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Some new representation problems involving primes

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

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Part I. On functions taking only prime values

Mills' Theorem

Any non-constant polynomial $P(x_1, \ldots, x_n)$ with integer coefficients cannot take primes for all $x_1, \ldots, x_n \in \mathbb{Z}$.

Theorem (Mills, 1947). There is a real number A such that $M(n) = \lfloor A^{3^n} \rfloor$ takes only prime values.

Sketch of the Proof. Since $p_{n+1} - p_n = O(p_n^{5/8})$ (A. E. Ingham, 1937), one can construct infinitely many primes $P_0, P_1, P_2, \ldots$ with

$$P_n^3 < P_{n+1} < (P_n + 1)^3 - 1.$$

Then the sequence $u_n = P_n^{3^{-n}}$ is increasing while the sequence $v_n = (P_n + 1)^{3^{-n}}$ is decreasing. As $u_n < v_n$, we see that $A = \lim_{n\to\infty} u_n \leqslant B = \lim_{n\to\infty} v_n$, hence

$$P_n = u_n^{3^n} < A^{3^n} < P_n + 1 = v_n^{3^n}.$$

So $\lfloor A^{3^n} \rfloor = P_n$ is a prime for all $n = 1, 2, 3, \ldots$.

My problems on central binomial coefficients

Let p be an odd prime. For $k = 0, \ldots, (p-1)/2$ we have

$$\binom{2k}{k} = \frac{(2k)!}{k!^2} \not\equiv 0 \pmod{p}.$$

Conjecture (Sun, Feb. 20, 2012). Let $p > 5$ be a prime. Then

$$\left\{\pm\binom{2k}{k}:\ k = 1, \ldots, \frac{p-1}{2}\right\}$$

cannot be a reduced system of residues modulo p. If $p > 11$, then $\binom{2k}{k}$ $(0 < k < (p-1)/2)$ cannot be pairwise distinct mod p.

Conjecture (Z. W. Sun, Feb. 21, 2012). For $n = 1, 2, 3, \ldots$ define $s(n)$ as the least integer $m > 1$ such that $\binom{2k}{k}$ $(k = 1, \ldots, n)$ are pairwise distinct modulo m. Then $s(n)$ is always a prime!

I also guessed that $s(n) < n^2$ for $n = 2, 3, 4, \ldots$. I calculated $s(n)$ for $n = 1, \ldots, 2065$. For example,

$$s(1) = 2,\ s(2) = 3,\ s(3) = 5,\ s(4) = s(5) = s(6) = 11,$$
$$s(7) = s(8) = s(9) = 23,\ s(10) = 31,\ s(11) = \cdots = s(14) = 43.$$

Discriminators for powers of x

Theorem 1 (L. K. Arnold, S. J. Benkoski and B. J. McCabe, 1985). For $n > 4$ the least positive integer m (which is called the *discriminator*, denoted by $D(n)$) such that $1^2, 2^2, \ldots, n^2$ are distinct modulo m, is

$$\min\{m \geqslant 2n : \ m = p \text{ or } m = 2p \ \text{ with } p \text{ an odd prime}\}.$$

Remark. The range of $D(n)$ does not contain those primes $p = 2q + 1$ with q an odd prime.

Theorem 2 (P. S. Bremser, P. D. Schumer and L. C. Washington, 1990). Let $k > 2$ and $n > 0$ be integers, and let $D_k(n)$ denote the the least positive integer m such that $1^k, 2^k, \ldots, n^k$ are distinct modulo m.

(i) If k is odd and n is sufficiently large, then

$$D_k(n) = \min\{m \geqslant n : \ m \text{ is squarefree, and } (k, \varphi(m)) = 1\}.$$

(ii) If k is even and n is sufficiently large, then

$$D_k(n) = \min\{m \geqslant 2n : m = p \text{ or } 2p \text{ with } p \text{ a prime, and } (k, \varphi(m)) = 2\}.$$

Generate all primes in a combinatorial manner

Theorem 1 (Sun, Feb. 29, 2012) For $n \in \mathbb{Z}^+$ let $S(n)$ denote the smallest integer $m > 1$ such that those $2k(k-1)$ mod m for $k = 1, \ldots, n$ are pairwise distinct. Then $S(n)$ is the least prime greater than $2n - 2$.

Remark.

(a) **The range of $S(n)$ is exactly the set of all primes!**

(b) I also showed that for any $d \in \mathbb{Z}^+$ whenever $n \geqslant d + 2$ the least prime $p \geqslant 2n + d$ is just the smallest $m \in \mathbb{Z}^+$ such that $2k(k+d)$ $(k = 1, \ldots, n)$ are pairwise distinct modulo m.

(c) I proved that the least positive integer m such that those $\binom{k}{2} = k(k-1)/2$ $(k = 1, \ldots, n)$ are pairwise distinct modulo m, is just the least power of two not smaller than n.

Another theorem

Theorem 2 (Sun, March 2012) (i) Let $d \in \{2, 3\}$ and $n \in \mathbb{Z}^+$. Then the smallest positive integer m such that those $k(dk-1)$ $(k = 1, \ldots, n)$ are pairwise distinct modulo m, is the least power of d not smaller than n.

(ii) Let $n \in \{4, 5, \ldots\}$. Then the least positive integer m such that $18k(3k-1)$ $(k = 1, \ldots, n)$ are pairwise distinct modulo m, is just the least prime $p > 3n$ with $p \equiv 1 \pmod 3$.

Remark. We are also able to prove some other similar results including the following one:

For $n > 5$ the least $m \in \mathbb{Z}^+$ such that those $18k(3k+1)$ $(k = 1, \ldots, n)$ are pairwise distinct modulo m, is just the first prime $p \equiv -1 \pmod 3$ after $3n$.

Generating primes in arithmetic progressions

Theorem (Sun, April 2013). Let $d \geqslant 4$ and $c \in (-d, d)$ be relatively prime integers, and let $r(d)$ be the product of all distinct prime divisors of d. Then, for any sufficiently large integer n, the least positive integer m with $2r(d)k(dk - c)$ $(k = 1, \ldots, n)$ pairwise distinct modulo m is just the first prime $p \equiv c \pmod{d}$ after $(d(2n-1) - c)/(d-1)$. Moreover, $n > 24310$ suffices for all $d = 4, 5, \ldots, 36$.

Corollary. For each integer $n \geqslant 6$, the least positive integer m such that $4k(4k-1)$ (or $4k(4k+1)$) for $k = 1, \ldots, n$ are pairwise distinct modulo m, is just the least prime $p \equiv 1 \pmod{4}$ after $(8n-4)/3$ (resp., $p \equiv -1 \pmod{4}$ after $(8n-2)/3$).

Alternating sums of primes

Let p_n be the nth prime and define

$$s_n = p_n - p_{n-1} + \cdots + (-1)^{n-1} p_1.$$

For example,

$$s_5 = p_5 - p_4 + p_3 - p_2 + p_1 = 11 - 7 + 5 - 3 + 2 = 8.$$

Note that

$$s_{2n} = \sum_{k=1}^{n} (p_{2k} - p_{2k-1}) > 0, \quad s_{2n+1} = \sum_{k=1}^{n} (p_{2k+1} - p_{2k}) + p_1 > 0.$$

Let $1 \leqslant k < n$. If $n - k$ is even, then

$$s_n - s_k = (p_n - p_{n-1}) + \cdots + (p_{k+2} - p_{k+1}) > 0.$$

If $n - k$ is odd, then

$$s_n - s_k = \sum_{l=k+1}^{n} (-1)^{n-l} p_l - 2 \sum_{j=1}^{k} (-1)^{k-j} p_j \equiv n - k \equiv 1 \pmod{2}.$$

So, $s_1, s_2, s_3, \ldots$ are pairwise distinct.

An amazing recurrence for primes

We may compute the $(n+1)$-th prime p_{n+1} in terms of $p_1, \ldots, p_n$.

Conjecture (Z. W. Sun, J. Number Theory 2013). For any positive integer $n \neq 1, 2, 4, 9$, the $(n+1)$-th prime p_{n+1} is the least positive integer m such that

$$2s_1^2, \ldots, 2s_n^2$$

are pairwise distinct modulo m.

Remark. I have verified the conjecture for $n \leqslant 10^5$, and proved that $2s_1^2, \ldots, 2s_n^2$ **are indeed pairwise distinct modulo** p_{n+1}.

Let $1 \leqslant j < k \leqslant n$. Then

$$0 < |s_k - s_j| \leqslant \max\{s_k, s_j\} \leqslant \max\{p_k, p_j\} \leqslant p_n < p_{n+1}.$$

Also, $s_k + s_j \leqslant p_k + p_j < 2p_{n+1}$. If $2 \nmid k - j$, then

$$s_k + s_j = p_k - p_{k-1} + \ldots + p_{j+1} \leqslant p_k < p_{n+1}.$$

If $2 \mid k - j$, then $s_k \equiv s_j \pmod 2$ and hence $s_k + s_j \neq p_{n+1}$. Thus $2s_k^2 - 2s_j^2 = 2(s_k - s_j)(s_k + s_j) \not\equiv 0 \pmod{p_{n+1}}$.

Conjecture on alternating sums of consecutive primes

Conjecture (Z. W. Sun, J. Number Theory, 2013). For any positive integer m, there are consecutive primes $p_k, \ldots, p_n$ $(k < n)$ not exceeding $2m + 2.2\sqrt{m}$ such that

$$m = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k.$$

(Moreover, we may even require that $m < p_n < m + 4.6\sqrt{m}$ if $2 \nmid m$ and $2m - 3.6\sqrt{m+1} < p_n < 2m + 2.2\sqrt{m}$ if $2 \mid m$.)

Examples.

$$10 = 17 - 13 + 11 - 7 + 5 - 3;$$

$$20 = 41 - 37 + 31 - 29 + 23 - 19 + 17 - 13 + 11 - 7 + 5 - 3;$$

$$303 = p_{76} - p_{75} + \cdots + p_{52},$$

$$p_{76} = 383 = \lfloor 303 + 4.6\sqrt{303} \rfloor, \ p_{52} = 239;$$

$$2382 = p_{652} - p_{651} + \cdots + p_{44} - p_{43},$$

$$p_{652} = 4871 = \lfloor 2 \cdot 2382 + 2.2\sqrt{2382} \rfloor, \ p_{43} = 191.$$

The conjecture has been verified for m up to 10^7.

Prize. I would like to offer 1000 US dollars for the first proof.

More conjectures involving $s_n = \sum_{k=1}^{n}(-1)^{n-k}p_k$

The following conjecture is an analogy of Dirichlet's theorem for primes in arithmetic progressions.

Conjecture (Sun, 2013). For any integers $m > 0$ and r, there are infinitely many $n \in \mathbb{Z}^+$ with $s_n \equiv r \pmod{m}$.

Conjecture (Sun, 2013). (i) For each $\lambda = 1, 2, 3$, any integer $n > \lambda$ can be written as $s_k + \lambda s_l$ with $k, l \in \mathbb{Z}^+$.
(ii) Any integer $n > 1$ can be written as $j(j+1)/2 + s_k$ with $j, k \in \mathbb{Z}^+$.

Recall that a prime p is called a Sophie Germain prime if $2p + 1$ is also prime. It is conjectured that there are infinitely many Sophie Germain primes.

Conjecture (Sun, 2013). Each $n = 3, 4, \ldots$ can be written as $p + s_k$, where p is a Sophie Germain prime and k is a positive integer.

Part II. A general hypothesis on representations involving primes

Goldbach's conjecture

Goldbach's Conjecture (1742): Every even number $n > 2$ can be written in the form $p + q$ with p and q both prime.

Goldbach's weak Conjecture [proved by I. M. Vinorgradov (1937) and H. Helfgott (2013)]. Each odd number $n > 6$ can be written as a sum of three primes.

Goldbach's conjecture implies that for any $n > 2$ there is a prime $p \in [n, 2n]$ since $2n \neq p + q$ if p and q are smaller than n.

Lemoine's Conjecture (1894). Any odd integer $n > 6$ can be written as $p + 2q$, where p and q are primes.

Conjectures for twin primes, cousin primes and sexy primes

Conjecture (Sun, 2012-12-22) Any integer $n \geqslant 12$ can be written as $p + q$ with $p, p + 6, 6q \pm 1$ all prime.

Remark. I have verified this for n up to 10^9.

Conjecture (2013-01-03) Let

$$A = \{x \in \mathbb{Z}^+ : \ 6x - 1 \text{ and } 6x + 1 \text{ are both prime}\},$$
$$B = \{x \in \mathbb{Z}^+ : \ 6x + 1 \text{ and } 6x + 5 \text{ are both prime}\},$$
$$C = \{x \in \mathbb{Z}^+ : \ 2x - 3 \text{ and } 2x + 3 \text{ are both prime}\}.$$

Then

$$A+B = \{2, 3, \ldots\}, \ B+C = \{5, 6, \ldots\}, \ A+C = \{5, 6, \ldots\} \setminus \{161\}.$$

Also, if we set $2X := X + X$ then

$$2A \supseteq \{702, 703, \ldots\}, \ 2B \supseteq \{492, 493, \ldots\}, \ 2C \supseteq \{4006, 4007, \ldots\}.$$

Ordowski's conjecture and Ming-Zhi Zhang's problem

As conjectured by Fermat and proved by Euler, any prime $p \equiv 1 \pmod 4$ can be written uniquely as a sum of two squares.

Tomasz Ordowski's Conjecture (Oct. 3-4, 2012) For any prime $p \equiv 1 \pmod 4$ write $p = a_p^2 + b_p^2$ with $a_p > b_p > 0$. Then

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv 1 \pmod 4} a_p}{\sum_{p\leqslant N,\, p\equiv 1 \pmod 4} b_p} = 1+\sqrt{2}$$

and

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv 1 \pmod 4} a_p^2}{\sum_{p\leqslant N,\, p\equiv 1 \pmod 4} b_p^2} = \frac{9}{2}.$$

Ming-Zhi Zhang's Problem (1990s) Whether any odd integer $n > 1$ can be written as $a+b$ with $a, b \in \mathbb{Z}^+$ and a^2+b^2 prime ? (Cf. http://oeis.org/A036468)

My conjecture involving $x^2 + xy + y^2$

It is known that any prime $p \equiv 1 \pmod 3$ can be written uniquely in the form $x^2 + xy + y^2$ with $x > y > 0$.

Conjecture (Sun, Nov. 3, 2012). (i) For any prime $p \equiv 1 \pmod 3$ write $p = x_p^2 + x_p y_p + y_p^2$ with $x_p > y_p > 0$. Then

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv 1 \pmod 3} x_p}{\sum_{p\leqslant N,\, p\equiv 1 \pmod 3} y_p} = 1 + \sqrt{3}$$

and

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv 1 \pmod 3} x_p^2}{\sum_{p\leqslant N,\, p\equiv 1 \pmod 3} y_p^2} = \frac{52}{9}.$$

(ii) Any integer $n > 1$ with $n \neq 8$ can be written as $x + y$, with $x, y \in \mathbb{Z}^+$ and $x^2 + xy + y^2$ prime.

My conjectures involving $x^2 + 3xy + y^2$

Conjecture (Sun, Nov. 4, 2012). (i) For any prime $p \equiv \pm 1 \pmod 5$ write $p = x_p^2 + 3x_py_p + y_p^2$ with $x_p > y_p > 0$. Then

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv \pm 1 \pmod 5} x_p}{\sum_{p\leqslant N,\, p\equiv \pm 1 \pmod 5} y_p} = 1 + \sqrt{5}.$$

(ii) Any integer $n > 1$ can be written as $x + y$, with $x, y \in \mathbb{Z}^+$ and $x^2 + 3xy + y^2$ prime.

Conjecture (Sun, Nov. 5, 2012). For any integer $n \geqslant 1188$, there are primes p and q with $p^2 + 3pq + q^2$ prime such that $p + (1 + \{n\}_2)q = n$, where $\{n\}_m$ denotes the least nonnegative residue of n mod m.

I have some other similar conjectures.

My conjecture on prime differences

Polignac's Conjecture (1849). For any positive even integer $2m$, there are infinitely many prime pairs $\{p, q\}$ with $p - q = 2m$.

My following conjecture "explains" why Polignac's Conjecture should be correct.

Conjecture (Sun, 2012). For each $m \in \mathbb{N} = \{0, 1, 2, \ldots\}$, any sufficiently large integer n can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $x \pm m$ and $2xy + 1$ all prime. In particular, any integer $n > 7$ can be written as $p + q$, where q is a positive integer, and p and $2pq + 1$ are both prime.

Remark. For $m = 1, 2, 3, 4, 5$ it suffices to require that n is greater than 623, 28, 151, 357, 199 respectively.

A conjecture refining Bertrand's Postulate

Bertrand's Postulate (proved by Chebyshev in 1850). For any positive integer n, the interval $[n, 2n]$ contains at least a prime.

Conjecture (Sun, 2012-12-18) For each positive integer n, there is an integer $k \in \{0, \ldots, n\}$ such that $n + k$ and $n + k^2$ are both prime. Moreover, for any integer $n > 971$, there is a positive integer $k < \sqrt{n} \log n$ such that $n + k$ and $n + k^2$ are both prime. Also, for any integer $n > 43181$ there is a positive integer $k \leqslant \sqrt{n}$ such that $n + k^2$ is prime.

Conjecture (Sun, 2013-04-15) For any positive integer n there is a positive integer $k \leqslant 4\sqrt{n+1}$ such that $n^2 + k^2$ is prime.

My conjecture on Heath-Brown primes

Heath-Brown's Theorem (2001). There are infinitely many primes of the form $x^3 + 2y^3$ where x and y are positive integers.

Conjecture (Sun, 2012-12-14). Any positive integer n can be written as $x + y$ ($x, y \in \mathbb{N} = \{0, 1, \ldots\}$) with $x^3 + 2y^3$ prime. In general, for each positive *odd* integer m, any sufficiently large integer can be written as $x + y$ ($x, y \in \mathbb{N}$) with $x^m + 2y^m$ prime.

Conjecture (Sun, 2013-04-15) For any integer $n > 4$ there is a positive integer $k < n$ such that $2n + k$ and $2n^3 + k^3$ are both prime.

My conjecture on primes of the form $x^m + 3y^m$

Conjecture (Sun, 2012-12-16). Let m be a positive integer. Then any sufficiently large odd integer n can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $x^m + 3y^m$ prime (and hence there are infinitely many primes of the form $x^m + 3y^m$), and any sufficiently large even integer n can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $x^m + 3y^m + 1$ prime (and hence there are infinitely many primes of the form $x^m + 3y^m + 1$). In particular, if $m \leqslant 6$ or $m = 18$, then each positive odd integer can be written as $x + y$ ($x, y \in \mathbb{N}$) with $x^m + 3y^m$ prime.

Example. 5 can be written as $1 + 4$ with

$$1^{18} + 3 \times 4^{18} = 206158430209$$

prime.

A general hypothesis on representations

Let's recall

Schinzel's Hypothesis H. If $f_1(x), \ldots, f_k(x)$ are irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing the product $f_1(q)f_2(q)...f_k(q)$ for all $q \in \mathbb{Z}$, then there are infinitely many $n \in \mathbb{Z}^+$ such that $f_1(n), f_2(n), \ldots, f_k(n)$ are all primes.

The following general hypothesis is somewhat similar to Schinzel's hypothesis.

General Conjecture on Representations (Sun, 2012-12-28) Let $f_1(x, y), \ldots, f_m(x, y)$ be non-constant polynomials with integer coefficients. Suppose that for large $n \in \mathbb{Z}^+$, those $f_1(x, n-x), \ldots, f_m(x, n-x)$ are irreducible, and there is no prime dividing all the products $\prod_{k=1}^m f_k(x, n-x)$ with $x \in \mathbb{Z}$. If $n \in \mathbb{Z}^+$ is large enough, then we can write $n = x + y$ $(x, y \in \mathbb{Z}^+)$ such that $|f_1(x, y)|, \ldots, |f_m(x, y)|$ are all prime.

Examples illustrating the general hypothesis

(i) Goldbach's conjecture says that any integer $n > 2$ can be written as $x + y(x, y > 0)$ with $2x + 1$ and $2y - 1$ both prime. (Note that $2(x + y) = (2x + 1) + (2y - 1)$.)

(ii) Each $n > 1$ can be written as $x + y$ $(x, y > 0)$ with $x^3 + 2y^3$ prime. But, for any integer $d > 2$, not every sufficiently large n can be written as $x + y$ $(x, y > 0)$ with $x^3 + dy^3$ prime. For, if n is a multiple of a prime divisor p of $d - 1$, then

$$x^3 + d(n - x)^3 \equiv (1 - d)x^3 \equiv 0 \pmod{p}.$$

(iii) Any even integer $2n > 2$ can be written as $p + q$ $(q > 0)$, where p a Sophie Germain prime and $(p - 1)^2 + q^2$ is also prime. This can be restated as follows: Any integer $n > 1$ can be written as $x + y$ $(x, y > 0)$ with

$$2x + 1,\ 2(2x + 1) = 4x + 3,\ (2x)^2 + (2y - 1)^2$$

all prime.

Two curious conjectures on primes

Recall that those $T_n = n(n+1)/2$ ($n \in \mathbb{N}$) are called triangular numbers. In 2008 I conjectured that any $n \in \mathbb{N}$ with $n \neq 216$ can be written in the form $p + T_k$ with p zero or prime.

Conjecture (Sun, 2013-01-05). Any integer $n > 48624$ with $n \neq 76106$ can be written as $x + T_y$ ($x, y \in \mathbb{N}$) with $\{6x-1, 6x+1\}$ a twin prime pair.

Conjecture (Sun, 2012-12-09). Any integer $n > 2$ can be written as $x^2 + y$ ($x, y \in \mathbb{Z}^+$) with $2xy - 1$ prime. In other words, for each $n = 3, 4, \ldots$ there is a prime in the form $2k(n-k^2) - 1$ with $k \in \mathbb{Z}^+$.

Remark. I verified the conjecture for n up to 3×10^9. When $n = 1691955723$, the number 411 is the only positive integer k with $2k(n-k^2) - 1$ prime, and $411/\log^2 n \approx 0.910246$. I ever thought that one may require $x < \log^2 n$ in part (i), but Jack Brennen found that $n = 4630581798$ is a counterexample, and the least $k \in \mathbb{Z}^+$ with $2k(n-k^2) - 1$ prime is $500 \approx 1.00943 \log^2 n$.

Part III. Various representation problems involving practical numbers

Practical numbers

A positive integer n is called a *practical* number if every $m = 1, \ldots, n$ can be written as a sum of some distinct divisors of n, i.e., there are distinct divisors $d_1, \ldots, d_k$ of n such that

$$\frac{m}{n} = \sum_{i=1}^{k} \frac{1}{d_i}.$$

For example, 6 is practical since $1, 2, 3, 6$ divides 6, and also $4 = 1 + 3$ and $5 = 2 + 3$. As any positive integer has a unique representation in base 2 with digits in $\{0, 1\}$, powers of 2 are all practical. 1 is the only odd practical number.

Practical numbers below 50:

1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48.

Goldbach-type results for practical numbers

Theorem (Stewart [Amer. J. Math., 76(1954)]). If $p_1 < \cdots < p_r$ are distinct primes and $a_1, \ldots, a_r$ are positive integers then $m = p_1^{q_1} \cdots p_r^{a_r}$ is practical if and only if $p_1 = 2$ and

$$p_{s+1} - 1 \leqslant \sigma\big(p_1^{a_1} \cdots p_s^{a_s}\big) \quad \text{for all} \;\; 0 < s < r,$$

where $\sigma(n)$ stands for the sum of all divisors of n.

The behavior of practical numbers is quite similar to that of primes. G. Melfi proved the following Goldbach-type conjecture of M. Margenstern.

Theorem (G. Melfi [J. Number Theory 56(1996)]). Each positive even integer is a sum of two practical numbers, and there are infinitely many practical numbers m with $m - 2$ and $m + 2$ also practical.

Conjecture (Sun, 2013). Any integer $n > 4$ can be written as $p + q/2$, where p and q are practical numbers smaller than n.

More conjectures involving practical numbers

Conjecture (Sun, 2013-01-14) (i) Every positive integer can be written as the sum of a practical number and a triangular number.
(ii) Any odd number greater than one can be written as the sum of a Sophie Germain prime and a practical number.
(iii) Any odd number $n > 1$ can be written as $p + q$, where p is prime, q is practical, and $p^4 + q^4$ is prime. We may also replace $p^4 + q^4$ by $p^2 + q^2$.

Conjecture (Sun, 2013) (i) For any integer $n > 2$, there is a practical number $p < n$ such that $n - p$ and $n + p$ are both prime or both practical.
(ii) Any even integer $2n > 4$ can be written as $p + q = (p + 1) + (q - 1)$, where p and q are primes with $p + 1$ and $q - 1$ both practical.
(iii) Any integer m can be written in the form $p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k$ with $k < n$ and $p_n \leqslant 3m$, and $p_n + 1$ and $p_k - 1$ both practical.

Two kinds of sandwiches

I introduced two kinds of sandwiches.

First kind of sandwiches: $\{p-1, p, p+1\}$ with p prime and $p \pm 1$ practical.

Second kind of sandwiches: $\{q-1, q, q+1\}$ with q practical and $q \pm 1$ prime.

Conjecture (Sun, 2013) (1) Each $n = 4, 5, \ldots$ can be written as $p + q$, where $\{p-1, p, p+1\}$ is a sandwich of the first kind, and q is either prime or practical.

(2) Each even number $n > 8$ can be written as $p + q + r$, where $\{p-1, p, p+1\}$ and $\{q-1, q, q+1\}$ are sandwiches of the first kind, and $\{r-1, r, r+1\}$ is a sandwich of the second kind.

(3) Each integer $n > 5$ can be written as the sum of a prime p with $p \pm 1$ both practical, a prime q with $q+2$ also prime, and a Fibonacci number.

Collatz-type problems

Conjecture (Sun, Feb. 2013). (i) For $n \in \mathbb{Z}^+$ define

$$f(n) = \begin{cases} (p+1)/2 & \text{if } 4 \mid p+1, \\ p & \text{otherwise,} \end{cases}$$

where p is the least prime greater than n with $2(n+1) - p$ prime. If $a_1 \in \{3, 4, \ldots\}$ and $a_{k+1} = f(a_k)$ for $k = 1, 2, 3, \ldots$, then $a_N = 4$ for some positive integer N.

(ii) For $n \in \mathbb{Z}^+$ define

$$g(n) = \begin{cases} q/2 & \text{if } 4 \mid q, \\ q & \text{if } 4 \mid q-2, \end{cases}$$

where q is the least practical number greater than n with $2(n+1) - q$ practical. If $b_1 \in \{4, 5, \ldots\}$ and $b_{k+1} = g(b_k)$ for $k = 1, 2, 3, \ldots$, then $b_N = 4$ for some positive integer N.

Example. In part (i) if we start from $a_1 = 45$ then we get the sequence $45,\ 61,\ 36,\ 37,\ 24,\ 16,\ 17,\ 10,\ 6,\ 4,\ 5,\ 4,\ \ldots$.

Part IV. Some conjectures involving admissible sets

Admissible sets and Yitang Zhang's work

A finite set S of k distinct integers is said to be *admissible* if for any prime p it contains *no* complete system of residues mod p.

The diameter of an admissible set S is defined by
$H(S) = \max S - \min S$ (an even number).

Example (Yitang Zhang): Let $k > 1$ be an integer and let

$$S = \{\text{the first } k \text{ primes after } k\}.$$

For any prime p, if $p \leqslant k$ then S contains no multiple of p; if $p > k = |S|$ then S obviously cannot contain a complete systems of residues mod p. Thus S is admissible. Note that

$$H(S) = p_{\pi(k)+k} - p_{\pi(k)+1} \sim k \log k$$

by the Prime Number Theorem, where $\pi(k) = \sum_{p \leqslant k} 1$.

For $k = 2, 3, \ldots$ define $H(k) = \min\{H(S) : |S| = k\}$.

Yitang Zhang noted that if $k = 3.5 \times 10^6$ then $H(k) < 7 \times 10^7$. Based on this, he proved in 2013 that there are infinitely many n such that $p_{n+1} - p_n < 7 \times 10^7$. This is a great achievement on primes!

My conjectures involving $H(k)$

$H(k)$ $(k = 2, \ldots, 342)$ were determined by Engelsma. The best known upper bounds for $H(k)$ $(342 < k \leqslant 5000)$ is available from http://math.mit.edu/∼primegaps/

Conjecture (Z. W. Sun, June 28, 2013).
(i) $(\sqrt[k]{H(k)})_{k\geqslant 3}$ is strictly decreasing.
(ii) For any integer $k > 4$ we have

$$0 < \frac{H(k)}{k} - H_k < \frac{\gamma+2}{\log k},$$

where H_k is the harmonic number $\sum_{0<j\leqslant k} 1/j$ and $\gamma = 0.5772\ldots$ is the Euler constant.

Conjecture (Z. W. Sun). (i) (June 30, 2013) Any integer $n > 4$ can be written in the form $H(j) + H(k)/2$ with $j, k \in \{2, 3, \ldots\}$.
(ii) (July 2, 2013) Each positive integer $n \neq 23$ can be written in the form $x^2 + H(k)$, where x and $k > 1$ are integers.
(iii) (July 3, 2013) Every integer $n > 4$ can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $xy = H(k)$ for some $k > 1$.

For sources of my conjectures, you may visit my homepage
http://math.nju.edu.cn/∼zwsun

You are welcome to solve my conjectures!

Thank you!