A talk given at Zhejiang Univ. (April 29, 2016)
and Xiamen Univ. (May 12, 2016) and Ningbo Univ. (June 12, 2016)
and Shenzhen Univ. (June 17, 2016) and Workshop on Analytic
and Number Theory (Xi'an Jiaotong Univ. August 18-20, 2016)
and Nankai Univ. (Dec. 2, 2016)

Refining Lagrange's Four-square Theorem

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December 2, 2016

Abstract

Lagrange's four-square theorem asserts that any natural number can be written as $x^2 + y^2 + z^2 + w^2$ with x, y, z, w integers. The speaker recently found that this can be refined in various ways. For example, we show that we may require additionally that $x + y + z$ (or $x + 2y$, or $x + y + 2z$) is a square (or a cube). Moreover, we have formulated lots of surprising conjectures on this topic; for example, we conjecture that any natural number can be written as $x^2 + y^2 + z^2 + w^2$ with x, y, z, w nonnegative integers such that $x + 3y + 5z$ is a square. Another mysterious conjecture of the speaker asserts that any natural number can be written as $w^2 + x^2 + y^2 + z^2 + w^2$ with w, x, y, z nonnegative integers such that $(10w + 5x)^2 + (12y + 36z)^2$ is a square. This reveals a surprising connection between Lagrange's theorem and Pythagorean triples. In this talk we will tell the story of such discoveries as well as related new results on partitions of integers motivated by our refinements of Lagrange's theorem.

Part I. Waring's Problem

Lagrange's theorem

Lagrange's Theorem. Each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as the sum of four squares.

Examples. $3 = 1^2 + 1^2 + 1^2 + 0^2$ and $7 = 2^2 + 1^2 + 1^2 + 1^2$.

A. Diophantus (AD 299-215, or AD 285-201) was aware of this theorem as indicated by examples given in his book *Arithmetica*.

In 1621 Bachet translated Diophantus' book into Latin and stated the theorem in the notes of his translation.

In 1748 L. Euler found the four-square identity

$$\begin{aligned}
&(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) \\
=&(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2 \\
&+ (x_1y_3 - x_3y_1 + x_2y_4 - x_4y_2)^2 + (x_1y_4 - x_4y_1 - x_2y_3 + x_3y_2)^2.
\end{aligned}$$

and hence reduced the theorem to the case with n prime.

The theorem was first proved by J. L. Lagrange in 1770.

The representation function $r_4(n)$

It is known that only the following numbers have a unique representation as the sum of four unordered squares:

$$1,\ 3,\ 5,\ 7,\ 11,\ 15,\ 23$$

and

$$2^{2k+1}m \ (k = 0, 1, 2, \ldots \text{ and } m = 1, 3, 7).$$

Jacobi considered the fourth power of the theta function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and this led him to show that

$$r_4(n) = 8 \sum_{d|n \ \& \ 4\nmid d} d \quad \text{for all } n = 1, 2, 3, \ldots,$$

where

$$r_4(n) := |\{(w, x, y, z) \in \mathbb{Z}^4 : \ w^2 + x^2 + y^2 + z^2 = n\}|.$$

Representations as sums of polygonal numbers

For $m = 3, 4, 5, \ldots$, the *polygonal numbers of order m* (or *m-gonal numbers*) are given by

$$p_m(n) := (m-2)\binom{n}{2} + n \quad (n = 0, 1, 2, \ldots).$$

Clearly, $p_4(n) = n^2$, $p_5(n) = n(3n-1)/2$ and $p_6(n) = n(2n-1)$.

Fermat's Claim. Let $m \geqslant 3$ be an integer. Then any $n \in \mathbb{N}$ can be written as the sum of m polygonal numbers of order m.

This was proved by Lagrange in the case $m = 4$, by Gauss in the case $m = 3$, and by Cauchy in the case $m \geqslant 5$.

Conjecture (Z.-W. Sun, March 14, 2015). Each $n \in \mathbb{N}$ can be written as

$$p_5(x_1) + p_5(x_2) + p_5(x_3) + 2p_5(x_4) \quad (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

Theorem (conjectured by the speaker and proved by X.-Z. Meng and Z.-W. Sun (arxiv:1608.02022)) Any $n \in \mathbb{N}$ can be written as

$$p_6(x_1) + p_6(x_2) + 2p_6(x_3) + 4p_6(x_4) \quad (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

Waring's Problem

In 1770 E. Waring proposed the following famous problem.

Waring's Problem. Whether for each integer $k > 1$ there is a positive integer $g(k) = r$ (as small as possible) such that every $n \in \mathbb{N}$ can be written as

$$x_1^k + x_2^k + \ldots + x_r^k \quad \text{with } x_1, \ldots, x_r \in \mathbb{N}.$$

In 1909 D. Hilbert proved that $g(k)$ always exists.

If we write $2^k\lfloor(3/2)^k\rfloor - 1 < 3^k$ as a sum of nonnegative k-th powers, the most economical way is to use $\lfloor(3/2)^k\rfloor - 1$ terms of 2^k and $2^k - 1$ terms of 1^k. So

$$g(k) \geqslant 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2.$$

J. A. Euler (a son of Leonhard Euler), conjectured in about 1772 that, in fact,

$$g(k) = 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2.$$

Known results on $g(k)$

It is known that

$$g(2) = 4 \text{ (Lagrange, 1770)},$$
$$g(3) = 9 \text{ (Wieferich and A. J. Kempner, 1909-1912)},$$
$$g(4) = 19 \text{ (R. Balasubramanian, F. Dress, J.-M. Deshouillers, 1986)},$$
$$g(5) = 37 \text{ (Jingrun Chen, 1964)},$$
$$g(6) = 73 \text{ (Pillai, 1940)},$$
$$g(k) = 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2 \text{ for large } k \quad \text{(K. Mahler, 1957)}.$$

Weighted sums of five cubes

In 1917 S. Ramanujan conjectured that for 54 quadruple (a, b, c, d) with $a, b, c, d \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and $a \leqslant b \leqslant c \leqslant d$ the sum $aw^2 + bx^2 + cy^2 + dz^2$ is universal, i.e.,

$$\{aw^2 + bx^2 + cy^2 + dz^2 : \ w, x, y, z \in \mathbb{N}\} = \mathbb{N}.$$

This was confirmed by L. E. Dickson in 1927.

Conjecture (Z.-W. Sun, April 2, 2016). We have

$$\{x_1^3 + ax_2^3 + bx_3^3 + cx_4^3 + dx_5^3 : \ x_1, \ldots, x_5 \in \mathbb{N}\} = \mathbb{N}$$

whenever (a, b, c, d) is among the following 32 quadruples:

$(1, 2, 2, 3), (1, 2, 2, 4), (1, 2, 3, 4), (1, 2, 4, 5), (1, 2, 4, 6), (1, 2, 4, 9),$
$(1, 2, 4, 10), (1, 2, 4, 11), (1, 2, 4, 18), (1, 3, 4, 6), (1, 3, 4, 9), (1, 3, 4, 10),$
$(2, 2, 4, 5), (2, 2, 6, 9), (2, 3, 4, 5), (2, 3, 4, 6), (2, 3, 4, 7), (2, 3, 4, 8),$
$(2, 3, 4, 9), (2, 3, 4, 10), (2, 3, 4, 12), (2, 3, 4, 15), (2, 3, 4, 18), (2, 3, 5, 6),$
$(2, 3, 6, 12), (2, 3, 6, 15), (2, 4, 5, 6), (2, 4, 5, 8), (2, 4, 5, 9), (2, 4, 5, 10),$
$(2, 4, 6, 7), (2, 4, 7, 10).$

Upgrade Waring's Problem

Recently the speaker proposed the following problem upgrading Waring's problem.

New Problem (Z.-W. Sun, March 30-31, 2016). Determine $s(k)$ and $t(k)$ for any integer $k > 1$, where $s(k)$ is the smallest positive integer s such that

$$\{a_1x_1^k + a_2x_2^k + \ldots + a_sx_s^k :\ x_1, \ldots, x_s \in \mathbb{N}\} = \mathbb{N}$$

for some $a_1, \ldots, a_s \in \mathbb{Z}^+$, and $t(k)$ is the smallest positive integer t such that

$$\{a_1x_1^k + a_2x_2^k + \ldots + a_tx_t^k :\ x_1, \ldots, x_t \in \mathbb{N}\} = \mathbb{N}$$

for some $a_1, \ldots, a_t \in \mathbb{Z}^+$ with $a_1 + a_2 + \ldots + a_t = g(k)$.

Clearly $s(k) \leqslant t(k) \leqslant g(k)$ for all $k = 2, 3, 4, \ldots$. It is easy to see that $s(2) = t(2) = 4$. With the help of a computer, the speaker found that

$$s(3) \geqslant 5,\ s(4) \geqslant 7,\ s(5) \geqslant 8 \text{ and } t(6) \geqslant 10.$$

A conjecture

Conjecture (Z.-W. Sun, March 30-31, 2016). (i) $s(3) = t(3) = 5$. In fact,

$$\{u^3 + v^3 + 2x^3 + 2y^3 + 3z^3 : \ u, v, x, y, z \in \mathbb{N}\} = \mathbb{N}.$$

(ii) $s(4) = t(4) = 7$. In fact, we have

$$\{x_1^4 + x_2^4 + 2x_3^4 + 2x_4^4 + 3x_5^4 + 3x_6^4 + 7x_7^4 : \ x_1, \ldots, x_7 \in \mathbb{N}\} = \mathbb{N},$$

$$\{x_1^4 + x_2^4 + 2x_3^4 + 2x_4^4 + 3x_5^4 + 4x_6^4 + 6x_7^4 : \ x_1, \ldots, x_7 \in \mathbb{N}\} = \mathbb{N}.$$

(iii) $s(5) = t(5) = 8$. In fact,

$$\{x_1^5+x_2^5+2x_3^5+3x_4^5+4x_5^5+5x_6^5+7x_7^5+14x_8^5 : \ x_1, \ldots, x_8 \in \mathbb{N}\} = \mathbb{N},$$

$$\{x_1^5+x_2^5+2x_3^5+3x_4^5+4x_5^5+6x_6^5+8x_7^5+12x_8^5 : \ x_1, \ldots, x_8 \in \mathbb{N}\} = \mathbb{N}.$$

(iv) $s(6) = t(6) = 10$. In fact,

$$\{x_1^6+x_2^6+x_3^6+2x_4^6+3x_5^6+5x_6^6+6x_7^6+10x_8^6+18x_9^6+26x_{10}^6 : \ x_i \in \mathbb{N}\} = \mathbb{N}.$$

(v) In general, $s(k) = t(k) \leqslant 2k - 1$ for any integer $k > 2$.

Part II. Refining Lagrange's Four-Square Theorem

Gauss-Legendre Theorem

For $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we define

$$E(a, b, c) := \{n \in \mathbb{N} : \ n \neq ax^2 + by^2 + cz^2 \text{ for any } x, y, z \in \mathbb{N}\}.$$

Gauss-Legendre Theorem. $E(1, 1, 1) = \{4^k(8l + 7) : \ k, l \in \mathbb{N}\}$.

Triangular numbers are those $T_n = \sum_{r=0}^{n} r = n(n+1)/2$ with $n \in \mathbb{N}$. Note that $T_{-n-1} = T_n$ for all $n \in \mathbb{N}$.

Corollary (Gauss). Each $n \in \mathbb{N}$ can be written as $T_x + T_y + T_z$ with $x, y, z \in \mathbb{Z}$.

Proof. By the Gauss-Legendre theorem, there are $u, v, w \in \mathbb{Z}$ such that $8n + 3 = u^2 + v^2 + w^2$. As $u^2 + v^2 + w^2 \equiv 3 \pmod 4$, we must have $2 \nmid uvw$. So we may write $u = 2x + 1$, $v = 2y + 1$ and $w = 2z + 1$ with $x, y, z \in \mathbb{Z}$. Hence

$$n = \frac{u^2 - 1}{8} + \frac{v^2 - 1}{8} + \frac{w^2 - 1}{8} = T_x + T_y + T_z.$$

Sums of a fourth power and three squares

Theorem (Z.-W. Sun, March 27, 2016). Each $n \in \mathbb{N}$ can be written as $w^4 + x^2 + y^2 + z^2$ with $w, x, y, z \in \mathbb{N}$.

Proof. For $n = 0, 1, 2, \ldots, 15$, the result can be verified directly. Now let $n \geqslant 16$ be an integer and assume that the result holds for smaller values of n.

Case 1. $16 \mid n$.
By the induction hypothesis, we can write

$$\frac{n}{16} = x^4 + y^2 + z^2 + w^2 \ \text{ with } x, y, z, w \in \mathbb{N}.$$

It follows that $n = (2x)^4 + (4y)^2 + (4z)^2 + (4w)^2$.

Case 2. $n = 4^k q$ with $k \in \{0, 1\}$ and $q \equiv 7 \pmod 8$.
In this case, $n - 1 \notin E(1, 1, 1)$, and hence $n = 1^4 + y^2 + z^2 + w^2$ for some $y, z, w \in \mathbb{N}$.

Case 3. $16 \nmid n$ and $n \neq 4^k(8l + 7)$ for any $k \in \{0, 1\}$ and $l \in \mathbb{N}$.
In this case, $n \notin E(1, 1, 1)$ and hence there are $y, z, w \in \mathbb{N}$ such that $n = 0^4 + y^2 + z^2 + w^2$.

$aw^k + x^2 + y^2 + z^2$ with $a \in \{1, 4\}$ and $k \in \{4, 5, 6\}$

Via a similar method, we have proved the following result.

Theorem (Z.-W. Sun, March-June, 2016). Let $a \in \{1, 4\}$ and $k \in \{4, 5, 6\}$. Then, each $n \in \mathbb{N}$ can be written as $aw^k + x^2 + y^2 + z^2$ with $w, x, y, z \in \mathbb{N}$.

Suitable polynomials

Definition (Z.-W. Sun, 2016). A polynomial $P(x, y, z, w)$ with integer coefficients is called *suitable* if any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z, w)$ is a square.

We have seen that x is a suitable polynomial. By a similar method, the speaker has shown that $2x, x - y$ and $2(x - y)$ are suitable. Moreover,

$$xy,\ 2xy,\ x^2 - y^2,\ 2(x^2 - y^2),\ 3(x^2 - y^2),\ x^2 - 3y^2,\ 3x^2 - 2y^2,$$
$$x^2 + ky^2\,(k = 2, 3, 5, 6, 8, 12),\ 2x^2 + 7y^2,\ 3x^2 + 4y^2,\ 4x^2 + 5y^2,$$
$$4x^2 + 9y^2,\ 5x^2 + 11y^2,\ 6x^2 + 10y^2,\ 7x^2 + 9y^2,$$
$$x^2y^2 + y^2z^2 + z^2x^2,\ x^2y^2 + 4y^2z^2 + 4z^2x^2$$

are all suitable.

$x - y$ and $2x - 2y$ are suitable

Let $a \in \{1, 2\}$. We claim that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $a(x - y)$ is a square, and want to prove this by induction.

For every $n = 0, 1, \ldots, 15$, we can verify the claim directly.

Now we fix an integer $n \geqslant 16$ and assume that the claim holds for smaller values of n.

Case 1. $16 \mid n$.
In this case, by the induction hypothesis, there are $x, y, z, w \in \mathbb{N}$ with $a(x - y)$ a square such that $n/16 = x^2 + y^2 + z^2 + w^2$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $a(4x - 4y)$ a square.

Case 2. $16 \nmid n$ and $n \notin E(1, 1, 2)$.
In this case, there are $x, y, z, w \in \mathbb{N}$ with $x = y$ and $n = x^2 + y^2 + z^2 + w^2$, thus $a(x - y) = 0^2$ is a square.

$x - y$ and $2x - 2y$ are suitable

Case 3. $16 \nmid n$ and $n \in E(1,1,2) = \{4^k(16l+14) : \ k, l \in \mathbb{N}\}$.

In this case, $n = 4^k(16l+14)$ for some $k \in \{0,1\}$ and $l \in \mathbb{N}$. Note that $n/2 - (2/a)^2 \notin E(1,1,1)$. So, $n/2 - (2/a)^2 = t^2 + u^2 + v^2$ for some $t, u, v \in \mathbb{N}$ with $t \geqslant u \geqslant v$. As $n/2 - (2/a)^2 \geqslant 8 - 4 > 3$, we have $t > 1$. Thus

$$
\begin{aligned}
n =& 2\left(\left(\frac{2}{a}\right)^2 + t^2\right) + 2(u^2+v^2) \\
=& \left(\frac{2}{a} + t\right)^2 + \left(\frac{2}{a} - t\right)^2 + (u+v)^2 + (u-v)^2
\end{aligned}
$$

with

$$
a\left(\left(\frac{2}{a} + t\right) - \left|\frac{2}{a} - t\right|\right) = a\left(\frac{2}{a} + t - \left(t - \frac{2}{a}\right)\right) = 2^2.
$$

This proves that $x - y$ and $2x - 2y$ are both suitable.

Suitable polynomials of the form $ax \pm by$

Conjecture (Z.-W. Sun, April 14,2016) Let $a, b \in \mathbb{Z}^+$ with $\gcd(a, b)$ squarefree.

(i) The polynomial $ax + by$ is suitable if and only if $\{a, b\} = \{1, 2\}, \{1, 3\}, \{1, 24\}$.

(ii) The polynomial $ax - by$ is suitable if and only if (a, b) is among the ordered pairs

$$(1, 1),\ (2, 1),\ (2, 2),\ (4, 3),\ (6, 2).$$

Remark. Though the speaker is unable to show that $x + 2y$ or $2x - y$ is suitable, he has proved that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + 2y$ is a square (or a cube).

Write $n = x^2 + y^2 + z^2 + w^2$ with $x + y$ a cube

Theorem (Z.-W. Sun, May 24,2016) Any $n \in \mathbb{N}$ can be written as $n = x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y$ a cube.

Proof. We can easily verify the desired result for all $n = 0, 1, \ldots, 63$.

Now let $n \geqslant 64$ and assume that any $r = 0, 1, \ldots, n - 1$ can be written as

$$x^2 + y^2 + z^2 + w^2 \ (x, y, z, w \in \mathbb{Z})$$

with $x + y$ a cube. If $64 \mid n$, then $n/64$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y = t^3$ for some $t \in \mathbb{Z}$, hence $n = (8x)^2 + (8y)^2 + (8z)^2 + (8w)^2$ with $8x + 8y = (2t)^3$.

Now we consider the case $64 \nmid n$. We claim that $\{2n, 2n - 1, 2n - 64\} \not\subseteq E(1, 1, 1)$. If $\{2n, 2n - 1\} \subseteq E(1, 1, 1)$, then $2n = 4^k(8l + 7)$ for some $k \in \{2, 3\}$ and $l \in \mathbb{N}$, hence $2n - 64$ is $4^2(8l + 3)$ or $4^3(8l + 6)$, and thus $2n - 64 \notin E(1, 1, 1)$. So the claim is true.

Write $n = x^2 + y^2 + z^2 + w^2$ with $x + y$ a cube (continued)

By the claim, for some $\delta \in \{0, 1, 8\}$, we can write $2n - \delta^2$ as the sum of three squares two of which have the same parity. Hence we may write $2n - \delta^2 = (2x - \delta)^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$ and $y \equiv z \pmod 2$. It follows that

$$\begin{aligned} n =& \frac{(2x-\delta)^2 + \delta^2}{2} + \frac{y^2 + z^2}{2} \\ =& x^2 + (\delta - x)^2 + \left(\frac{y+z}{2}\right)^2 + \left(\frac{y-z}{2}\right)^2 \end{aligned}$$

with $x + (\delta - x) = \delta \in \{t^3 : \ t = 0, 1, 2\}$. This concludes the induction step.

$n = x^2 + y^2 + z^2 + w^2$ with $x + y + z$ a square (or a cube)

Theorem (Z.-W. Sun, April-May, 2016) Let $c \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + cz = t^m$ for some $t \in \mathbb{Z}$.

Proof for the Case $c = 1$. For $n = 0, \ldots, 4^m - 1$ we can easily verify the desired result directly.

Now let $n \in \mathbb{N}$ with $n \geqslant 4^m$. Assume that any $r \in \{0, \ldots, n-1\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + z \in \{t^m : \ t \in \mathbb{Z}\}$. If $4^m \mid n$, then there are $x, y, z, w \in \mathbb{Z}$ with $x^2 + y^2 + z^2 + w^2 = n/4^m$ such that $x + y + z = t^m$ for some $t \in \mathbb{Z}$, and hence

$$n = (2^m x)^2 + (2^m y)^2 + (2^m z)^2 + (2^m w)^2$$

with $2^m x + 2^m y + (2^m z) = 2^m (x + y + z) = (2t)^m$. Below we suppose that $4^m \nmid n$.

Continued the proof

It suffices to show that there are $x, y, z \in \mathbb{Z}$ and $\delta \in \{0, 1, 2^m\}$ such that

$$n = x^2 + (y+z)^2 + (z-y)^2 + (\delta - 2z)^2 = x^2 + 2y^2 + 6z^2 - 4\delta z + \delta^2.$$

(Note that $(y+z) + (z-y) + (\delta - 2z) = \delta \in \{t^m : \ t \in \mathbb{Z}\}$.)
Suppose that this fails for $\delta = 0$. As

$$E(1,2,6) = \{4^k(8l+5) : \ k, l \in \mathbb{N}\},$$

$n = 4^k(8l+5)$ for some $k, l \in \mathbb{N}$ with $k < m$. Clearly,

$$3n - 1 = \begin{cases} 3(8l+5) - 1 = 2(12l+7) & \text{if } k = 0, \\ 3 \times 4(8l+5) - 1 = 8(12l+7) + 3 & \text{if } k = 1. \end{cases}$$

Thus, if $k \in \{0, 1\}$, then $3n - 1$ does not belong to

$$E(2,3,6) = \{3q+1 : \ q \in \mathbb{N}\} \cup \{4^k(8l+7) : \ k, l \in \mathbb{N}\},$$

Continue the proof

hence for some $x, y, z \in \mathbb{Z}$ we have

$$3n - 1 = 3x^2 + 6y^2 + 2(3z-1)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 2z)) + 2$$

and thus

$$n = x^2 + 2y^2 + 6z^2 - 4z + 1 = x^2 + (y+z)^2 + (z-y)^2 + (1-2z)^2$$

as desired.

When $k = 2$ and $m = 3$, we have

$$3n - 64 = 3 \times 16(8l + 5) - 64 = 4^2(8(3l+1) + 3) \notin E(2, 3, 6),$$

and hence there are $x, y, z \in \mathbb{Z}$ such that

$$3n - 4^3 = 3x^2 + 6y^2 + 2(3z-8)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 16z)) + 2 \times 4^3$$

and thus

$$n = x^2 + 2y^2 + 6z^2 - 32z + 64 = x^2 + (y+z)^2 + (z-y)^2 + (2^3 - 2z)^2$$

as desired.

Write $n = x^2 + y^2 + z^2 + w^2$ with $x + 3y$ a square

In 1916 Ramanujan conjectured that

(1) *the only positive even numbers not of the form* $x^2 + y^2 + 10z^2$ *are those* $4^k(16l + 6)$ $(k, l \in \mathbb{N})$

and

(2) *sufficiently large odd numbers are of the form* $x^2 + y^2 + 10z^2$.

In 1927 L. E. Dickson [Bull. AMS] proved (1). In 1990 W. Duke and R. Schulze-Pillot [Invent. Math.] confirmed (2). In 1997 K. Ono and K. Soundararajan [Invent. Math.] proved that under the GRH (Generalized Riemann Hypothesis) any odd number greater than 2719 has the form $x^2 + y^2 + 10z^2$.

With the help of the Ono-Soundararajan result, the speaker has proved the following result.

Theorem (Z.-W. Sun, 2016) Under the GRH, any $n \in \mathbb{N}$ can be written as $n = x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 3y$ a square.

1-3-5-Conjecture

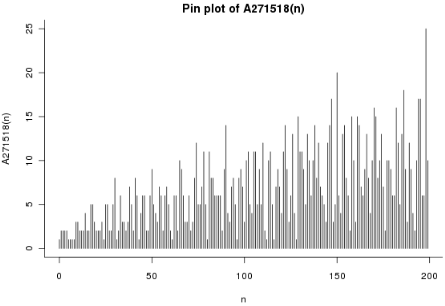
1-3-5-Conjecture (Z.-W. Sun, April 9, 2016): The polynomial $x+3y+5z$ is suitable. In other words, any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ such that $x+3y+5z$ is a square. Moreover, such a representation is unique only for

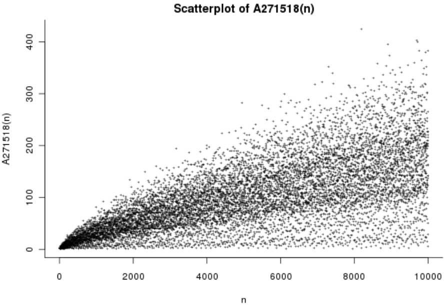
$$n=0,\ 4^k\times 6\,(k\in\mathbb{N}),\ 16^k\times m\,(k\in\mathbb{N},\ m\in\{5,7,8,31,43,61,116\}).$$

Examples.

$$\begin{aligned}
6 =&1^2+1^2+0^2+2^2 \text{ with } 1+3\times 1+5\times 0=2^2,\\
7 =&1^2+1^2+1^2+2^2 \text{ with } 1+3\times 1+5\times 1=3^2,\\
8 =&0^2+2^2+2^2+0^2 \text{ with } 0+3\times 2+5\times 2=4^2,\\
24 =&0^2+2^2+2^2+4^2 \text{ with } 0+3\times 2+5\times 2=4^2,\\
31 =&5^2+2^2+1^2+1^2 \text{ with } 5+3\times 2+5\times 1=4^2,\\
43 =&1^2+5^2+4^2+1^2 \text{ with } 1+3\times 5+5\times 4=6^2,\\
61 =&0^2+0^2+5^2+6^2 \text{ with } 0+3\times 0+5\times 5=5^2,\\
116 =&4^2+0^2+0^2+10^2 \text{ with } 4+3\times 0+5\times 0=2^2.
\end{aligned}$$

Graph for the number of such representations of n





A general theorem joint with Yu-Chen Sun

Theorem (Yu-Chen Sun and Z.-W. Sun, 2016) Let $a, b, c, d \in \mathbb{Z}$ with a, b, c, d not all zero. Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$ Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}/(a^2 + b^2 + c^2 + d^2)$ such that $ax + by + cz + dw = \lambda r^m$ for some $r \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. By a result of Z.-W. Sun, we can write $(a^2 + b^2 + c^2 + d^2)n$ as $(\lambda r^m)^2 + t^2 + u^2 + v^2$ with $r, t, u, v \in \mathbb{N}$. Set $s = \lambda r^m$, and define x, y, z, w by

$$\begin{cases} x = & \frac{as-bt-cu-dv}{a^2+b^2+c^2+d^2}, \\ y = & \frac{bs+at+du-cv}{a^2+b^2+c^2+d^2}, \\ z = & \frac{cs-dt+au+bv}{a^2+b^2+c^2+d^2}, \\ w = & \frac{ds+ct-bu+av}{a^2+b^2+c^2+d^2}. \end{cases}$$

Proof of the general theorem

Then

$$\begin{cases} ax + by + cz + dw = & s, \\ ay - bx + cw - dz = & t, \\ az - bw - cx + dy = & u, \\ aw + bz - cy - dx = & v. \end{cases}$$

With the help of Euler's four-square identity,

$$x^2 + y^2 + z^2 + w^2 = \frac{s^2 + t^2 + u^2 + v^2}{a^2 + b^2 + c^2 + d^2} = n$$

and

$$ax + by + cz + dw = s = \lambda r^m.$$

This concludes the proof.

Progress on the 1-3-5-Conjecture

Theorem (Yu-Chen Sun and Z.-W. Sun, 2016) (i) Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + z + 2w = \lambda r^m$ for some $r \in \mathbb{N}$.

(ii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 2y + 3z$ a square (or twice a square).

(iii) Let $\lambda \in \{1, 2\}$, $m \in \{2, 3\}$ and $n \in \mathbb{N}$. Then we can write n as $x^2 + y^2 + z^2 + w^2$ with $x, y, 5z, 5w \in \mathbb{Z}$ such that $x + 3y + 5z \in \{\lambda r^m : \ r \in \mathbb{N}\}$. Also, any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}/7$ such that $x + 3y + 5z \in \{\lambda r^m : \ r \in \mathbb{N}\}$.

Similar to part (ii), we are also able to show that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + 3z$ a square (or twice a square).

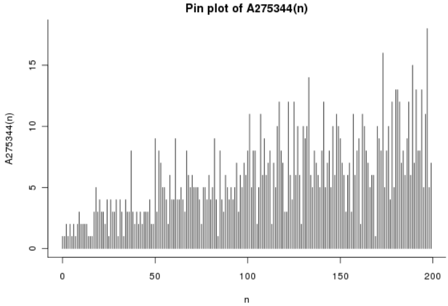
1-2-3-Conjecture (Companion of 1-3-5-Conjecture)

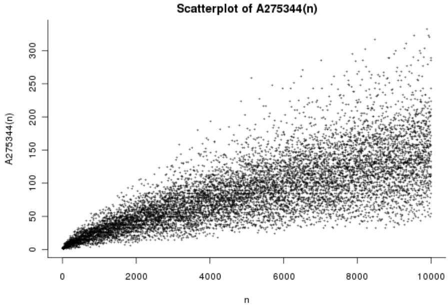
1-2-3-Conjecture (Z.-W. Sun, July 24, 2016): Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y + 3z$ is a square. Moreover, such a representation is unique only for $n \in \{0, 1, 3, 5, 7, 14, 15, 16, 25, 30, 84, 169, 225\}$.

Examples.

$$
\begin{aligned}
14 =& 1^2 + 1^2 + 2^2 + 2 \times 2^2 \ \text{ with } 1 + 2 \times 1 + 3 \times 2 = 3^2,\\
15 =& 3^2 + 0^2 + 2^2 + 2 \times 1^2 \ \text{ with } 3 + 2 \times 0 + 3 \times 2 = 3^2,\\
16 =& 4^2 + 0^2 + 0^2 + 2 \times 0^2 \ \text{ with } 4 + 2 \times 0 + 3 \times 0 = 2^2,\\
25 =& 1^2 + 4^2 + 0^2 + 2 \times 2^2 \ \text{ with } 1 + 2 \times 4 + 3 \times 0 = 3^2,\\
30 =& 3^2 + 2^2 + 3^2 + 2 \times 2^2 \ \text{ with } 3 + 2 \times 2 + 3 \times 3 = 4^2,\\
33 =& 1^2 + 0^2 + 0^2 + 2 \times 4^2 \ \text{ with } 1 + 2 \times 0 + 3 \times 0 = 1^2,\\
84 =& 4^2 + 6^2 + 0^2 + 2 \times 4^2 \ \text{ with } 4 + 2 \times 6 + 3 \times 0 = 4^2,\\
169 =& 10^2 + 6^2 + 1^2 + 2 \times 4^2 \ \text{ with } 10 + 2 \times 6 + 3 \times 1 = 5^2,\\
225 =& 10^2 + 6^2 + 9^2 + 2 \times 2^2 \ \text{ with } 10 + 2 \times 6 + 3 \times 9 = 7^2.
\end{aligned}
$$

Graph for the number of such representations of n





Conjectures involving higher powers

Conjecture (Z.-W. Sun, May 23, 2016): For each $c = 1, 2, 4$, any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $2x + y - z = ct^3$ for some $t \in \mathbb{N}$.

Conjecture (Z.-W. Sun, May 31, 2016): Any positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ with $x \in \mathbb{Z}^+$, $y \in \mathbb{N}$ and $z, w \in \mathbb{Z}$ such that $xy + yz + zw$ is a fourth power.

Example. $1016 = 2^2 + 20^2 + 6^2 + (-24)^2$ with $2 \times 20 + 20 \times 6 + 6 \times (-24) = 2^4$.

Conjecture (Z.-W. Sun, June 6, 2016): Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $xy + yz + 2zw + 2wx$ is a fifth power.

Two related theorems on partitions of integers

Theorem 1 (Z.-W. Sun, April 11, 2016). Let $n > 2$ be an integer.

(i) We can write $n = x + y + z$ with $x, y, z \in \mathbb{Z}^+$ such that $x + 11y + 13z$ is a square.

(ii) We can write $n = x + y + z$ with $x, y, z \in \mathbb{Z}^+$ such that $x + 240y + 720z$ is a square.

Theorem 2 (Z. W. Sun, April 2016). Let $a, b, c, m \in \mathbb{Z}^+$ with $a < b \leqslant c$ and $\gcd(b - a, c - a) = 1$. Then any sufficiently large integer can be written as $x + y + z$ with $x, y, z \in \mathbb{Z}^+$ such that $ax + by + cz = p^m$ for some prime number p.

P. Dusart [Math. Comp. 68(1999)] proved that for $x \geqslant 3275$ there is a prime p with $x \leqslant p \leqslant x + x/(2\log^2 x)$. With the help of this, we can modify our proof of Theorem 2 to show some concrete results (e.g., any integer $n \geqslant 6$ can be written as $x + y + z$ $(x, y, z \in \mathbb{Z}^+)$ with $x + 3y + 6z = p^2$ for some prime p).

Suitable polynomials of the form $ax - by - cz$ or $ax + by - cz$

Conjecture (Z.-W. Sun, April 14, 2016): Let $a, b, c \in \mathbb{Z}^+$ with $b \leqslant c$ and $\gcd(a, b, c)$ squarefree. Then $ax - by - cz$ is suitable if and only if (a, b, c) is among the five triples

$$(1,1,1),\ (2,1,1),\ (2,1,2),\ (3,1,2),\ (4,1,2).$$

We conjecture that there are totally 52 concrete triples (a, b, c) with $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b$, and $\gcd(a, b, c)$ squarefree such that $ax + by - cz$ is suitable. Two of them are $(48, 49, 48)$ and $(48, 121, 48)$, and the other 50 triples satisfying $a, b, c \leqslant 32$.

Conjecture (Z.-W. Sun, April 14, 2016): If (a, b, c) is among the triples

$$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,3),\ (1,3,1),$$
$$(1,3,3),\ (1,4,4),\ (1,5,1),\ (1,6,6),\ (1,8,6),\ (2,2,2),\ (2,2,4),$$
$$(2,3,2),\ (2,3,3),\ (2,4,1),\ (2,4,2),\ (2,6,1),\ (2,6,6),\ (2,7,4),$$

then $ax + by - cz$ is suitable.

A theorem joint with Yu-Chen Sun

Conjecture (Z.-W. Sun, May 2016) (i) Any positive integer can be written as $w^2 + x^2 + y^2 + z^2$ with $w + x + y - z$ a square, where $w \in \mathbb{Z}$ and $x, y, z \in \mathbb{N}$ with $|w| \leqslant x \geqslant y \leqslant z < x + y$.

(ii) Each $n \in \mathbb{N}$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w + x + y - z$ a nonnegative cube, where w, x, y, z are integers with $|x| \leqslant y \geqslant z \geqslant 0$.

Theorem (Y.-C. Sun and Z.-W. Sun, arXiv:1605.03074). (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + z + w$ is a square (or a cube).

(ii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + 2y + 2z$ is a square (or a cube).

In the case $2 \nmid n$, we actually proved part (i) of the Theorem by showing that n can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + z + w = 1$.

Suitable polynomials of the form $ax+by+cz-dw$ or $ax+by-cz-dw$

Conjecture (Z.-W. Sun, April 14, 2016): Let $a,b,c,d\in\mathbb{Z}^+$ with $a\leqslant b\leqslant c$, and $\gcd(a,b,c,d)$ squarefree. Then $ax+by+cz-dw$ is suitable if and only if (a,b,c,d) is among the 12 quadruples

$$(1,1,2,1),\ (1,2,3,1),\ (1,2,3,3),\ (1,2,4,2),$$
$$(1,2,4,4),\ (1,2,5,5),\ (1,2,6,2),\ (1,2,8,1),$$
$$(2,2,4,4),\ (2,4,6,4),\ (2,4,6,6),\ (2,4,8,2).$$

Conjecture (Z.-W. Sun, April 14, 2016): Let $a,b,c,d\in\mathbb{Z}^+$ with $a\leqslant b$ and $c\leqslant d$, and $\gcd(a,b,c,d)$ squarefree. Then $ax+by-cz-dw$ is suitable if and only if (a,b,c,d) is among the 9 quadruples

$$(1,2,1,1),\ (1,2,1,2),\ (1,3,1,2),\ (1,4,1,3),$$
$$(2,4,1,2),\ (2,4,2,4),\ (8,16,7,8),\ (9,11,2,9),\ (9,16,2,7).$$

Suitable polynomials of the form $ax^2 + by^2 + cz^2$

Conjecture (Z.-W. Sun, April 9, 2016): (i) Any natural number can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x \geqslant y$ such that $ax^2 + by^2 + cz^2$ is a square, provided that the triple (a, b, c) is among

$$(1, 8, 16),\ (4, 21, 24),\ (5, 40, 4),\ (9, 63, 7),\ (16, 80, 25),$$
$$(16, 81, 48),\ (20, 85, 16),\ (36, 45, 40),\ (40, 72, 9).$$

(ii) $ax^2 + by^2 + cz^2$ is suitable if (a, b, c) is among the triples

$$(1, 3, 12),\ (1, 3, 18),\ (1, 3, 21),\ (1, 3, 60),\ (1, 5, 15),$$
$$(1, 8, 24),\ (1, 12, 15),\ (1, 24, 56),\ (3, 4, 9),\ (3, 9, 13),$$
$$(4, 5, 12),\ (4, 5, 60),\ (4, 9, 60),\ (4, 12, 21),\ (4, 12, 45),\ (5, 36, 40).$$

(iii) If a, b, c are positive integers with $ax^2 + by^2 + cz^2$ suitable, then a, b, c cannot be pairwise coprime.

On $ax^2-by^2-cz^2$ and $ax^2+by^2-cz^2-dw^2$

Conjecture (Z.-W. Sun, April 14, 2016): (i) Any natural number can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ and $x\geqslant y$ such that $ax^2-by^2-cz^2$ is a square, provided that the triple (a,b,c) is among

$$(21,5,15),\ (36,3,8),\ (48,8,39),\ (64,7,8),$$
$$(40,15,144),\ (45,20,144),\ (69,20,60).$$

(ii) $ax^2+by^2-cz^2-dw^2$ is suitable if (a,b,c,d) is among the quadruples

$$(3,9,3,20),\ (5,9,5,20),\ (5,25,4,5),\ (9,81,9,20),\ (12,16,3,12),$$
$$(16,64,15,16),\ (20,25,4,20),\ (27,81,20,27),\ (30,64,15,30).$$

It seems that there are infinitely many quadruples (a,b,c,d) with $a,b,c,d\in\mathbb{Z}^+$ and $\gcd(a,b,c,d)$ squarefree such that $ax^2+by^2+cz^2-dw^2$ is suitable. For example, we conjecture that $x^2+3y^2+5z^2-8w^2$ is suitable.

Suitable polynomials related to Pythagorean triples

Conjecture (Z.-W. Sun, April 16, 2016). Both $(x+2y)^2+8z^2+40w^2$ and $9(x+2y)^2+16z^2+24w^2$ are all suitable.

Conjecture (Z.-W. Sun, April 12, 2016). Any $n \in \mathbb{Z}^+$ can be written as $w^2+x^2+y^2+z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $(10w+5x)^2+(12y+36z)^2$ is a square.

Examples.

$3 = 1^2+1^2+0^2+1^2$ with $(10\times 1+5\times 1)^2+(12\cdot 0+36\cdot 1)^2 = 39^2$,

$4 = 2^2+0^2+0^2+0^2$ with $(10\cdot 2+5\cdot 0)^2+(12\cdot 0+36\cdot 0)^2 = 20^2$,

$7 = 1^2+2^2+1^2+1^2$ with $(10\cdot 1+5\cdot 2)^2+(12\cdot 1+36\cdot 1)^2 = 52^2$,

$19 = 3^2+0^2+3^2+1^2$ with $(10\cdot 3+5\cdot 0)^2+(12\cdot 3+36\cdot 1)^2 = 78^2$,

$133 = 9^2+0^2+6^2+4^2$ with $(10\cdot 9+5\cdot 0)^2+(12\cdot 6+36\cdot 4)^2 = 234^2$,

and $589 = 17^2+10^2+2^2+14^2$ with

$$(10\cdot 17+5\cdot 10)^2+(12\cdot 2+36\cdot 14)^2 = 220^2+528^2 = 572^2.$$

More things related to Pythagorean triples

Conjecture (Z.-W. Sun, May 15, 2016). (i) Any positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > z$ such that $(x + y)^2 + (4z)^2$ is a square.

(ii) Any integer $n > 5$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $8x + 12y$ and $15z$ are the two legs of a right triangle with positive integer sides.

Theorem (Z.-W. Sun, May 16, 2016). Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > 0$ such that $x + 4y + 4z$ and $9x + 3y + 3z$ are the two legs of a right triangle with positive integer sides.

An easier conjecture related to Pythagorean triples

Conjecture (Z.-W. Sun, April 12, 2016). Any integer $n > 10$ can be written as $x + y + z$ with $x, y, z \in \mathbb{Z}^+$, $x \geqslant y$ and $\gcd(x, y, z) = 1$ such that $x^2 + (2y + z)^2$ is a square.

Examples. We have

$$\begin{aligned}
11 =& 6 + 3 + 2 \text{ with } \gcd(6,3,2) = 1,\ 6^2 + (2 \times 3 + 2)^2 = 10^2,\\
14 =& 5 + 3 + 6 \text{ with } \gcd(5,3,6) = 1,\ 5^2 + (2 \times 3 + 6)^2 = 13^2,\\
24 =& 7 + 7 + 10 \text{ with } \gcd(7,7,10) = 1,\ 7^2 + (2 \times 7 + 10)^2 = 25^2,\\
54 =& 28 + 19 + 7 \text{ with } \gcd(28,19,7) = 1,\ 28^2 + (2 \times 19 + 7)^2 = 53^2.
\end{aligned}$$

When $n = x^2 + y^2 + z^2 + w^2$ with $x + y = z$?

A Lemma. Let $n \in \mathbb{N}$. Then $n \notin E(1,2,6)$ if and only if $n = x^2 + y^2 + z^2 + w^2$ for some $x, y, z, w \in \mathbb{N}$ with $x + y = z$.

Proof. Assume that $n \notin E(1,2,6)$. Then, there are $x, y, z \in \mathbb{N}$ for which $n = x^2 + 2y^2 + 6z^2 = x^2 + (y+z)^2 + |y-z|^2 + (2z)^2$. Clearly $(y+z) + |y-z| = 2z$ if $y \leqslant z$, and $|y-z| + 2z = y + z$ if $y > z$. Therefore $n = x^2 + u^2 + v^2 + w^2$ for some $u, v, w \in \mathbb{N}$ with $u + v = w$.

Now suppose that $n = x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x + y = z$. If $x \equiv y \pmod 2$, then

$$n = w^2 + 2\left(\frac{x-y}{2}\right)^2 + 6\left(\frac{x+y}{2}\right)^2$$

and hence $n \notin E(1,2,6)$. When $x \not\equiv y \pmod 2$, without loss of generality we assume that $y \equiv z \pmod 2$, hence

$$n = w^2 + 2\left(\frac{y+z}{2}\right)^2 + 6\left(\frac{y-z}{2}\right)^2$$

and thus $n \notin E(1,2,6)$.

$x^2y^2 + y^2z^2 + z^2x^2$ is suitable

Theorem (Z.-W. Sun, May 6, 2016). Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $x^2y^2 + y^2z^2 + z^2x^2$ is a square.

Proof. If n can be written as the sum of three squares, then $n = x^2 + y^2 + 0^2 + w^2$ for some $x, y \in \mathbb{N}$ and $w \in \mathbb{Z}^+$. Clearly $x^2y^2 + y^2 0^2 + 0^2x^2 = (xy)^2$ is a square.
If $n \in E(1,1,1)$, then n has the form $4^k(8l + 7)$ with $k, l \in \mathbb{N}$, and hence $n \notin E(1,2,6) = \{4^k(8l+5): \ k, l \in \mathbb{N}\}$. Thus, by the lemma, there are $x, y, z, w \in \mathbb{N}$ with $x + y = z$ such that $n = x^2 + y^2 + z^2 + w^2$. Clearly $w \neq 0$. Observe that

$$\begin{aligned} x^2y^2 + y^2z^2 + z^2x^2 =& (xy)^2 + (x^2+y^2)(x+y)^2 \\ =& (xy)^2 + (x^2+xy+y^2-xy)(x^2+xy+y^2+xy) \\ =& (x^2+xy+y^2)^2. \end{aligned}$$

In contrast with the Theorem, we conjecture that the polynomial $x^2y^2 + 9y^2z^2 + 9z^2x^2$ is suitable.

More results and conjectures

Theorem (Z.-W. Sun, May 2016). For $(b, c) = (8, 8), (16, 64)$, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $x^4 + by^3z + cyz^3$ is a fourth power.

Conjecture (Z.-W. Sun, May 2016) (i) If (a, b) is among the ordered pairs

$$(1, 1),\ (1, 15),\ (1, 20),\ (1, 36),\ (1, 60),\ (9, 260),$$

then any positive integer can be written as $x^2 + y^2 + z^2 + w^2$ with $ax^4 + by^3z$ a square, where $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$.

(ii) For each triple $(a, b, c) = (1, 20, 60)$, (1,24,56), (9,20,60), (9,32,96), any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $ax^4 + by^3z + cyz^3$ is a square.

Theorem (Conjectured by Z.-W. Sun and essentially proved by You-Ying Deng and Yu-Chen Sun) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x^2 + 4yz$ (or $x^2 + 8yz$) a square.

Other interesting suitable polynomials

Conjecture (Z. W. Sun). (i) (April 11, 2016) Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $wx + 2xy + 2yz$ is a square.

(ii) (April 12, 2016) Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $w^2 + 4xy + 8yz + 32zx$ is a square.

(iii) (April 13, 2016) Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $x \in \mathbb{Z}^+$ and $w, y, z \in \mathbb{N}$ such that $w(x + 2y + 3z)$ is a square.

(iv) (April 17, 2016) The polynomial $w(x^2 + 8y^2 - z^2)$ is suitable.

(v) (April 17, 2016) The polynomials $w^2x^2 + 3x^2y^2 + 2y^2z^2$ and $36x^2y + 12y^2z + z^2x$ are suitable.

(vi) (April 19, 2016) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $z < w$ such that $4x^2 + 5y^2 + 20zw$ is a square.

Other interesting suitable polynomials (continued)

(vii) (April 17, 2016) Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $w^2x^2 + 5x^2y^2 + 80y^2z^2 + 20z^2w^2$ is a square.

(viii) (April 30, 2016) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y \geqslant z$ such that $xyz(x + 3y + 13z)$ is a square.

(ix) (May 1, 2016) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $xyz(x+9y+11z+10w)$ is a square.

(x) (May 4, 2016) For each triple $(a, b, c) = (1, 2, 4), (1, 2, 9), (1, 3, 4), (2, 3, 4), (2, 4, 6), (4, 8, 10)$, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $w(25w + 24(ax + by + cz))$ is a square.

Two more conjectures

Conjecture (Z.-W. Sun, May 7, 2016). (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $xy + 2zw$ or $xy - 2zw$ is a square.

(ii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $\max\{x, y\} \geqslant \min\{z, w\}$ such that $xy + zw/2$ or $xy - zw/2$ is a square.

Conjecture (Z.-W. Sun, May 12, 2016). (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$, $x \geqslant z$ and $w \in \mathbb{Z} \setminus \{0\}$ such that $3x^2y + z^2w$ is a square.

(ii) For each ordered pair $(a, b) = (7, 1),\ (8, 1),\ (9, 2)$, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z} \setminus \{0\}$ such that $ax^2y + bz^2w$ is a square.

Remark. It seems hopeless to prove the above two conjectures.

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Conjecture (Z.-W. Sun, August 7, 2016). Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2(1 + y^2 + z^2)$ with $w, x, y, z \in \mathbb{N}$, $x > 0$ and $y \equiv z \pmod 2$. Moreover, any $n \in \mathbb{Z}^+$ with $n \neq 449$ can be written as $4^k(1 + x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$ and $x \equiv y \pmod 2$.

Theorem (i) The above conjecture holds provided that for any integer $n > 432$ we can write $16n + 1 = 32x^2 + 32y^2 + z^2$ with $x, y, z \in \mathbb{Z}$.
(ii) Any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + 4x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$.
(iii) Under the GRH, any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + 5x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$, and also any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + x^2 + y^2) + 5z^2$ with $k, x, y, z \in \mathbb{N}$.

Our proof of part (iii) of the Theorem uses the main result of Ben Kane and Zhi-Wei Sun [Trans. AMS 362(2010), 6425–6455] where the authors determined for what $a, b, c \in \mathbb{Z}^+$ sufficiently large integers can be expressed as $ax^2 + by^2 + cT_z$ with $x, y, z \in \mathbb{Z}$.

References

For the main sources of my above conjectures and related results, you may look at two recent preprints:

1. Zhi-Wei Sun, *Refining Lagrange's four-square theorem*, `arXiv:1604.06723`, `http://arxiv.org/abs/1604.06723`.

2. Yu-Chen Sun and Zhi-Wei Sun, *Two refinements of Lagrange's four-square theorem*, `arXiv:1605.03074`, `http://arxiv.org/abs/1605.03074`.

Thank you!