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Historical Remarks on my Conjectural Congruences and Series for $1/\pi$

Zhi-Wei Sun

Nanjing University
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

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My initial contact with π-series (1984-86)

When I was an undergraduate at Nanjing University, I learned from calculus the following classical results :

Leibniz:

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

Euler:

$$\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

But I did not know any other π-series then.

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$, $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$, and $|x| < 1$.

My first impression on Ramanujan-type series

During 2003-2006 I happened to see a paper on Ramanujan-type series. Here is one of Ramanujan series for $1/\pi$:

$$\sum_{k=0}^{\infty}(28k+3)\frac{(-27)^k}{512^k}\cdot\frac{(1/2)_k(1/6)_k(5/6)_k}{(1)_k^3}=\frac{32\sqrt{2}}{\pi}.$$

At that time I felt that this is too complicated! I did not like it at all. I only enjoy simple and beautiful results! Thus this paper gave me almost no impression. Consequently, I could not remember what paper it is.

During that period I also saw Mortenson's 2003 paper in J. Number Theory in which he used character sums and the p-adic Gamma functions to prove the following congruence conjectured by F. Rodriguez-Villegas:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2}\quad\text{for any prime } p>2.$$

But at that time I had no feeling about such a congruence.

My joint work on congruences modulo prime powers

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod p\quad(d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod p\quad\text{for } p>3.$$

Sun & R. Tauraso [AAM 45(2010); IJNT 7(2011)].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\quad\text{for } p>3,$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers given by

$$B_0=1,\quad\sum_{k=0}^{n}\binom{n+1}{k}B_k=0\quad(n=1,2,3,\ldots).$$

My result on $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod p^2

Sun [Sci. China Math. 53(2010)]: Let p be an odd prime and let $m \in \mathbb{Z}$ with $p \nmid m$. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m^2-4m}{p}\right) + u_{p-(\frac{m^2-4m}{p})} \pmod{p^2},$$

where $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0 = 0,\ u_1 = 1,\ \text{and}\ u_{n+1} = (m-2)u_n - u_{n-1}\ (n = 1, 2, 3, \ldots).$$

In particular,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} \pmod{p^2}.$$

Remark. Later I found that the last congruence p^3 is related to Euler numbers.

Multinomial coefficients

Multinomial coefficients:

$$\binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n} = \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}.$$

Note that $\binom{2k}{k} = \binom{2k}{k,k}$. So, a natural extension of $\binom{2k}{k}$ is

$$\binom{kn}{k, k, \ldots, k} = \frac{(kn)!}{(k!)^n}.$$

Clearly,

$$\binom{3k}{k, k, k} = \binom{2k}{k}\binom{3k}{k}$$

and

$$\binom{4k}{k, k, k, k} = \binom{2k}{k}^2\binom{4k}{2k}.$$

My result and conjecture on multinomial coefficients

Theorem (Sun [Acta Arith. 148(2011)]). An integer $p > 1$ is a prime if and only if

$$\sum_{k=0}^{p-1} \binom{(p-1)k}{k,\ldots,k} \equiv 0 \pmod{p}.$$

Conjecture (Sun [Acta Arith. 148(2011)]). For any odd prime p and positive integer n,

$$\frac{1}{n\binom{2n}{n}} \sum_{k=0}^{n-1} \binom{(p-1)k}{k,\ldots,k}$$

is always a p-adic integer.

Remark. When $p = 3$, Strauss, Shallit and Zagier [Amer. Math. Monthly 99(1992)] show that $\sum_{k=0}^{n-1}\binom{2k}{k}/(n^2\binom{2n}{n})$ is a 3-adic integer for any $n = 1, 2, 3, \ldots$.

What happened in November, 2009

During Nov. 6-7, 2009 both Zhi-Hong and I attended the 1st National Conference on Combinatorial Number Theory held at Nanjing Normal University. After the conference, Zhi-Hong did not return home and came to our univ. to copy some books.

On Nov. 10, 2009 I had a supper with Zhi-Hong who brought a copy of Ken Ono's book *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series* (Amer. Math. Soc., 2004). I had a glance at the last page of the book and found a list of few supercongruences conjectured by F. Rodriguez-Villegas and proved by Mortenson including

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k,k,k}}{27^k}\equiv\left(\frac{-3}{p}\right)\pmod{p^2}.$$

I knew such things before. But, on that day, as I had determined $\sum_{k=0}^{p-1}\binom{2k}{k,k}/m^k$ mod p^2, I suddenly realized that I should check $\sum_{k=0}^{p-1}\binom{3k}{k,k,k}/m^k$ mod p^2 via `Mathematica` which I just began to learn. I wished to go home immediately (for secret computation).

What happened in November, 2009

But Zhi-Hong insisted that I should live with him in the guest room at the New Era Hotel. So, on Nov. 10, 2009 I brought my computer to the hotel and found that $\sum_{k=0}^{p-1}\binom{3k}{k,k,k}/24^k\equiv 0\pmod{p}$ for any odd prime $p\equiv 2\pmod{3}$. Later I figured out the pattern and conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k,k,k}}{24^k}\equiv\begin{cases}\binom{2(p-1)/3}{(p-1)/3}\pmod{p^2} & \text{if } p\equiv 1\pmod{3},\\ p/\binom{2(p+1)/3}{(p+1)/3}\pmod{p^2} & \text{if } p\equiv 2\pmod{3}.\end{cases}$$

(See Conj. 5.13 of Sun [Sci. China Math. 54(2011)].)

I also noted that

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{81^k}\equiv\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv 0\pmod{p^2}$$

for half of the primes:

$p=5,13,17,19,31,41,59,61,73,83,89,97,101,103,131,139,157,\ldots$

But I could not find the pattern for these primes.

What happened in November, 2009

In the afternoon of Nov. 11 (Wednesday) we had a seminar. First I reported my discovery and asked if anybody (my students and Zhi-Hong) can figure out the pattern of those primes. Nobody gave an answer. Then I left for a meeting in our dept and Zhi-Hong gave a talk on his results on Euler numbers. When I got to the dept there are still few minutes left, so I opened my computer and searched the sequence 5,13,17,19,31,41 via google and this led me to find that these primes are quadratic nonresidues modulo 7. I immediately called my student Yong Zhang or Hao Pan in the seminar to inform this news.

On Nov. 11 I wrote a draft and posted it to arXiv. The results in the paper include

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p \pmod p & \text{if } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod p & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Several days later I learned that this is not new, it has been proved to hold mod p^2.

What happened in November, 2009

If $(\frac{p}{7}) = 1$, what about $\sum_{k=0}^{p-1}\binom{2k}{k}^3$ mod p^2 ? I was puzzled by this. On Friday afternoon (Nov. 13), I attended another meeting in out dept, and suddenly remembered that $\mathbb{Q}(\sqrt{-7})$ is an Euclidean domain and hence an PID as I often taught undergraduates in the course Modern Algebra. If $(\frac{p}{7}) = 1$, i.e., $(\frac{-7}{p}) = 1$, then p splits by algebraic number theory and thus p can be written in the form

$$\frac{x+y\sqrt{-7}}{2} \times \frac{x-y\sqrt{-7}}{2} = \frac{x^2+7y^2}{4},$$

as both x and y must be even we have $p = (x/2)^2 + 7(y/2)^2$. After the meeting I immediately went back and verified this observation from Cox's book *Primes of the Form* $x^2 + ny^2$. Thus this led me to find that if $(\frac{p}{7}) = 1$ and $p = x^2 + 7y^2$ then $\sum_{k=0}^{p-1}\binom{2k}{k}^2 \equiv 4x^2 - 2p \pmod{p^2}$. I updated my arXiv article to add this immediately.

I also found patterns for $\sum_{k=0}^{p-1}\binom{2k}{k}^2/m^k$ mod p^2 with $m = 8, -16, 32$.

What happened in November, 2009

On Nov. 14 (Saturday) I called Zhi-Hong and informed my discovery. He said that he just wanted to make computations to determine $\sum_{k=0}^{p-1}\binom{2k}{k}^3 \bmod p^2$ in the case $(\frac{p}{7})=1$, and he complained that his student was too lazy and did not compute for him.

Lesson. If one has not yet formulated a complete conjecture, better not inform others to avoid potential competition.

On Nov. 11 I also conjectured that if $(\frac{p}{7})=-1$ then $\sum_{k=0}^{p-1}k\binom{2k}{k}^3\equiv 0 \pmod p$. On Nov. 27, 2009 I posted *Open Conjectures on Congruences* to collect my conjectural congruences. After reading this material, on Nov. 28 Bilgin Ali and Bruno Mishutka guessed that if $p=x^2+7y^2$ then

$$\sum_{k=0}^{p-1}k\binom{2k}{k}^3\equiv\begin{cases}11y^2/3-x^2 \pmod p & \text{if } 3\mid y,\\ 4(y^2-x^2)/3 \pmod p & \text{if } 3\nmid y.\end{cases}$$

What happened in November, 2009

Inspired this I immediately realized that

$$\sum_{k=0}^{p-1} k\binom{2k}{k}^3 \equiv \begin{cases} \frac{8}{21}(3-4x^2) \pmod{p^2} & \text{if } p = x^2+7y^2, \\ \frac{8}{21}p \pmod{p^2}. \end{cases}$$

and circulated this via a message to Number Theory Mailing List.

Thus, in Nov. 2009 I formulated complete conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3 \bmod p^2$ and $\sum_{k=0}^{p-1} k\binom{2k}{k}^3 \bmod p^2$.

Prof. Ken Ono was very interested in this and he and one of his students worked on my conjecture. They claimed that they had a proof but in Jan. 2010 they replied me that they met real difficulties.

What happened in Jan.-Feb., 2010

I visited India during Jan.-Feb. 2010. On Jan. 23 I suddenly realized that I should combine the congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ and $\sum_{k=0}^{p-1}k\binom{2k}{k}^3/m^k$ mod p^2. This led me to conjecture that

$$\frac{1}{p}\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3 \equiv 8+16p^3B_{p-3} \pmod{p^4} \qquad (*)$$

and that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3 \in \mathbb{Z}.$$

After reading my message to Number Theory List on Feb. 10, Kasper Andersen found on Feb. 11 that

$$\frac{1}{n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3 = \sum_{k=0}^{n-1}\binom{n+k-1}{k}^2$$

via Sloane's OEIS (Online Encyclopedia of Integer Sequences). Inspired by this I finally proved $(*)$.

van Hamme's conjecture

After I found $\sum_{k=0}^{p-1}\binom{2k}{k}^3/4096^k \bmod p^2$ and conjectured the congruence

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right)-p^3E_{p-3}\pmod{p^4},$$

I got to know that van Hamme had the conjecture

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right)\pmod{p^3}$$

motivated by Ramanujan's identity

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi}.$$

Thus I became interested in Ramanujan-type series and wrote to several mathematicians to get Hamme's paper.

Ramanujan-type series for $1/\pi$

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are totally 36 known Ramanujan-type series for $1/\pi$ with a, b, m rational.

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Remark. This yielded the record for the calculation of π during 1989-1994.

My Philosophy about Series for $1/\pi$

Part I of the Philosophy. Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we must have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k}\equiv 0 \pmod n$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1,2,2^2,2^3,\ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d>1$ and for any prime $p>3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k}\equiv\begin{cases}(\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p=x^2+dy^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p})=-1,\end{cases}$$

or $d=1$, $\gcd(15,m)>1$, and for any prime $p\equiv 3 \pmod 4$ with $p\nmid 3m$ we have $\sum_{k=0}^{p-1}a_k/m^k\equiv 0 \pmod{p^2}$.

Philosophy about Series for $1/\pi$ (continued)

Part II of the Philosophy. Let $b, c, m, a_0, a_1, \ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k} \equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right) \pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

Illustrating the Philosophy by an Example

For $b, c \in \mathbb{Z}$ let $T_k(b,c)$ be the coefficient of x^k in $(x^2+bx+c)^k$.

Conjecture (Sun). Let $p > 5$ be a prime. Then

$$\left(\frac{15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}$$

$$\equiv\begin{cases} x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=1 \ \&\ 4p=x^2+91y^2, \\ 2p-7x^2 \pmod{p^2} & \text{if } (\frac{p}{7})=(\frac{p}{13})=-1 \ \&\ 4p=7x^2+13y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{91})=-1. \end{cases}$$

$$\sum_{k=0}^{p-1}(1638k+277)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}$$

$$\equiv\frac{p}{40}\left(8701\left(\frac{-\mathbf{105}}{p}\right)+2379\left(\frac{735}{p}\right)\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}.$$

Another Example Illustrating the Philosophy

Recall my following conjectural series

$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi}.$$

Actually this identity was motivated by the following conjecture.

Conjecture (Sun). Let $p>3$ be a prime with $p\neq 7$. Then

$$\left(\frac{-42}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)}{(-168^2)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2,\\ 12x^2-2p \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1.\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)$$

$$\equiv p\left(3\left(\frac{-\mathbf{42}}{p}\right)+10\left(\frac{-\mathbf{210}}{p}\right)\right) \pmod{p^2}.$$

The 3rd Example Illustrating the Philosophy

Conjecture (Sun). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{T_k(10,121)^3}{(2^{11}3^3)^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \ \& \ p = x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{66k+17}{(2^{11}3^3)^k} T_k^3(10,121)$$

$$\equiv \frac{p}{11}\left(195\left(\frac{-2}{p}\right) - 8\left(\frac{-6}{p}\right)\right) \pmod{p^2}.$$

Also,

$$\sum_{k=0}^{\infty} \frac{66k+17}{(2^{11}3^3)^k} T_k^3(10,11^2) = \frac{540\sqrt{2}}{11\pi}.$$

The 4th Example Illustrating the Philosophy

I would like to offer \$90 for the first proof of the identity in the following conjecture and \$1050 for the first proof of congruences in the conjecture.

Conjecture (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

The 4th Example Illustrating the Philosophy (continued)

And

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } 2p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 2p-6x^2 \pmod{p^2} & \text{if } 2p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 10x^2-2p \pmod{p^2} & \text{if } 2p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 28x^2-2p \pmod{p^2} & \text{if } p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 14x^2-2p \pmod{p^2} & \text{if } 2p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p})=-1.\end{cases}$$

Remark. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

My conjectural series of a special type

I have 18 conjectural series like the following five.

$$\sum_{k=0}^{\infty}\frac{340k+59}{(-480^2)^k}\binom{2k}{k}^2 T_{2k}(62,1)=\frac{120}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{13940k+1559}{(-5760^2)^k}\binom{2k}{k}^2 T_{2k}(322,1)=\frac{4320}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{14280k+899}{39200^{2k}}\binom{2k}{k}^2 T_{2k}(198,1)=\frac{1155\sqrt{6}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{57720k+3967}{439280^{2k}}\binom{2k}{k}^2 T_{2k}(5778,1)=\frac{2890\sqrt{19}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{1615k-314}{243360^{2k}}\binom{2k}{k}^2 T_{2k}(54758,1)=\frac{1989\sqrt{95}}{4\pi}.$$

Remark. I conjectured that my list of the 18 series of that type is complete! Prof. G. Almkvist asked me why I thought so.

My criterion for existence of series for $1/\pi$ of a special type

Hypothesis (Sun, 2011). (i) Suppose that

$$\sum_{k=0}^{\infty}\frac{a_0+a_1k}{m^k}\binom{2k}{k}^2T_{2k}(b,1)=\frac{C}{\pi}$$

with $a_0,a_1,b,m\in\mathbb{Z}$, $b>0$ and $C^2\in\mathbb{Q}\setminus\{0\}$. Then $\sqrt{|m|}$ is an integer dividing $16(b^2-4)$. Also, $b=7$ or $b\equiv 2\pmod 4$.

(ii) Let $\varepsilon\in\{\pm1\}$, $b,m\in\mathbb{Z}^+$ and $m\mid 16(b^2-4)$. Then, there are $a_0,a_1\in\mathbb{Z}$ such that

$$\sum_{k=0}^{\infty}\frac{a_0+a_1k}{(\varepsilon m^2)^k}\binom{2k}{k}^2T_{2k}(b,1)=\frac{C}{\pi}$$

for some $C\neq0$ with C^2 rational, if and only if $m>4(b+2)$ and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(b,1)}{(\varepsilon m^2)^k}\equiv\left(\frac{\varepsilon(b^2-4)}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(b,1)}{(\varepsilon\bar{m}^2)^k}\pmod{p^2}$$

for all odd primes $p\nmid b^2-4$, where $\bar{m}=16(b^2-4)/m$.

My general philosophy about p-adic congruences

Given a *natural* sequence $a_0, a_1, a_2, \ldots$ of p-adic integers, we may consider $\sum_{k=0}^{p-1} a_k$ or $\sum_{k=0}^{(p-1)/2} a_k$ modulo power of p because such a sum usually behaves better than a general term. When a_{p-1} and $a_{(p-1)/2}$ modulo powers of p obey certain patterns, the partial sum $\sum_{k=0}^{p-1} a_k$ or $\sum_{k=0}^{(p-1)/2} a_k$ should also have patterns modulo powers of p.

Example. Let $p > 3$ be a prime. Then

$$\binom{2p}{p} = 2\binom{2p-1}{p-1} \equiv 2 \pmod{p^3} \ \text{(J. Wolstenholme, 1863)},$$

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3} \ \text{(F. Morley, 1895)},$$

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2} \ \text{(Z. W. Sun and R. Tauraso, 2011)}.$$

Another example involving Apéry numbers

In his proof of the irrationality of $\zeta(3)$, Apéry introduced

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \ (n = 0, 1, 2, \ldots).$$

Beukers' Conjecture (1985) [proved by S. Ahlgren and K. Ono in 2000]. For any prime $p > 3$ we have the super congruence

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2},$$

where $a(n)$ $(n = 1, 2, 3, \ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n) q^n.$$

Conjecture (Z. W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Arithmetic means involving Apéry numbers

Theorem(Z. W. Sun, 2010). Let n be any positive integer. Then

$$\sum_{k=0}^{n-1}(2k+1)A_k \equiv 0 \pmod{n}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x) = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}\binom{n+k}{2k+1}\binom{2k}{k}x^k \in \mathbb{Z}[x].$$

where

$$A_n(x) := \sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 x^k \ (n = 0, 1, 2, \ldots).$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p + \frac{7}{6}p^4 B_{p-3} \pmod{p^5}.$$

Apéry polynomails

Richard Penner (June 2011) pointed out an application of my proof of (i):

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k = \text{the trace of the inverse of } nH_n,$$

where H_n refers to the Hilbert matrix $(\frac{1}{i+j-1})_{1\leqslant i,j\leqslant n}$.

Theorem (Conjectured by Z. W. Sun in 2010 and proved by V.J.W. Guo and J. Zeng in 2011) For any positive integer n we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k(x) \in \mathbb{Z}[x].$$

Recall that T. Sato announced in 2002 the following series for $1/\pi$:

$$\sum_{k=0}^{\infty}(20n+10-3\sqrt{5})\left(\frac{\sqrt{5}-1}{2}\right)^{12n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2 = \frac{20\sqrt{3}+9\sqrt{15}}{6\pi}.$$

An inspiration

My experience with Apéry numbers and Apéry polynomials led me to the following idea.

Idea. If there are Ramanujan-type series

$$\sum_{n=0}^{\infty}(bn+c)\sum_{k=0}^{n}a_k=\frac{C}{\pi}$$

or

$$\sum_{n=0}^{\infty}(bn+c)\binom{2n}{n}\sum_{k=0}^{n}a_k=\frac{C}{\pi},$$

then we should also seek for identities of the form

$$\sum_{n=0}^{\infty}(bn+c)\sum_{k=0}^{n}a_kx^k=\frac{C}{\pi}$$

or

$$\sum_{n=0}^{\infty}(bn+c)\binom{2n}{n}\sum_{k=0}^{n}a_kx^k=\frac{C}{\pi}.$$

An example

Example. Let

$$g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \quad \text{and} \quad g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k.$$

Since there are Ramanujan-type series

$$\sum_{n=0}^{\infty} (bk+c) \binom{2k}{k} g_k = \frac{C}{\pi},$$

I found some Ramanujan-type identities of the form

$$\sum_{k=0}^{\infty} (bk+c) g_k(x) = \frac{C}{\pi}$$

such as

$$\sum_{k=0}^{\infty} \frac{944607040k + 86734691}{33385284^k} \binom{2k}{k} g_k(5776) = \frac{1071111195\sqrt{95}}{38\pi}.$$

More on Apéry polynomails

Theorem 1 (Z. W. Sun, 2011) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(-1)^k A_k(-2) \equiv \sum_{k=0}^{p-1}(-1)^k A_k\left(\frac{1}{4}\right)$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

Remark. A lemma states that for any odd prime p we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k} \equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4).\end{cases}$$

This was first conjectured by the author in 2009 and later confirmed by his twin brother Z.-H. Sun in 2010.

Apéry polynomials

Theorem (Z. W. Sun, 2011). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(-1)^k A_k(x) \equiv \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}x^k \pmod{p^2}.$$

Also, for any p-adic integer $x \not\equiv 0 \pmod{p}$ we have

$$\sum_{k=0}^{p-1}A_k(x) \equiv \left(\frac{x}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p}.$$

A Key Lemma (Z. W. Sun, 2011). If x is a p-adic integer with $x \equiv 2k \pmod{p}$ where $k \in \{0, \ldots, (p-1)/2\}$, then we have

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^2 \equiv (-1)^k\binom{x}{k} \pmod{p^2}.$$

My problems for x^2 mod p^2 with $4p = x^2 + dy^2$

Problem 1. Given a squarefree positive integer d, find *integers* $a_0, a_1, a_2, \ldots$ such that for sufficiently large primes p we have

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1\text{)}, \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1. \end{cases}$$

If one thinks that the integral condition of $a_0, a_1, a_2, \ldots$ in Problem 1 is too harsh, we may study the following easier problem.

Problem 2. Given a squarefree positive integer d, find *rational numbers* $a_0, a_1, a_2, \ldots$ with denominators not divisible by large primes such that for large primes p we have

$$\sum_{k=0}^{p-1} a_k \equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1\text{)}, \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1. \end{cases}$$

We find that Problems 1 and 2 have affirmative answers for most of those $d \in \mathbb{Z}^+$ with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ having class number 1 or 2 or 4.

An example for $d = 21$

Recall that $T_k(b, c)$ denotes the coefficient of x^k in $(x^2 + bx + c)^k$.

Conjecture (Sun, 2011). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k} T_k^2(3,-3)}{(-108)^k} \equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k} g_k}{(-108)^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{-1}{p}) = (\frac{p}{3}) = (\frac{p}{7}) = 1,\ p = x^2 + 21y^2, \\ 12x^2 - 2p \pmod{p^2} & \text{if } (\frac{-1}{p}) = (\frac{p}{7}) = -1,\ (\frac{p}{3}) = 1,\ p = 3x^2 + 7y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } (\frac{-1}{p}) = (\frac{p}{3}) = -1,\ (\frac{p}{7}) = 1,\ 2p = x^2 + 21y^2 \\ 6x^2 - 2p \pmod{p^2} & \text{if } (\frac{-1}{p}) = 1, (\frac{p}{3}) = (\frac{p}{7}) = -1,\ 2p = 3x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-21}{p}) = -1. \end{cases}$$

We also have

$$\sum_{k=0}^{\infty} \frac{56k + 19}{(-108)^k} \binom{2k}{k} T_k^2(3,-3) = \frac{9\sqrt{7}}{\pi}.$$

Progress on the problems

Problem 1 for $d = 1$ already has a positive answer.

We suggest positive answers to Problem 1 for

$$d \in \{2, 3, 5, 6, 7, 10, 13, 15, 22, 30, 37, 58, 70, 85, 130, 190\}.$$

We also formulate many conjectures concerning Problem 2; in particular, we give explicit conjectural positive answers for those squarefree positive integers d with $\mathbb{Q}(\sqrt{-d})$ having class number at most two except for $d = 187, 403$.

Note that $\mathbb{Q}(\sqrt{-d})$ has class number two if and only if

$$d \in \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58,\\ 91, 115, 123, 187, 235, 267, 403, 427\}.$$

Connections of Problems 1 and 2 to series for $1/\pi$ that I discovered are very mysterious!

For more detailed survey, the reader may consult my paper *Conjectures and results on x^2 mod p^2 with $4p = x^2 + dy^2$* available from `http://arxiv.org/abs/1103.4325`

Connections between $A_n(x)$ and $g_n(x)$

It is known that $A_n = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}g_k$.

Theorem (Sun, Dec. 2011). (i) For any positive integer n we have

$$A_n(x) = \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}g_k(x).$$

For any odd prime p and integer x, we have

$$\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)A_k(x) \equiv \sum_{k=0}^{p-1}g_k(x) \pmod{p^2}.$$

(ii) For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1}g_k \equiv 0 \pmod{p^2},\ \sum_{k=0}^{p-1}g_k(-1) \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}g_k(-3) \equiv p\sum_{k=0}^{p-1}\frac{(-3)^k}{2k+1} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

Franel numbers

It is well known that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \ (n = 0, 1, 2, \ldots).$$

In 1895 J. Franel noted that the numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots)$$

satisfy the recurrence relation:

$$(n+1)^2 f_{n+1} = (7n(n+1)+2)f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \ldots).$$

Such numbers are now called *Franel numbers*. D. Callan found combinatorial interpretations of Franel numbers and Barrucand's identity $\sum_{k=0}^{n} \binom{n}{k} f_k = g_n$. V. Strehl proved that Apéry numbers can be expressed in terms of Franel numbers as follows:

$$A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} f_k.$$

Congruences for Franel numbers

Theorem (Sun, arXiv:1112.1034) Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(-1)^k k f_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k} f_{k-1} \equiv 3q_p(2) + 3p\,q_p(2)^2 \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k} f_k \equiv 0 \pmod{p},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Conjecture (later proved by myself). Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k} f_k \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1}\frac{(-1)^k}{k^2} f_k \equiv 0 \pmod{p}.$$

Thank you!