Historical Remarks on my Conjectural Congruences and Series for $1/\pi$

Zhi-Wei Sun

Nanjing University
zwsun@nju.edu.cn
http://math.nju.edu.cn/~zwsun

Dec. 25, 2011
My initial contact with $\pi$-series (1984-86)

When I was an undergraduate at Nanjing University, I learned from calculus the following classical results:

**Leibniz:**

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} x^{2k+1}, \quad \sum_{k=0}^{\infty} \frac{(-1)^k}{2k + 1} = \frac{\pi}{4}.$$  

**Euler:**

$$\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$  

But I did not know any other $\pi$-series then.
The rising factorial (or Pochhammer symbol):

\[(a)_n = a(a + 1) \cdots (a + n - 1) = \frac{\Gamma(a + n)}{\Gamma(a)}.
\]

Note that \((1)_n = n!\).

**Classical Gaussian hypergeometric series:**

\[
{r+1 \choose r} F_r (\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n (\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},
\]

where \(0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_r < 1\), \(0 \leq \beta_1 \leq \cdots \leq \beta_r < 1\), and \(|x| < 1\).
My first impression on Ramanujan-type series

During 2003-2006 I happened to see a paper on Ramanujan-type series. Here is one of Ramanujan series for $1/\pi$:

$$\sum_{k=0}^{\infty} \frac{(28k + 3)(-27)^k}{512^k} \cdot \frac{(1/2)_k(1/6)_k(5/6)_k}{(1)_3^k} = \frac{32\sqrt{2}}{\pi}.$$  

At that time I felt that this is too complicated! I did not like it at all. I only enjoy simple and beautiful results! Thus this paper gave me almost no impression. Consequently, I could not remember what paper it is.

During that period I also saw Mortenson’s 2003 paper in J. Number Theory in which he used character sums and the $p$-adic Gamma functions to prove the following congruence conjectured by F. Rodriguez-Villegas:

$$\sum_{k=0}^{p-1} \frac{(2k)_k^2}{16^k} \equiv \left( \frac{-1}{p} \right) \quad \text{(mod } p^2) \quad \text{for any prime } p > 2.$$  

But at that time I had no feeling about such a congruence.
My joint work on congruences modulo prime powers

**H. Pan and Z. W. Sun** [Discrete Math. 2006].

\[
\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \binom{p-d}{3} \pmod{p} \quad (d = 0, \ldots, p),
\]

\[
\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv 0 \pmod{p} \quad \text{for } p > 3.
\]

**Sun & R. Tauraso** [AAM 45(2010); IJNT 7(2011)].

\[
\sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \binom{p^a}{3} \pmod{p^2},
\]

\[
\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv \frac{8}{9} p^2 B_{p-3} \pmod{p^3} \quad \text{for } p > 3,
\]

where \(B_0, B_1, B_2, \ldots\) are Bernoulli numbers given by

\[
B_0 = 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad (n = 1, 2, 3, \ldots).
\]
My result on $\sum_{k=0}^{p-1} \binom{2k}{k} / m^k \mod p^2$

Sun [Sci. China Math. 53(2010)]: Let $p$ be an odd prime and let $m \in \mathbb{Z}$ with $p \nmid m$. Then

$$\sum_{k=0}^{p-1} \binom{2k}{k} / m^k \equiv \left( \frac{m^2 - 4m}{p} \right) + u_{p-\left( \frac{m^2 - 4m}{p} \right)} \pmod{p^2},$$

where $\{u_n\}_{n \geq 0}$ is the Lucas sequence given by

$$u_0 = 0, \quad u_1 = 1, \quad \text{and} \quad u_{n+1} = (m - 2)u_n - u_{n-1} \quad (n = 1, 2, 3, \ldots).$$

In particular,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} \pmod{p^2}.$$

Remark. Later I found that the last congruence $p^3$ is related to Euler numbers.
Multinomial coefficients:

\[
\binom{k_1 + \cdots + k_n}{k_1, \ldots, k_n} = \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}.
\]

Note that \((2k\ k) = (2k\ k, k)\). So, a natural extension of \((2k\ k)\) is

\[
\binom{kn}{k, k, \ldots, k} = \frac{(kn)!}{(k!)^n}.
\]

Clearly,

\[
\binom{3k}{k, k, k} = \binom{2k}{k} \binom{3k}{k}
\]

and

\[
\binom{4k}{k, k, k, k} = \left(\frac{2k}{k}\right)^2 \binom{4k}{2k}.
\]
My result and conjecture on multinomial coefficients

**Theorem** (Sun [Acta Arith. 148(2011)]). An integer $p > 1$ is a prime if and only if

$$
\sum_{k=0}^{p-1} \binom{(p-1)k}{k, \ldots, k} \equiv 0 \pmod{p}.
$$

**Conjecture** (Sun [Acta Arith. 148(2011)]). For any odd prime $p$ and positive integer $n$,

$$
\frac{1}{n\binom{2n}{n}} \sum_{k=0}^{n-1} \binom{(p-1)k}{k, \ldots, k}
$$

is always a $p$-adic integer.

**Remark.** When $p = 3$, Strauss, Shallit and Zagier [Amer. Math. Monthly 99(1992)] show that $\sum_{k=0}^{n-1} \frac{\binom{2k}{k}}{n^2 \binom{2n}{n}}$ is a 3-adic integer for any $n = 1, 2, 3, \ldots$. 
What happened in November, 2009

During Nov. 6-7, 2009 both Zhi-Hong and I attended the 1st National Conference on Combinatorial Number Theory held at Nanjing Normal University. After the conference, Zhi-Hong did not return home and came to our univ. to copy some books. On Nov. 10, 2009 I had a supper with Zhi-Hong who brought a copy of Ken Ono’s book *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-Series* (Amer. Math. Soc., 2004). I had a glance at the last page of the book and found a list of few supercongruences conjectured by F. Rodriguez-Villegas and proved by Mortenson including

\[
\sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{27^k} \equiv \left( \frac{-3}{p} \right) \pmod{p^2}.
\]

I knew such things before. But, on that day, as I had determined \(\sum_{k=0}^{p-1} \binom{2k}{k,k} / m^k \pmod{p^2}\), I suddenly realized that I should check \(\sum_{k=0}^{p-1} \binom{3k}{k,k,k} / m^k \pmod{p^2}\) via Mathematica which I just began to learn. I wished to go home immediately (for secret computation).
What happened in November, 2009

But Zhi-Hong insisted that I should live with him in the guest room at the New Era Hotel. So, on Nov. 10, 2009 I brought my computer to the hotel and found that
\[
\sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{24^k} \equiv 0 \pmod{p}
\]
for any odd prime \( p \equiv 2 \pmod{3} \).
Later I figured out the pattern and conjectured that
\[
\sum_{k=0}^{p-1} \frac{\binom{3k}{k,k,k}}{24^k} \equiv \begin{cases} 
\left(\frac{2(p-1)^3}{(p-1)^3}\right) \pmod{p^2} & \text{if } p \equiv 1 \pmod{3}, \\
\frac{p}{2}\left(\frac{2(p+1)^3}{(p+1)^3}\right) \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}.
\end{cases}
\]
(See Conj. 5.13 of Sun [Sci. China Math. 54(2011)].)
I also noted that
\[
\sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{81^k} \equiv \sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv 0 \pmod{p^2}
\]
for half of the primes:
\[p = 5, 13, 17, 19, 31, 41, 59, 61, 73, 83, 89, 97, 101, 103, 131, 139, 157, \ldots\]
But I could not find the pattern for these primes.
What happened in November, 2009

In the afternoon of Nov. 11 (Wednesday) we had a seminar. First I reported my discovery and asked if anybody (my students and Zhi-Hong) can figure out the pattern of those primes. Nobody gave an answer. Then I left for a meeting in our dept and Zhi-Hong gave a talk on his results on Euler numbers. When I got to the dept there are still few minutes left, so I opened my computer and searched the sequence 5,13,17,19,31,41 via google and this led me to find that these primes are quadratic nonresidues modulo 7. I immediately called my student Yong Zhang or Hao Pan in the seminar to inform this news.

On Nov. 11 I wrote a draft and posted it to arXiv. The results in the paper include

\[
\sum_{k=0}^{p-1} \frac{(2k)^3}{64^k} \equiv \begin{cases} 
4x^2 - 2p \pmod{p} & \text{if } p = x^2 + y^2 \text{ (}2 \nmid x\text{)}, \\
0 \pmod{p} & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

Several days later I learned that this is not new, it has been proved to hold mod \( p^2 \).
What happened in November, 2009

If \((\frac{p}{7}) = 1\), what about \(\sum_{k=0}^{p-1} \binom{2k}{k}^3 \mod p^2\)? I was puzzled by this. On Friday afternoon (Nov. 13), I attended another meeting in out dept, and suddenly remembered that \(\mathbb{Q}(\sqrt{-7})\) is an Euclidean domain and hence an PID as I often taught undergraduates in the course Modern Algebra. If \((\frac{p}{7}) = 1\), i.e., \((\frac{-7}{p}) = 1\), then \(p\) splits by algebraic number theory and thus \(p\) can be written in the form

\[
\frac{x + y\sqrt{-7}}{2} \times \frac{x - y\sqrt{-7}}{2} = \frac{x^2 + 7y^2}{4},
\]

as both \(x\) and \(y\) must be even we have \(p = (x/2)^2 + 7(y/2)^2\).

After the meeting I immediately went back and verified this observation from Cox’s book *Primes of the Form \(x^2 + ny^2\)*. Thus this led me to find that if \((\frac{p}{7}) = 1\) and \(p = x^2 + 7y^2\) then

\[
\sum_{k=0}^{p-1} \binom{2k}{k}^2 \equiv 4x^2 - 2p \pmod{p^2}.
\]

I updated my arXiv article to add this immediately.

I also found patterns for \(\sum_{k=0}^{p-1} \binom{2k}{k}^2 / m^k \mod p^2\) with \(m = 8, -16, 32\).
What happened in November, 2009

On Nov. 14 (Saturday) I called Zhi-Hong and informed my discovery. He said that he just wanted to make computations to determine \( \sum_{k=0}^{p-1} \binom{2k}{k}^3 \mod p^2 \) in the case \( \left( \frac{p}{7} \right) = 1 \), and he complained that his student was too lazy and did not compute for him.

**Lesson.** If one has not yet formulated a complete conjecture, better not inform others to avoid potential competition.

On Nov. 11 I also conjectured that if \( \left( \frac{p}{7} \right) = -1 \) then

\[
\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv 0 \pmod{p}.
\]

On Nov. 27, 2009 I posted *Open Conjectures on Congruences* to collect my conjectural congruences. After reading this material, on Nov. 28 Bilgin Ali and Bruno Mishutka guessed that if \( p = x^2 + 7y^2 \) then

\[
\sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv \begin{cases} 
11y^2/3 - x^2 & \text{if } 3 \mid y, \\
4(y^2 - x^2)/3 & \text{if } 3 \nmid y.
\end{cases} \pmod{p}.
\]
What happened in November, 2009

Inspired this I immediately realized that

\[ \sum_{k=0}^{p-1} k \binom{2k}{k}^3 \equiv \begin{cases} \frac{8}{21}(3 - 4x^2) \pmod{p^2} & \text{if } p = x^2 + 7y^2, \\ \frac{8}{21}p \pmod{p^2} & \end{cases} \]

and circulated this via a message to Number Theory Mailing List.

Thus, in Nov. 2009 I formulated complete conjectures on

\[ \sum_{k=0}^{p-1} \binom{2k}{k}^3 \pmod{p^2} \text{ and } \sum_{k=0}^{p-1} k \binom{2k}{k}^3 \pmod{p^2}. \]

Prof. Ken Ono was very interested in this and he and one of his students worked on my conjecture. They claimed that they had a proof but in Jan. 2010 they replied me that they met real difficulties.
What happened in Jan.-Feb., 2010

I visited India during Jan.-Feb. 2010. On Jan. 23 I suddenly realized that I should combine the congruences for \( \sum_{k=0}^{p-1} \binom{2k}{k}^3 / m^k \) and \( \sum_{k=0}^{p-1} k \binom{2k}{k}^3 / m^k \mod p^2 \). This led me to conjecture that

\[
\frac{1}{p} \sum_{k=0}^{p-1} (21k + 8) \binom{2k}{k}^3 \equiv 8 + 16p^3 B_{p-3} \pmod{p^4} \quad (\ast)
\]

and that

\[
\frac{1}{n(\binom{2n}{n})} \sum_{k=0}^{n-1} (21k + 8) \binom{2k}{k}^3 \in \mathbb{Z}.
\]

After reading my message to Number Theory List on Feb. 10, Kasper Andersen found on Feb. 11 that

\[
\frac{1}{n(\binom{2n}{n})} \sum_{k=0}^{n-1} (21k + 8) \binom{2k}{k}^3 = \sum_{k=0}^{n-1} \binom{n+k-1}{k}^2
\]

via Sloane’s OEIS (Online Encyclopedia of Integer Sequences). Inspired by this I finally proved (\ast).
van Hamme’s conjecture

After I found \( \sum_{k=0}^{p-1} \binom{2k}{k}^3 / 4096^k \mod p^2 \) and conjectured the congruence

\[
\sum_{k=0}^{p-1} (42k + 5) \frac{(2k)^3}{4096^k} \equiv 5p \left( -\frac{1}{p} \right) - p^3 E_{p-3} \pmod{p^4},
\]

I got to know that van Hamme had the conjecture

\[
\sum_{k=0}^{p-1} (42k + 5) \frac{(2k)^3}{4096^k} \equiv 5p \left( -\frac{1}{p} \right) \pmod{p^3}
\]

motivated by Ramanujan’s identity

\[
\sum_{k=0}^{\infty} (42k + 5) \frac{(2k)^3}{4096^k} = \frac{16}{\pi}.
\]

Thus I became interested in Ramanujan-type series and wrote to several mathematicians to get Hamme’s paper.
Ramanujan-type series for $1/\pi$

**General forms of Ramanujan-type series:**

$$
\sum_{k=0}^{\infty} (ak + b) \frac{(2k)^3}{m^k}, \quad \sum_{k=0}^{\infty} (ak + b) \frac{(2k)^2 (3k)}{m^k},
$$

$$
\sum_{k=0}^{\infty} (ak + b) \frac{(2k)^2 (4k)}{m^k}, \quad \sum_{k=0}^{\infty} (ak + b) \frac{(2k) (3k) (6k)}{m^k}.
$$

There are totally 36 known Ramanujan-type series for $1/\pi$ with $a, b, m$ rational.

**D. V. Chudnovsky and G. V. Chudnovsky (1987):**

$$
\sum_{k=0}^{\infty} \frac{545140134k + 13591409}{(-640320)^{3k}} \frac{(6k)(3k)(2k)}{3k \binom{k}{3k} k} = \frac{3 \times 53360^2}{2\pi \sqrt{10005}}.
$$

*Remark.* This yielded the record for the calculation of $\pi$ during 1989-1994.
My Philosophy about Series for $1/\pi$

Part I of the Philosophy. Given a *regular* identity of the form

$$\sum_{k=0}^{\infty} (bk + c) \frac{a_k}{m^k} = \frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, $bm$ is nonzero and $C^2$ is rational, we must have

$$\sum_{k=0}^{n-1} (bk + c)a_k m^{n-1-k} \equiv 0 \pmod{n}$$

for any positive integer $n$. Furthermore, there exist an integer $m'$ and a squarefree positive integer $d$ with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, 2^3, \ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing $dm$ we have

$$\sum_{k=0}^{p-1} \frac{a_k}{m^k} \equiv \begin{cases} 
(m') \left( (x^2 - 2p) \pmod{p^2} \right) & \text{if } 4p = x^2 + dy^2, \\
0 \pmod{p^2} & \text{if } (-d/p) = -1,
\end{cases}$$

or $d = 1$, $\gcd(15, m) > 1$, and for any prime $p \equiv 3 \pmod{4}$ with $p \nmid 3m$ we have $\sum_{k=0}^{p-1} a_k / m^k \equiv 0 \pmod{p^2}$. 
Part II of the Philosophy. Let \( b, c, m, a_0, a_1, \ldots \) be integers with \( bm \) nonzero and the series \( \sum_{k=0}^{\infty} (bk + c) a_k / m^k \) convergent. Suppose that there are \( d \in \mathbb{Z}^+, d' \in \mathbb{Z}, \) and rational numbers \( c_0 \) and \( c_1 \) such that

\[
\sum_{k=0}^{p-1} (bk + c) \frac{a_k}{m^k} \equiv p \left( c_0 \left( \frac{-d}{p} \right) + c_1 \left( \frac{d'}{p} \right) \right) \pmod{p^2}
\]

for all sufficiently large primes \( p \). If \( d' \geq 0 \), then

\[
\sum_{k=0}^{\infty} (bk + c) \frac{a_k}{m^k} = C \frac{\pi}{\sqrt{d}}
\]

for some \( C \) with \( C^2 \) rational (and with \( C/\sqrt{d} \) rational if \( c_0 \neq 0 \)). If \( d' = -d_1 < 0 \), then there are rational numbers \( \lambda_0 \) and \( \lambda_1 \) such that

\[
\sum_{k=0}^{\infty} (bk + c) \frac{a_k}{m^k} = \frac{\lambda_0 \sqrt{d} + \lambda_1 \sqrt{d_1}}{\pi}.
\]
Illustrating the Philosophy by an Example

For \(b, c \in \mathbb{Z}\) let \(T_k(b, c)\) be the coefficient of \(x^k\) in \((x^2 + bx + c)^k\).

**Conjecture (Sun).** Let \(p > 5\) be a prime. Then

\[
\left(\frac{15}{p}\right) \sum_{k=0}^{p-1} \frac{2^k \binom{3k}{k} T_{3k}(62, 1)}{(-240)^{3k}} \equiv \begin{cases} 
  x^2 - 2p \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = \left(\frac{p}{13}\right) = 1 \text{ & } 4p = x^2 + 91y^2, \\
  2p - 7x^2 \pmod{p^2} & \text{if } \left(\frac{p}{7}\right) = \left(\frac{p}{13}\right) = -1 \text{ & } 4p = 7x^2 + 13y^2, \\
  0 \pmod{p^2} & \text{if } \left(\frac{p}{91}\right) = -1.
\end{cases}
\]

\[
\sum_{k=0}^{p-1} (1638k + 277) \frac{2^k \binom{3k}{k} T_{3k}(62, 1)}{(-240)^{3k}} \equiv \frac{p}{40} \left( 8701 \left(\frac{-105}{p}\right) + 2379 \left(\frac{735}{p}\right) \right) \pmod{p^2}.
\]

We also have

\[
\sum_{k=0}^{\infty} \frac{1638k + 277}{(-240)^{3k}} \binom{2k}{k} \binom{3k}{k} T_{3k}(62, 1) = \frac{44\sqrt{105}}{\pi}.
\]
Another Example Illustrating the Philosophy

Recall my following conjectural series

\[
\sum_{k=0}^{\infty} \frac{80k + 13}{(-168^2)^k} \binom{4k}{2k} \binom{2k}{k} T_k(7, 4096) = \frac{14\sqrt{210} + 21\sqrt{42}}{8\pi}.
\]

Actually this identity was motivated by the following conjecture.

**Conjecture** (Sun). Let \( p > 3 \) be a prime with \( p \neq 7 \). Then

\[
\left( \frac{-42}{p} \right) \sum_{k=0}^{p-1} \frac{(4k) \binom{2k}{k} T_k(7, 4096)}{(-168^2)^k}
\]

\[
\equiv \begin{cases} 
4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \text{ and } p = x^2 + 15y^2, \\
12x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \text{ and } p = 3x^2 + 5y^2, \\
0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1.
\end{cases}
\]

\[
\sum_{k=0}^{p-1} \frac{80k + 13}{(-168^2)^k} \binom{4k}{2k} \binom{2k}{k} T_k(7, 4096)
\]

\[
\equiv p \left( 3 \left( \frac{-42}{p} \right) + 10 \left( \frac{-210}{p} \right) \right) \pmod{p^2}.
\]
The 3rd Example Illustrating the Philosophy

**Conjecture** (Sun). Let \( p > 3 \) be a prime. Then

\[
\sum_{k=0}^{p-1} \frac{T_k(10, 121)^3}{(2^{11}3^3)^k} = \begin{cases} 
4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{3} \& p = x^2 + 3y^2, \\
0 \pmod{p^2} & \text{if } p \equiv 2 \pmod{3}.
\end{cases}
\]

\[
\sum_{k=0}^{p-1} \frac{66k + 17}{(2^{11}3^3)^k} T_k^3(10, 121) 
= \frac{p}{11} \left( 195 \left( \frac{-2}{p} \right) - 8 \left( \frac{-6}{p} \right) \right) \pmod{p^2}.
\]

Also,

\[
\sum_{k=0}^{\infty} \frac{66k + 17}{(2^{11}3^3)^k} T_k^3(10, 11^2) = \frac{540\sqrt{2}}{11\pi}.
\]
I would like to offer $90 for the first proof of the identity in the following conjecture and $1050 for the first proof of congruences in the conjecture.

**Conjecture** (Z. W. Sun, 2011). We have

\[
\sum_{n=0}^{\infty} \frac{357n + 103}{2160^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n + 2k}{2k} \binom{2k}{k} (-324)^{n-k} = \frac{90}{\pi}.
\]

For any prime \( p > 5 \), we have

\[
\sum_{n=0}^{p-1} \frac{357n + 103}{2160^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n + 2k}{2k} \binom{2k}{k} (-324)^{n-k} \equiv p \left( \frac{-1}{p} \right) \left( 54 + 49 \left( \frac{p}{15} \right) \right) \pmod{p^2}.
\]
The 4th Example Illustrating the Philosophy (continued)

And

\[
\sum_{n=0}^{p-1} \frac{\binom{2n}{n}}{2160^n} \sum_{k=0}^{n} \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k}
\]

\[
\begin{cases}
4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 105y^2 \; (x, y \in \mathbb{Z}), \\
2x^2 - 2p \pmod{p^2} & \text{if } 2p = x^2 + 105y^2 \; (x, y \in \mathbb{Z}), \\
2p - 12x^2 \pmod{p^2} & \text{if } p = 3x^2 + 35y^2 \; (x, y \in \mathbb{Z}), \\
2p - 6x^2 \pmod{p^2} & \text{if } 2p = 3x^2 + 35y^2 \; (x, y \in \mathbb{Z}), \\
20x^2 - 2p \pmod{p^2} & \text{if } p = 5x^2 + 21y^2 \; (x, y \in \mathbb{Z}), \\
10x^2 - 2p \pmod{p^2} & \text{if } 2p = 5x^2 + 21y^2 \; (x, y \in \mathbb{Z}), \\
28x^2 - 2p \pmod{p^2} & \text{if } p = 7x^2 + 15y^2 \; (x, y \in \mathbb{Z}), \\
14x^2 - 2p \pmod{p^2} & \text{if } 2p = 7x^2 + 15y^2 \; (x, y \in \mathbb{Z}), \\
0 \pmod{p^2} & \text{if } \left( \frac{-105}{p} \right) = -1.
\end{cases}
\]

Remark. The quadratic field \( \mathbb{Q}(\sqrt{-105}) \) has class number 8.
My conjectural series of a special type

I have 18 conjectural series like the following five.

\[
\sum_{k=0}^{\infty} \frac{340k + 59}{(-480^2)^k} \binom{2k}{k}^2 T_{2k}(62, 1) = \frac{120}{\pi},
\]

\[
\sum_{k=0}^{\infty} \frac{13940k + 1559}{(-5760^2)^k} \binom{2k}{k}^2 T_{2k}(322, 1) = \frac{4320}{\pi},
\]

\[
\sum_{k=0}^{\infty} \frac{14280k + 899}{39200^{2k}} \binom{2k}{k}^2 T_{2k}(198, 1) = \frac{1155\sqrt{6}}{\pi},
\]

\[
\sum_{k=0}^{\infty} \frac{57720k + 3967}{439280^{2k}} \binom{2k}{k}^2 T_{2k}(5778, 1) = \frac{2890\sqrt{19}}{\pi},
\]

\[
\sum_{k=0}^{\infty} \frac{1615k - 314}{243360^{2k}} \binom{2k}{k}^2 T_{2k}(54758, 1) = \frac{1989\sqrt{95}}{4\pi}.
\]

Remark. I conjectured that my list of the 18 series of that type is complete! Prof. G. Almkvist asked me why I thought so.
My criterion for existence of series for $1/\pi$ of a special type

**Hypothesis** (Sun, 2011). (i) Suppose that

$$\sum_{k=0}^{\infty} \frac{a_0 + a_1 k}{m^k} \binom{2k}{k}^2 T_{2k}(b, 1) = \frac{C}{\pi}$$

with $a_0, a_1, b, m \in \mathbb{Z}$, $b > 0$ and $C^2 \in \mathbb{Q} \setminus \{0\}$. Then $\sqrt{|m|}$ is an integer dividing $16(b^2 - 4)$. Also, $b = 7$ or $b \equiv 2 \pmod{4}$.

(ii) Let $\epsilon \in \{\pm 1\}$, $b, m \in \mathbb{Z}^+$ and $m \mid 16(b^2 - 4)$. Then, there are $a_0, a_1 \in \mathbb{Z}$ such that

$$\sum_{k=0}^{\infty} \frac{a_0 + a_1 k}{(\epsilon m^2)^k} \binom{2k}{k}^2 T_{2k}(b, 1) = \frac{C}{\pi}$$

for some $C \neq 0$ with $C^2$ rational, if and only if $m > 4(b + 2)$ and

$$\sum_{k=0}^{p-1} \frac{(2k)^2 T_{2k}(b, 1)}{(\epsilon m^2)^k} \equiv \left( \frac{\epsilon(b^2 - 4)}{p} \right) \sum_{k=0}^{p-1} \frac{(2k)^2 T_{2k}(b, 1)}{(\epsilon \tilde{m}^2)^k} \pmod{p^2}$$

for all odd primes $p \mid b^2 - 4$, where $\tilde{m} = 16(b^2 - 4)/m$. 
My general philosophy about $p$-adic congruences

Given a natural sequence $a_0, a_1, a_2, \ldots$ of $p$-adic integers, we may consider $\sum_{k=0}^{p-1} a_k$ or $\sum_{k=0}^{(p-1)/2} a_k$ modulo power of $p$ because such a sum usually behaves better than a general term. When $a_{p-1}$ and $a_{(p-1)/2}$ modulo powers of $p$ obey certain patterns, the partial sum $\sum_{k=0}^{p-1} a_k$ or $\sum_{k=0}^{(p-1)/2} a_k$ should also have patterns modulo powers of $p$.

Example. Let $p > 3$ be a prime. Then

\[
\binom{2p}{p} = 2 \binom{2p-1}{p-1} \equiv 2 \pmod{p^3} \quad \text{(J. Wolstenholme, 1863)},
\]

\[
\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3} \quad \text{(F. Morley, 1895)},
\]

\[
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2} \quad \text{(Z. W. Sun and R. Tauraso, 2011)}.
\]
Another example involving Apéry numbers

In his proof of the irrationality of \( \zeta(3) \), Apéry introduced

\[
A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (n = 0, 1, 2, \ldots).
\]

**Beukers’ Conjecture (1985)** [proved by S. Ahlgren and K. Ono in 2000]. For any prime \( p > 3 \) we have the super congruence

\[
A_{(p-1)/2} \equiv a(p) \pmod{p^2},
\]

where \( a(n) \ (n = 1, 2, 3, \ldots) \) are given by

\[
\eta^4(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n)q^n.
\]

**Conjecture** (Z. W. Sun, 2010). For any odd prime \( p \), we have

\[
\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 
4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\
0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}.
\end{cases}
\]
Arithmetic means involving Apéry numbers

**Theorem** (Z. W. Sun, 2010). Let $n$ be any positive integer. Then

$$
\sum_{k=0}^{n-1} (2k + 1)A_k \equiv 0 \pmod{n}.
$$

Moreover,

$$
\frac{1}{n} \sum_{k=0}^{n-1} (2k+1)A_k(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n+k}{k} \binom{n+k}{2k+1} \binom{2k}{k} x^k \in \mathbb{Z}[x].
$$

where

$$
A_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k \quad (n = 0, 1, 2, \ldots).
$$

If $p > 3$ is a prime, then

$$
\sum_{k=0}^{p-1} (2k + 1)A_k \equiv p + \frac{7}{6} p^4 B_{p-3} \pmod{p^5}.
$$
Apéry polynomials

Richard Penner (June 2011) pointed out an application of my proof of (i):

\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)A_k = \text{the trace of the inverse of } nH_n,
\]

where \( H_n \) refers to the Hilbert matrix \( \left( \frac{1}{i+j-1} \right)_{1 \leq i,j \leq n} \).

**Theorem** (Conjectured by Z. W. Sun in 2010 and proved by V.J.W. Guo and J. Zeng in 2011) For any positive integer \( n \) we have

\[
\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)(-1)^k A_k(x) \in \mathbb{Z}[x].
\]

Recall that T. Sato announced in 2002 the following series for \( 1/\pi \):

\[
\sum_{k=0}^{\infty} (20n+10-3\sqrt{5}) \left( \frac{\sqrt{5} - 1}{2} \right)^{12n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \frac{20\sqrt{3} + 9\sqrt{15}}{6\pi}.
\]
An inspiration

My experience with Apéry numbers and Apéry polynomials led me to the following idea.

**Idea.** If there are Ramanujan-type series

\[
\sum_{n=0}^{\infty} \left( bn + c \right) \sum_{k=0}^{n} a_k = \frac{C}{\pi}
\]

or

\[
\sum_{n=0}^{\infty} \left( bn + c \right) \binom{2n}{n} \sum_{k=0}^{n} a_k = \frac{C}{\pi},
\]

then we should also seek for identities of the form

\[
\sum_{n=0}^{\infty} \left( bn + c \right) \sum_{k=0}^{n} a_k x^k = \frac{C}{\pi}
\]

or

\[
\sum_{n=0}^{\infty} \left( bn + c \right) \binom{2n}{n} \sum_{k=0}^{n} a_k x^k = \frac{C}{\pi}.
\]
An example

Example. Let

\[ g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \quad \text{and} \quad g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k. \]

Since there are Ramanujan-type series

\[ \sum_{n=0}^{\infty} (bk + c) \binom{2k}{k} g_k = \frac{C}{\pi}, \]

I found some Ramanujan-type identities of the form

\[ \sum_{k=0}^{\infty} (bk + c) g_k(x) = \frac{C}{\pi} \]

such as

\[ \sum_{k=0}^{\infty} \frac{944607040k + 86734691}{33385284^k} \binom{2k}{k} g_k(5776) = \frac{1071111195\sqrt{95}}{38\pi}. \]
More on Apéry polynomials

**Theorem 1** (Z. W. Sun, 2011) Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(-2) \equiv \sum_{k=0}^{p-1} (-1)^k A_k \left(\frac{1}{4}\right) \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4} . \end{cases}$$

**Remark.** A lemma states that for any odd prime $p$ we have

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + y^2 \ (2 \nmid x), \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod{4} . \end{cases}$$

This was first conjectured by the author in 2009 and later confirmed by his twin brother Z.-H. Sun in 2010.
Apéry polynomials

**Theorem** (Z. W. Sun, 2011). Let $p$ be an odd prime. Then

$$\sum_{k=0}^{p-1} (-1)^k A_k(x) \equiv \sum_{k=0}^{p-1} \frac{(2k)^3}{16^k} x^k \pmod{p^2}.$$ 

Also, for any $p$-adic integer $x \not\equiv 0 \pmod{p}$ we have

$$\sum_{k=0}^{p-1} A_k(x) \equiv \left(\frac{x}{p}\right) \sum_{k=0}^{p-1} \frac{(4k)}{(k,k,k,k)} \pmod{p}.$$ 

**A Key Lemma** (Z. W. Sun, 2011). If $x$ is a $p$-adic integer with $x \equiv 2k \pmod{p}$ where $k \in \{0, \ldots, (p - 1)/2\}$, then we have

$$\sum_{r=0}^{p-1} (-1)^r \binom{x}{r}^2 \equiv (-1)^k \binom{x}{k} \pmod{p^2}.$$
My problems for $x^2 \mod p^2$ with $4p = x^2 + dy^2$

**Problem 1.** Given a squarefree positive integer $d$, find integers $a_0, a_1, a_2, \ldots$ such that for sufficiently large primes $p$ we have

$$
\sum_{k=0}^{p-1} a_k \equiv \begin{cases} 
  x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\
  0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1.
\end{cases}
$$

If one thinks that the integral condition of $a_0, a_1, a_2, \ldots$ in Problem 1 is too harsh, we may study the following easier problem.

**Problem 2.** Given a squarefree positive integer $d$, find rational numbers $a_0, a_1, a_2, \ldots$ with denominators not divisible by large primes such that for large primes $p$ we have

$$
\sum_{k=0}^{p-1} a_k \equiv \begin{cases} 
  x^2 - 2p \pmod{p^2} & \text{if } 4p = x^2 + dy^2 \text{ (and } 4 \nmid x \text{ if } d = 1), \\
  0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1.
\end{cases}
$$

We find that Problems 1 and 2 have affirmative answers for most of those $d \in \mathbb{Z}^+$ with the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ having class number 1 or 2 or 4.
An example for $d = 21$

Recall that $T_k(b, c)$ denotes the coefficient of $x^k$ in $(x^2 + bx + c)^k$.

**Conjecture** (Sun, 2011). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{(2k)}{k} \frac{T_k^2(3, -3)}{(-108)^k} \equiv \left( \frac{p}{3} \right) \sum_{k=0}^{p-1} \frac{(2k)g_k}{(-108)^k}$$

$$\equiv \begin{cases} 
4x^2 - 2p \pmod{p^2} & \text{if } \left( -\frac{1}{p} \right) = \left( \frac{p}{3} \right) = \left( \frac{p}{7} \right) = 1, \ p = x^2 + 21y^2, \\
12x^2 - 2p \pmod{p^2} & \text{if } \left( -\frac{1}{p} \right) = \left( \frac{p}{7} \right) = -1, \left( \frac{p}{3} \right) = 1, \ p = 3x^2 + 7y^2, \\
2x^2 - 2p \pmod{p^2} & \text{if } \left( -\frac{1}{p} \right) = \left( \frac{p}{3} \right) = -1, \left( \frac{p}{7} \right) = 1, \ 2p = x^2 + 21y^2, \\
6x^2 - 2p \pmod{p^2} & \text{if } \left( -\frac{1}{p} \right) = 1, \left( \frac{p}{3} \right) = \left( \frac{p}{7} \right) = -1, \ 2p = 3x^2 + 7y^2, \\
0 \pmod{p^2} & \text{if } \left( -\frac{21}{p} \right) = -1.
\end{cases}$$

We also have

$$\sum_{k=0}^{\infty} \frac{56k + 19}{(-108)^k} \binom{2k}{k} T_k^2(3, -3) = \frac{9\sqrt{7}}{\pi}.$$
Progress on the problems

Problem 1 for \( d = 1 \) already has a positive answer.

We suggest positive answers to Problem 1 for
\[
d \in \{2, 3, 5, 6, 7, 10, 13, 15, 22, 30, 37, 58, 70, 85, 130, 190\}.\]

We also formulate many conjectures concerning Problem 2; in particular, we give explicit conjectural positive answers for those squarefree positive integers \( d \) with \( \mathbb{Q}(\sqrt{-d}) \) having class number at most two except for \( d = 187, 403 \).

Note that \( \mathbb{Q}(\sqrt{-d}) \) has class number two if and only if
\[
d \in \{5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427\}.\]

Connections of Problems 1 and 2 to series for \( 1/\pi \) that I discovered are very mysterious!

For more detailed survey, the reader may consult my paper
*Conjectures and results on \( x^2 \) mod \( p^2 \) with \( 4p = x^2 + dy^2 \)* available from http://arxiv.org/abs/1103.4325
Connections between $A_n(x)$ and $g_n(x)$

It is known that $A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} g_k$.

**Theorem** (Sun, Dec. 2011). (i) For any positive integer $n$ we have

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} g_k(x).$$

For any odd prime $p$ and integer $x$, we have

$$\frac{1}{p} \sum_{k=0}^{p-1} (2k + 1) A_k(x) \equiv \sum_{k=0}^{p-1} g_k(x) \pmod{p^2}.$$

(ii) For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} g_k \equiv 0 \pmod{p^2}, \quad \sum_{k=0}^{p-1} g_k(-1) \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} g_k(-3) \equiv p \sum_{k=0}^{p-1} \frac{(-3)^k}{2k+1} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$
Franel numbers

It is well known that
\[ \sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \quad (n = 0, 1, 2, \ldots). \]

In 1895 J. Franel noted that the numbers
\[ f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots) \]
satisfy the recurrence relation:
\[ (n + 1)^2 f_{n+1} = (7n(n + 1) + 2)f_n + 8n^2 f_{n-1} \quad (n = 1, 2, 3, \ldots). \]

Such numbers are now called Franel numbers. D. Callan found combinatorial interpretations of Franel numbers and Barrucand’s identity
\[ \sum_{k=0}^{n} \binom{n}{k} f_k = g_n. \]
V. Strehl proved that Apéry numbers can be expressed in terms of Franel numbers as follows:
\[ A_n = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} f_k. \]
Congruences for Franel numbers

**Theorem** (Sun, arXiv:1112.1034) Let $p > 3$ be a prime. Then

$$
\sum_{k=0}^{p-1} (-1)^k f_k \equiv \left( \frac{p}{3} \right) \pmod{p^2},
$$

$$
\sum_{k=0}^{p-1} (-1)^k kf_k \equiv -\frac{2}{3} \left( \frac{p}{3} \right) \pmod{p^2},
$$

$$
\sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_{k-1} \equiv 3q_p(2) + 3p q_p(2)^2 \pmod{p^2},
$$

$$
\sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_k \equiv 0 \pmod{p},
$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$.

**Conjecture** (later proved by myself). Let $p > 3$ be a prime. Then

$$
\sum_{k=1}^{p-1} \frac{(-1)^k}{k} f_k \equiv 0 \pmod{p^2} \quad \text{and} \quad \sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} f_k \equiv 0 \pmod{p}.
$$
Thank you!