*

$$x^2 + 3y^2 + t_z,\ x^2 + 3t_y + t_z,\ x^2 + 6t_y + t_z,\ 3x^2 + 2t_y + t_z,\ 4x^2 + 2t_y + t_z.$$

In the proof of Theorem 1.4, the following identity of Jacobi plays an important role:

$$3(x^2 + y^2 + z^2) = (x + y + z)^2 + 2\left(\frac{x + y - 2z}{2}\right)^2 + 6\left(\frac{x - y}{2}\right)^2.$$

Now we state the first conjecture of Sun [Acta Arith. 2007] which is still open.

Conjecture 1.1 (Z. W. Sun). *Any positive integer n is a sum of a square, an odd square and a triangular number. In other words, each natural number can be written in the form $x^2 + 8t_y + t_z$ with $x, y, z \in \mathbb{Z}$.*

By Theorem 1.1(iii), Conjecture 1.1 is valid when $n \neq t_4, t_8, t_{12}, \ldots$. In 2005 Sun verified Conjecture 1.1 for $n \leqslant 15,000$, and in 2007 Weixiang Tang (an undergraduate at Nanjing Univ.) verified the conjecture for all $n \leqslant 10^7$.

Here is another conjecture of Sun which has also been verified for all $n \leqslant 10^7$.

Conjecture 1.2 (Sun). *Every $n \in \mathbb{N}$ can be written in the form $x^2 + 2y^2 + 3t_z$ (with $x, y, z \in \mathbb{Z}$) except $n = 23$, in the form $x^2 + 5y^2 + 2t_z$ (or the equivalent form $5x^2 + t_y + t_z$) except $n = 19$, in the form $x^2 + 6y^2 + t_z$ except $n = 47$, and in the form $2x^2 + 4y^2 + t_z$ except $n = 20$.*

The second statement in Conjecture 1.2 is related to an assertion of Ramanujan confirmed by Dickson [D1] which states that even natural numbers not of the form $4^k(16l + 6)$ (with $k, l \in \mathbb{N}$) can be written as $x^2 + y^2 + 10z^2$ with $x, y, z \in \mathbb{Z}$. Observe that

$$\begin{aligned}
&n = x^2 + 5y^2 + 2t_z \text{ for some } x, y, z \in \mathbb{Z} \\
\iff &4n + 1 = x^2 + 5y^2 + z^2 \text{ for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid z \\
\iff &8n + 2 = 2(x^2 + y^2) + 10z^2 = (x + y)^2 + (x - y)^2 + 10z^2 \\
&\qquad\qquad\qquad\qquad \text{for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid y \\
\iff &8n + 2 = x^2 + y^2 + 10z^2 \text{ for some } x, y, z \in \mathbb{Z} \text{ with } x \not\equiv y \pmod 4.
\end{aligned}$$

2. Rado numbers for linear equations

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $[a, b] = \{x \in \mathbb{N} : a \leqslant x \leqslant b\}$ for $a, b \in \mathbb{N}$. For $k, n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we call a function $\Delta : [1, n] \to [0, k - 1]$

a *k-coloring* of the set $[1,n]$, and $\Delta(i)$ the *color* of $i \in [1,n]$. Given a k-coloring of the set $[1,n]$, a solution to the linear diophantine equation

$$a_0x_0 + a_1x_1 + \cdots + a_mx_m = 0 \quad (a_0, a_1, \ldots, a_m \in \mathbb{Z})$$

with $x_0, x_1, \ldots, x_m \in [1,n]$ is called *monochromatic* if $\Delta(x_0) = \Delta(x_1) = \cdots = \Delta(x_m)$.

Let $k \in \mathbb{Z}^+$. In 1916, I. Schur [S] proved that if $n \in \mathbb{Z}^+$ is sufficiently large then for every *k-coloring* of the set $[1,n]$, there exists a monochromatic solution to

$$x_1 + x_2 = x_0$$

with $x_0, x_1, x_2 \in [1,n]$.

Let $k \in \mathbb{Z}^+$ and $a_0, a_1, \ldots, a_m \in \mathbb{Z} \setminus \{0\}$. Provided that $\sum_{i \in I} a_i = 0$ for some $\emptyset \neq I \subseteq \{0, 1, \ldots, m\}$, R. Rado showed that for sufficiently large $n \in \mathbb{Z}^+$ the equation $a_0x_0 + a_1x_1 + \cdots + a_mx_m = 0$ always has a monochromatic solution when a k-coloring of $[1,n]$ is given; the least value of such an n is called the *k-color Rado number* for the equation. Since $-1 + 1 = 0$, Schur's theorem is a particular case of Rado's result. The reader may consult the book [LR] by B. M. Landman and A. Robertson for a survey of results on Rado numbers.

In this paper, we are interested in precise values of 2-color Rado numbers. By a theorem of Rado [R], if $a_0, a_1, \ldots, a_m \in \mathbb{Z}$ contain both positive and negative integers and at least three of them are nonzero, then the homogeneous linear equation

$$a_0x_0 + a_1x_1 + \cdots + a_mx_m = 0$$

has a monochromatic solution with $x_0, \ldots, x_m \in [1, n]$ for any sufficiently large $n \in \mathbb{Z}^+$ and a 2-coloring of $[1, n]$. In particular, if $a_1, \ldots, a_m \in \mathbb{Z}^+$ $(m \geqslant 2)$ then there is a least positive integer $n_0 = R(a_1, \ldots, a_m)$ such that for any $n \geqslant n_0$ and a 2-coloring of $[1, n]$ the diophantine equation

$$a_1x_1 + \cdots + a_mx_m = x_0 \tag{2.0}$$

always has a monochromatic solution with $x_0, \ldots, x_m \in [1, n]$.

In 1982, A. Beutelspacher and W. Brestovansky [BB] proved that the 2-color Rado number $R(1, \ldots, 1)$ for the equation $x_1 + \cdots + x_m = x_0$ $(m \geqslant 2)$ is $m^2 + m - 1$. In 1991, H. L. Abbott [A] extended this by showing that for the equation

$$a(x_1 + \cdots + x_m) = x_0 \quad (a \in \mathbb{Z}^+ \text{ and } m \geqslant 2)$$

the corresponding 2-color Rado number $R(a, \ldots, a)$ is $a^3m^2 + am - a$; that $R(a, \ldots, a) \geqslant a^3m^2 + am - a$ was first obtained by L. Funar [F], who conjectured the equality. In 2001, S. Jones and D. Schaal [JS] proved that if $a_1, \ldots, a_m \in \mathbb{Z}^+$ $(m \geqslant 2)$ and $\min\{a_1, \ldots, a_m\} = 1$ then $R(a_1, \ldots, a_m) = b^2 + 3b + 1$ where $b = a_1 + \cdots + a_m - 1$; this result actually appeared earlier in Funar [F].

In 2005 B. Hopkins and D. Schaal [HS] showed the following result.

Theorem 1.0. *Let $m \geqslant 2$ be an integer and let $a_1, \ldots, a_m \in \mathbb{Z}^+$. Then*

$$R(a, b) \geqslant R(a_1, \ldots, a_m) \geqslant a(a + b)^2 + b, \tag{2.1}$$

where

$$a = \min\{a_1, \ldots, a_m\} \quad \textit{and} \quad b = \sum_{i=1}^{m} a_i - a. \tag{2.2}$$

Hopkins and Schaal obtained (2.1) by constructing a two-coloring $\Delta : [1, (a+b)^2 + b) \to [0, 1]$ in the following way:

$$\Delta(x) = \begin{cases} 0 & \text{if } x \in [1, a+b) \cup [(a+b)^2, a(a+b)^2 + b), \\ 1 & \text{if } x \in [a+b, (a+b)^2). \end{cases}$$

They showed that for this two-coloring, (2.0) has no monochromatic solution with $x_0, \ldots, x_m \in [1, a(a+b)^2 + b)$.

Hopkins and Schaal ([HS]) conjectured further that the two inequalities in (2.1) are actually equalities and verified this in the case $a = 2$.

Song Guo and the speaker confirmd the conjecture of Hopkins and Schaal; namely, we establish the following theorem.

Theorem 2.1 (S. Guo and Z.W. Sun, J. Combin. Theory Ser. A, to appear). *Let $m \geqslant 2$ be an integer and let $a_1, \ldots, a_m \in \mathbb{Z}^+$. Then*

$$R(a_1, \ldots, a_m) = a(a+b)^2 + b, \tag{2.3}$$

where $a = \min\{a_1, \ldots, a_m\}$ and $b = a_1 + \cdots + a_m - a$.

By Theorem 2.1, if $a_1, \ldots, a_m \in \mathbb{Z}^+$ and $n \geqslant av^2 + v - a$ with $a = \min\{a_1, \ldots, a_m\}$ and $v = a_1 + \cdots + a_m$, then for any $X \subseteq [1, n]$ either there are $x_1, \ldots, x_m \in X$ such that $\sum_{i=1}^m a_i x_i \in X$ or there are $x_1, \ldots, x_m \in [1, n] \setminus X$ such that $\sum_{i=1}^m a_i x_i \in [1, n] \setminus X$.

Guo and Sun first reduced Theorem 2.1 to the following weaker version, and then proved this weaker version in the cases $\delta = 0$ and $\delta = 1$ respectively.

Theorem 2.2. *Let* $a, b, n \in \mathbb{Z}^+$, $a \leqslant b$ *and* $n \geqslant av^2 + b$ *with* $v = a + b$. *Suppose that* $b(b-1) \not\equiv 0 \pmod{a}$ *and* $\Delta : [1, n] \to [0, 1]$ *is a* 2*-coloring of* $[1, n]$ *with* $\Delta(1) = 0$ *and* $\Delta(a) = \Delta(b) = \delta \in [0, 1]$. *Then there is a monochromatic solution to the equation*

$$ax + by = z \qquad (x, y, z \in [1, n]). \tag{2.4}$$