Some new representation problems involving primes

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Part I. On functions taking only prime values
Mills’ Theorem

Any non-constant polynomial \( P(x_1, \ldots, x_n) \) with integer coefficients cannot take primes for all \( x_1, \ldots, x_n \in \mathbb{Z} \).

**Theorem** (Mills, 1947). There is a real number \( A \) such that \( M(n) = \lfloor A^{3^n} \rfloor \) takes only prime values.

**Sketch of the Proof.** Since \( p_{n+1} - p_n = O(p_n^{5/8}) \) (A. E. Ingham, 1937), one can construct infinitely many primes \( P_0, P_1, P_2, \ldots \) with

\[
P_n^3 < P_{n+1} < (P_n + 1)^3 - 1.
\]

Then the sequence \( u_n = P_n^{3^{-n}} \) is increasing while the sequence \( v_n = (P_n + 1)^{3^{-n}} \) is decreasing. As \( u_n < v_n \), we see that \( A = \lim_{n \to \infty} u_n \leq B = \lim_{n \to \infty} v_n \), hence

\[
P_n = u_n^{3^n} < A^{3^n} < P_n + 1 = v_n^{3^n}.
\]

So \( \lfloor A^{3^n} \rfloor = P_n \) is a prime for all \( n = 1, 2, 3, \ldots \).
My problems on central binomial coefficients

Let $p$ be an odd prime. For $k = 0, \ldots, (p - 1)/2$ we have

$$\binom{2k}{k} = \frac{(2k)!}{k!^2} \not\equiv 0 \pmod{p}.$$  

**Conjecture** (Sun, Feb. 20, 2012). Let $p > 5$ be a prime. Then

$$\left\{ \pm \binom{2k}{k} : k = 1, \ldots, \frac{p - 1}{2} \right\}$$

cannot be a reduced system of residues modulo $p$. If $p > 11$, then

$$\binom{2k}{k} \ (0 < k < (p - 1)/2)$$
cannot be pairwise distinct mod $p$.

**Conjecture** (Z. W. Sun, Feb. 21, 2012). For $n = 1, 2, 3, \ldots$ define $s(n)$ as the least integer $m > 1$ such that $\binom{2k}{k} \ (k = 1, \ldots, n)$ are pairwise distinct modulo $m$. Then $s(n)$ is always a prime!

I also guessed that $s(n) < n^2$ for $n = 2, 3, 4, \ldots$. I calculated $s(n)$ for $n = 1, \ldots, 2065$. For example,

$$s(1) = 2, \ s(2) = 3, \ s(3) = 5, \ s(4) = s(5) = s(6) = 11, \ s(7) = s(8) = s(9) = 23, \ s(10) = 31, \ s(11) = \cdots = s(14) = 43.$$
Discriminators for powers of $x$

**Theorem 1** (L. K. Arnold, S. J. Benkoski and B. J. McCabe, 1985). For $n > 4$ the least positive integer $m$ (which is called the *discriminator*, denoted by $D(n)$) such that $1^2, 2^2, \ldots, n^2$ are distinct modulo $m$, is

$$\min\{m \geq 2n : m = p \text{ or } m = 2p \text{ with } p \text{ an odd prime}\}.$$ 

*Remark*. The range of $D(n)$ does not contain those primes $p = 2q + 1$ with $q$ an odd prime.

**Theorem 2** (P. S. Bremser, P. D. Schumer and L. C. Washington, 1990). Let $k > 2$ and $n > 0$ be integers, and let $D_k(n)$ denote the least positive integer $m$ such that $1^k, 2^k, \ldots, n^k$ are distinct modulo $m$.

(i) If $k$ is odd and $n$ is sufficiently large, then

$$D_k(n) = \min\{m \geq n : m \text{ is squarefree, and } (k, \varphi(m)) = 1\}.$$ 

(ii) If $k$ is even and $n$ is sufficiently large, then

$$D_k(n) = \min\{m \geq 2n : m = p \text{ or } 2p \text{ with } p \text{ a prime, and } (k, \varphi(m)) = 2\}.$$
Generate all primes in a combinatorial manner

**Theorem 1** (Sun, Feb. 29, 2012) For $n \in \mathbb{Z}^+$ let $S(n)$ denote the smallest integer $m > 1$ such that those $2k(k - 1) \mod m$ for $k = 1, \ldots, n$ are pairwise distinct. Then $S(n)$ is the least prime greater than $2n - 2$.

**Remark.**

(a) **The range of $S(n)$ is exactly the set of all primes!**

(b) I also showed that for any $d \in \mathbb{Z}^+$ whenever $n \geq d + 2$ the least prime $p \geq 2n + d$ is just the smallest $m \in \mathbb{Z}^+$ such that $2k(k + d)$ $(k = 1, \ldots, n)$ are pairwise distinct modulo $m$.

(c) I proved that the least positive integer $m$ such that those $\binom{k}{2} = k(k - 1)/2$ $(k = 1, \ldots, n)$ are pairwise distinct modulo $m$, is just the least power of two not smaller than $n$. 
Another theorem

**Theorem 2** (Sun, March 2012) (i) Let $d \in \{2, 3\}$ and $n \in \mathbb{Z}^+$. Then the smallest positive integer $m$ such that those $k(dk - 1) \ (k = 1, \ldots, n)$ are pairwise distinct modulo $m$, is the least power of $d$ not smaller than $n$.

(ii) Let $n \in \{4, 5, \ldots\}$. Then the least positive integer $m$ such that $18k(3k - 1) \ (k = 1, \ldots, n)$ are pairwise distinct modulo $m$, is just the least prime $p > 3n$ with $p \equiv 1 \pmod{3}$.

**Remark.** We are also able to prove some other similar results including the following one:

For $n > 5$ the least $m \in \mathbb{Z}^+$ such that those $18k(3k + 1) \ (k = 1, \ldots, n)$ are pairwise distinct modulo $m$, is just the first prime $p \equiv -1 \pmod{3}$ after $3n$. 
Theorem (Sun, April 2013). Let $d \geq 4$ and $c \in (-d, d)$ be relatively prime integers, and let $r(d)$ be the product of all distinct prime divisors of $d$. Then, for any sufficiently large integer $n$, the least positive integer $m$ with $2r(d)k(dk - c)$ ($k = 1, \ldots, n$) pairwise distinct modulo $m$ is just the first prime $p \equiv c \pmod{d}$ after $(d(2n - 1) - c)/(d - 1)$. Moreover, $n > 24310$ suffices for all $d = 4, 5, \ldots, 36$.

Corollary. For each integer $n \geq 6$, the least positive integer $m$ such that $4k(4k - 1)$ (or $4k(4k + 1)$) for $k = 1, \ldots, n$ are pairwise distinct modulo $m$, is just the least prime $p \equiv 1 \pmod{4}$ after $(8n - 4)/3$ (resp., $p \equiv -1 \pmod{4}$ after $(8n - 2)/3$).
Alternating sums of primes

Let $p_n$ be the $n$th prime and define

$$s_n = p_n - p_{n-1} + \cdots + (-1)^{n-1}p_1.$$ 

For example,

$$s_5 = p_5 - p_4 + p_3 - p_2 + p_1 = 11 - 7 + 5 - 3 + 2 = 8.$$ 

Note that

$$s_{2n} = \sum_{k=1}^{n}(p_{2k} - p_{2k-1}) > 0, \quad s_{2n+1} = \sum_{k=1}^{n}(p_{2k+1} - p_{2k}) + p_1 > 0.$$ 

Let $1 \leq k < n$. If $n - k$ is even, then

$$s_n - s_k = (p_n - p_{n-1}) + \cdots + (p_{k+2} - p_{k+1}) > 0.$$ 

If $n - k$ is odd, then

$$s_n - s_k = \sum_{l=k+1}^{n}(-1)^{n-l}p_l - 2\sum_{j=1}^{k}(-1)^{k-j}p_j \equiv n - k \equiv 1 \pmod{2}.$$ 

So, $s_1, s_2, s_3, \ldots$ are pairwise distinct.
An amazing recurrence for primes

We may compute the \((n+1)\)-th prime \(p_{n+1}\) in terms of \(p_1, \ldots, p_n\).

**Conjecture** (Z. W. Sun, J. Number Theory 2013). For any positive integer \(n \neq 1, 2, 4, 9\), the \((n+1)\)-th prime \(p_{n+1}\) is the least positive integer \(m\) such that

\[2s_1^2, \ldots, 2s_n^2\]

are pairwise distinct modulo \(m\).

**Remark.** I have verified the conjecture for \(n \leq 10^5\), and proved that \(2s_1^2, \ldots, 2s_n^2\) are indeed pairwise distinct modulo \(p_{n+1}\).

Let \(1 \leq j < k \leq n\). Then

\[0 < |s_k - s_j| \leq \max\{s_k, s_j\} \leq \max\{p_k, p_j\} \leq p_n < p_{n+1}.
\]

Also, \(s_k + s_j \leq p_k + p_j < 2p_{n+1}\). If \(2 \nmid k - j\), then

\[s_k + s_j = p_k - p_{k-1} + \cdots + p_{j+1} \leq p_k < p_{n+1}.
\]

If \(2 \mid k - j\), then \(s_k \equiv s_j \pmod{2}\) and hence \(s_k + s_j \neq p_{n+1}\). Thus

\[2s_k^2 - 2s_j^2 = 2(s_k - s_j)(s_k + s_j) \not\equiv 0 \pmod{p_{n+1}}.
\]
Conjecture on alternating sums of consecutive primes

**Conjecture** (Z. W. Sun, J. Number Theory, 2013). For any positive integer \( m \), there are consecutive primes \( p_k, \ldots, p_n \) \((k < n)\) not exceeding \( 2m + 2.2\sqrt{m} \) such that

\[
m = p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k.
\]

(Moreover, we may even require that \( m < p_n < m + 4.6\sqrt{m} \) if \( 2 \nmid m \) and \( 2m - 3.6\sqrt{m} + 1 < p_n < 2m + 2.2\sqrt{m} \) if \( 2 \mid m \).)

**Examples.**

\[
10 = 17 - 13 + 11 - 7 + 5 - 3;
\]
\[
20 = 41 - 37 + 31 - 29 + 23 - 19 + 17 - 13 + 11 - 7 + 5 - 3;
\]
\[
303 = p_{76} - p_{75} + \cdots + p_{52};
\]
\[
p_{76} = 383 = \lfloor 303 + 4.6\sqrt{303} \rfloor, \quad p_{52} = 239;
\]
\[
2382 = p_{652} - p_{651} + \cdots + p_{44} - p_{43};
\]
\[
p_{652} = 4871 = \lfloor 2 \cdot 2382 + 2.2\sqrt{2382} \rfloor, \quad p_{43} = 191.
\]

The conjecture has been verified for \( m \) up to \( 10^7 \).

**Prize.** I would like to offer 1000 US dollars for the first proof.
More conjectures involving $s_n = \sum_{k=1}^{n}(-1)^{n-k}p_k$

The following conjecture is an analogy of Dirichlet’s theorem for primes in arithmetic progressions.

**Conjecture** (Sun, 2013). For any integers $m > 0$ and $r$, there are infinitely many $n \in \mathbb{Z}^+$ with $s_n \equiv r \pmod{m}$.

**Conjecture** (Sun, 2013). (i) For each $\lambda = 1, 2, 3$, any integer $n > \lambda$ can be written as $s_k + \lambda s_l$ with $k, l \in \mathbb{Z}^+$.

(ii) Any integer $n > 1$ can be written as $j(j + 1)/2 + s_k$ with $j, k \in \mathbb{Z}^+$.

Recall that a prime $p$ is called a Sophie Germain prime if $2p + 1$ is also prime. It is conjectured that there are infinitely many Sophie Germain primes.

**Conjecture** (Sun, 2013). Each $n = 3, 4, \ldots$ can be written as $p + s_k$, where $p$ is a Sophie Germain prime and $k$ is a positive integer.
Part II. A general hypothesis on representations involving primes
Goldbach’s conjecture

**Goldbach’s Conjecture** (1742): Every even number $n > 2$ can be written in the form $p + q$ with $p$ and $q$ both prime.

**Goldbach’s weak Conjecture** [proved by I. M. Vinogradov (1937) and H. Helfgott (2013)]. Each odd number $n > 6$ can be written as a sum of three primes.

Goldbach’s conjecture implies that for any $n > 2$ there is a prime $p \in [n, 2n]$ since $2n \neq p + q$ if $p$ and $q$ are smaller than $n$.

**Lemoine’s Conjecture** (1894). Any odd integer $n > 6$ can be written as $p + 2q$, where $p$ and $q$ are primes.
Conjectures for twin primes, cousin primes and sexy primes

**Conjecture** (Sun, 2012-12-22) Any integer $n \geq 12$ can be written as $p + q$ with $p, p + 6, 6q \pm 1$ all prime.

**Remark.** I have verified this for $n$ up to $10^9$.

**Conjecture** (2013-01-03) Let

- $A = \{x \in \mathbb{Z}^+ : 6x - 1$ and $6x + 1$ are both prime\},
- $B = \{x \in \mathbb{Z}^+ : 6x + 1$ and $6x + 5$ are both prime\},
- $C = \{x \in \mathbb{Z}^+ : 2x - 3$ and $2x + 3$ are both prime\}.

Then

$A + B = \{2, 3, \ldots\}$, $B + C = \{5, 6, \ldots\}$, $A + C = \{5, 6, \ldots\} \setminus \{161\}$.

Also, if we set $2X := X + X$ then

$2A \supseteq \{702, 703, \ldots\}$, $2B \supseteq \{492, 493, \ldots\}$, $2C \supseteq \{4006, 4007, \ldots\}$. 
Ordowski’s conjecture and Ming-Zhi Zhang’s problem

As conjectured by Fermat and proved by Euler, any prime $p \equiv 1 \pmod{4}$ can be written uniquely as a sum of two squares.

**Tomasz Ordowski’s Conjecture** (Oct. 3-4, 2012) For any prime $p \equiv 1 \pmod{4}$ write $p = a_p^2 + b_p^2$ with $a_p > b_p > 0$. Then

$$\lim_{N \to \infty} \frac{\sum_{p \leq N, p \equiv 1 \pmod{4}} a_p}{\sum_{p \leq N, p \equiv 1 \pmod{4}} b_p} = 1 + \sqrt{2}$$

and

$$\lim_{N \to \infty} \frac{\sum_{p \leq N, p \equiv 1 \pmod{4}} a_p^2}{\sum_{p \leq N, p \equiv 1 \pmod{4}} b_p^2} = \frac{9}{2}.$$  

**Ming-Zhi Zhang’s Problem** (1990s) Whether any odd integer $n > 1$ can be written as $a + b$ with $a, b \in \mathbb{Z}^+$ and $a^2 + b^2$ prime? (Cf. http://oeis.org/A036468)
My conjecture involving $x^2 + xy + y^2$

It is known that any prime $p \equiv 1 \pmod{3}$ can be written uniquely in the form $x^2 + xy + y^2$ with $x > y > 0$.

**Conjecture** (Sun, Nov. 3, 2012). (i) For any prime $p \equiv 1 \pmod{3}$ write $p = x_p^2 + x_p y_p + y_p^2$ with $x_p > y_p > 0$. Then

$$\lim_{N \to \infty} \frac{\sum_{p \leq N, p \equiv 1 \pmod{3}} x_p}{\sum_{p \leq N, p \equiv 1 \pmod{3}} y_p} = 1 + \sqrt{3}$$

and

$$\lim_{N \to \infty} \frac{\sum_{p \leq N, p \equiv 1 \pmod{3}} x_p^2}{\sum_{p \leq N, p \equiv 1 \pmod{3}} y_p^2} = \frac{52}{9}.$$  

(ii) Any integer $n > 1$ with $n \neq 8$ can be written as $x + y$, with $x, y \in \mathbb{Z}^+$ and $x^2 + xy + y^2$ prime.
My conjectures involving $x^2 + 3xy + y^2$

**Conjecture** (Sun, Nov. 4, 2012). (i) For any prime $p \equiv \pm 1 \pmod{5}$ write $p = x_p^2 + 3x_py_p + y_p^2$ with $x_p > y_p > 0$. Then

$$
\lim_{N \to \infty} \frac{\sum_{p \leq N, \ p \equiv \pm 1 \pmod{5}} x_p}{\sum_{p \leq N, \ p \equiv \pm 1 \pmod{5}} y_p} = 1 + \sqrt{5}.
$$

(ii) Any integer $n > 1$ can be written as $x + y$, with $x, y \in \mathbb{Z}^+$ and $x^2 + 3xy + y^2$ prime.

**Conjecture** (Sun, Nov. 5, 2012). For any integer $n \geq 1188$, there are primes $p$ and $q$ with $p^2 + 3pq + q^2$ prime such that $p + (1 + \{n\}_2)q = n$, where $\{n\}_m$ denotes the least nonnegative residue of $n \pmod{m}$.

I have some other similar conjectures.
My conjecture on prime differences

**Polignac’s Conjecture** (1849). For any positive even integer $2m$, there are infinitely many prime pairs $\{p, q\}$ with $p - q = 2m$.

My following conjecture “explains” why Polignac’s Conjecture should be correct.

**Conjecture** (Sun, 2012). For each $m \in \mathbb{N} = \{0, 1, 2, \ldots\}$, any sufficiently large integer $n$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $x \pm m$ and $2xy + 1$ all prime. In particular, any integer $n > 7$ can be written as $p + q$, where $q$ is a positive integer, and $p$ and $2pq + 1$ are both prime.

*Remark.* For $m = 1, 2, 3, 4, 5$ it suffices to require that $n$ is greater than 623, 28, 151, 357, 199 respectively.
A conjecture refining Bertrand’s Postulate

**Bertrand’s Postulate** (proved by Chebyshev in 1850). For any positive integer $n$, the interval $[n, 2n]$ contains at least a prime.

**Conjecture** (Sun, 2012-12-18) For each positive integer $n$, there is an integer $k \in \{0, \ldots, n\}$ such that $n + k$ and $n + k^2$ are both prime. Moreover, for any integer $n > 971$, there is a positive integer $k < \sqrt{n \log n}$ such that $n + k$ and $n + k^2$ are both prime. Also, for any integer $n > 43181$ there is a positive integer $k \leq \sqrt{n}$ such that $n + k^2$ is prime.

**Conjecture** (Sun, 2013-04-15) For any positive integer $n$ there is a positive integer $k \leq 4\sqrt{n+1}$ such that $n^2 + k^2$ is prime.
My conjecture on Heath-Brown primes

**Heath-Brown’s Theorem** (2001). There are infinitely many primes of the form $x^3 + 2y^3$ where $x$ and $y$ are positive integers.

**Conjecture** (Sun, 2012-12-14). Any positive integer $n$ can be written as $x + y$ ($x, y \in \mathbb{N} = \{0, 1, \ldots\}$) with $x^3 + 2y^3$ prime. In general, for each positive odd integer $m$, any sufficiently large integer can be written as $x + y$ ($x, y \in \mathbb{N}$) with $x^m + 2y^m$ prime.

**Conjecture** (Sun, 2013-04-15) For any integer $n > 4$ there is a positive integer $k < n$ such that $2n + k$ and $2n^3 + k^3$ are both prime.
My conjecture on primes of the form $x^m + 3y^m$

**Conjecture** (Sun, 2012-12-16). Let $m$ be a positive integer. Then any sufficiently large odd integer $n$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $x^m + 3y^m$ prime (and hence there are infinitely many primes of the form $x^m + 3y^m$), and any sufficiently large even integer $n$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $x^m + 3y^m + 1$ prime (and hence there are infinitely many primes of the form $x^m + 3y^m + 1$). In particular, if $m \leq 6$ or $m = 18$, then each positive odd integer can be written as $x + y$ ($x, y \in \mathbb{N}$) with $x^m + 3y^m$ prime.

*Example.* 5 can be written as $1 + 4$ with

$$1^{18} + 3 \times 4^{18} = 206158430209$$

prime.
A general hypothesis on representations

Let’s recall

**Schinzel’s Hypothesis H.** If \( f_1(x), \ldots, f_k(x) \) are irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing the product \( f_1(q)f_2(q)\ldots f_k(q) \) for all \( q \in \mathbb{Z} \), then there are infinitely many \( n \in \mathbb{Z}^+ \) such that \( f_1(n), f_2(n), \ldots, f_k(n) \) are all primes.

The following general hypothesis is somewhat similar to Schinzel’s hypothesis.

**General Conjecture on Representations** (Sun, 2012-12-28) Let \( f_1(x, y), \ldots, f_m(x, y) \) be non-constant polynomials with integer coefficients. Suppose that for large \( n \in \mathbb{Z}^+ \), those \( f_1(x, n-x), \ldots, f_m(x, n-x) \) are irreducible, and there is no prime dividing all the products \( \prod_{k=1}^m f_k(x, n-x) \) with \( x \in \mathbb{Z} \). If \( n \in \mathbb{Z}^+ \) is large enough, then we can write \( n = x + y \) \((x, y \in \mathbb{Z}^+)\) such that \( |f_1(x, y)|, \ldots, |f_m(x, y)| \) are all prime.
Examples illustrating the general hypothesis

(i) Goldbach’s conjecture says that any integer \( n > 2 \) can be written as \( x + y(x, y > 0) \) with \( 2x + 1 \) and \( 2y - 1 \) both prime. (Note that \( 2(x + y) = (2x + 1) + (2y - 1) \).)

(ii) Each \( n > 1 \) can be written as \( x + y \ (x, y > 0) \) with \( x^3 + 2y^3 \) prime. But, for any integer \( d > 2 \), not every sufficiently large \( n \) can be written as \( x + y \ (x, y > 0) \) with \( x^3 + dy^3 \) prime. For, if \( n \) is a multiple of a prime divisor \( p \) of \( d - 1 \), then

\[
x^3 + d(n - x)^3 \equiv (1 - d)x^3 \equiv 0 \pmod{p}.
\]

(iii) Any even integer \( 2n > 2 \) can be written as \( p + q \ (q > 0) \), where \( p \) a Sophie Germain prime and \((p - 1)^2 + q^2\) is also prime. This can be restated as follows: Any integer \( n > 1 \) can be written as \( x + y \ (x, y > 0) \) with

\[
2x + 1, \ 2(2x + 1) = 4x + 3, \ (2x)^2 + (2y - 1)^2
\]

all prime.
Two curious conjectures on primes

Recall that those $T_n = n(n + 1)/2$ ($n \in \mathbb{N}$) are called triangular numbers. In 2008 I conjectured that any $n \in \mathbb{N}$ with $n \neq 216$ can be written in the form $p + T_k$ with $p$ zero or prime.

**Conjecture** (Sun, 2013-01-05). Any integer $n > 48624$ with $n \neq 76106$ can be written as $x + T_y$ ($x, y \in \mathbb{N}$) with $\{6x - 1, 6x + 1\}$ a twin prime pair.

**Conjecture** (Sun, 2012-12-09). Any integer $n > 2$ can be written as $x^2 + y$ ($x, y \in \mathbb{Z}^+$) with $2xy - 1$ prime. In other words, for each $n = 3, 4, \ldots$ there is a prime in the form $2k(n - k^2) - 1$ with $k \in \mathbb{Z}^+$.

*Remark.* I verified the conjecture for $n$ up to $3 \times 10^9$. When $n = 1691955723$, the number 411 is the only positive integer $k$ with $2k(n - k^2) - 1$ prime, and $411/\log^2 n \approx 0.910246$. I ever thought that one may require $x < \log^2 n$ in part (i), but Jack Brennen found that $n = 4630581798$ is a counterexample, and the least $k \in \mathbb{Z}^+$ with $2k(n - k^2) - 1$ prime is $500 \approx 1.00943 \log^2 n$. 

Part III. Various representation problems involving practical numbers
Practical numbers

A positive integer \( n \) is called a \textit{practical} number if every \( m = 1, \ldots, n \) can be written as a sum of some distinct divisors of \( n \), i.e., there are distinct divisors \( d_1, \ldots, d_k \) of \( n \) such that

\[
\frac{m}{n} = \sum_{i=1}^{k} \frac{1}{d_i}.
\]

For example, 6 is practical since 1, 2, 3, 6 divides 6, and also \( 4 = 1 + 3 \) and \( 5 = 2 + 3 \). As any positive integer has a unique representation in base 2 with digits in \{0, 1\}, powers of 2 are all practical. 1 is the only odd practical number.

Practical numbers below 50:
1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48.
Goldbach-type results for practical numbers

**Theorem** (Stewart [Amer. J. Math., 76(1954)]). If $p_1 < \cdots < p_r$ are distinct primes and $a_1, \ldots, a_r$ are positive integers then $m = p_1^{a_1} \cdots p_r^{a_r}$ is practical if and only if $p_1 = 2$ and

$$p_{s+1} - 1 \leq \sigma(p_1^{a_1} \cdots p_s^{a_s}) \quad \text{for all } 0 < s < r,$$

where $\sigma(n)$ stands for the sum of all divisors of $n$.

The behavior of practical numbers is quite similar to that of primes. G. Melfi proved the following Goldbach-type conjecture of M. Margenstern.

**Theorem** (G. Melfi [J. Number Theory 56(1996)]). Each positive even integer is a sum of two practical numbers, and there are infinitely many practical numbers $m$ with $m - 2$ and $m + 2$ also practical.

**Conjecture** (Sun, 2013). Any integer $n > 4$ can be written as $p + q/2$, where $p$ and $q$ are practical numbers smaller than $n$. 
More conjectures involving practical numbers

**Conjecture** (Sun, 2013-01-14) (i) Every positive integer can be written as the sum of a practical number and a triangular number. (ii) Any odd number greater than one can be written as the sum of a Sophie Germain prime and a practical number. (iii) Any odd number $n > 1$ can be written as $p + q$, where $p$ is prime, $q$ is practical, and $p^4 + q^4$ is prime. We may also replace $p^4 + q^4$ by $p^2 + q^2$.

**Conjecture** (Sun, 2013) (i) For any integer $n > 2$, there is a practical number $p < n$ such that $n - p$ and $n + p$ are both prime or both practical. (ii) Any even integer $2n > 4$ can be written as $p + q = (p + 1) + (q - 1)$, where $p$ and $q$ are primes with $p + 1$ and $q - 1$ both practical. (iii) Any integer $m$ can be written in the form $p_n - p_{n-1} + \cdots + (-1)^{n-k} p_k$ with $k < n$ and $p_n \leq 3m$, and $p_n + 1$ and $p_k - 1$ both practical.
Two kinds of sandwiches

I introduced two kinds of sandwiches.

First kind of sandwiches: \( \{ p - 1, p, p + 1 \} \) with \( p \) prime and \( p \pm 1 \) practical.

Second kind of sandwiches: \( \{ q - 1, q, q + 1 \} \) with \( q \) practical and \( q \pm 1 \) prime.

**Conjecture** (Sun, 2013) (1) Each \( n = 4, 5, \ldots \) can be written as \( p + q \), where \( \{ p - 1, p, p + 1 \} \) is a sandwich of the first kind, and \( q \) is either prime or practical.

(2) Each even number \( n > 8 \) can be written as \( p + q + r \), where \( \{ p - 1, p, p + 1 \} \) and \( \{ q - 1, q, q + 1 \} \) are sandwiches of the first kind, and \( \{ r - 1, r, r + 1 \} \) is a sandwich of the second kind.

(3) Each integer \( n > 5 \) can be written as the sum of a prime \( p \) with \( p \pm 1 \) both practical, a prime \( q \) with \( q + 2 \) also prime, and a Fibonacci number.
Collatz-type problems

Conjecture (Sun, Feb. 2013). (i) For \( n \in \mathbb{Z}^+ \) define

\[
f(n) = \begin{cases} 
  (p + 1)/2 & \text{if } 4 \mid p + 1, \\
  p & \text{otherwise},
\end{cases}
\]

where \( p \) is the least prime greater than \( n \) with \( 2(n + 1) - p \) prime. If \( a_1 \in \{3, 4, \ldots\} \) and \( a_{k+1} = f(a_k) \) for \( k = 1, 2, 3, \ldots \), then \( a_N = 4 \) for some positive integer \( N \).

(ii) For \( n \in \mathbb{Z}^+ \) define

\[
g(n) = \begin{cases} 
  q/2 & \text{if } 4 \mid q, \\
  q & \text{if } 4 \mid q - 2,
\end{cases}
\]

where \( q \) is the least practical number greater than \( n \) with \( 2(n + 1) - q \) practical. If \( b_1 \in \{4, 5, \ldots\} \) and \( b_{k+1} = g(b_k) \) for \( k = 1, 2, 3, \ldots \), then \( b_N = 4 \) for some positive integer \( N \).

Example. In part (i) if we start from \( a_1 = 45 \) then we get the sequence 45, 61, 36, 37, 24, 16, 17, 10, 6, 4, 5, 4, \ldots
Part IV. Some conjectures involving admissible sets
Admissible sets and Yitang Zhang’s work

A finite set $S$ of $k$ distinct integers is said to be admissible if for any prime $p$ it contains no complete system of residues mod $p$. The diameter of an admissible set $S$ is defined by $H(S) = \max S - \min S$ (an even number).

**Example** (Yitang Zhang): Let $k > 1$ be an integer and let

$$S = \{\text{the first } k \text{ primes after } k\}.$$ 

For any prime $p$, if $p \leq k$ then $S$ contains no multiple of $p$; if $p > k = |S|$ then $S$ obviously cannot contain a complete system of residues mod $p$. Thus $S$ is admissible. Note that

$$H(S) = p_{\pi(k)+k} - p_{\pi(k)+1} \sim k \log k$$

by the Prime Number Theorem, where $\pi(k) = \sum_{p \leq k} 1$.

For $k = 2, 3, \ldots$ define $H(k) = \min\{H(S) : |S| = k\}$.

Yitang Zhang noted that if $k = 3.5 \times 10^6$ then $H(k) < 7 \times 10^7$. Based on this, he proved in 2013 that there are infinitely many $n$ such that $p_{n+1} - p_n < 7 \times 10^7$. This is a great achievement on primes!
My conjectures involving $H(k)$

$H(k) (k = 2, \ldots, 342)$ were determined by Engelsma. The best known upper bounds for $H(k) (342 < k \leq 5000)$ is available from http://math.mit.edu/~primegaps/

**Conjecture** (Z. W. Sun, June 28, 2013).
(i) $(\sqrt[k]{H(k)})_{k \geq 3}$ is strictly decreasing.
(ii) For any integer $k > 4$ we have

$$0 < \frac{H(k)}{k} - H_k < \frac{\gamma + 2}{\log k},$$

where $H_k$ is the harmonic number $\sum_{0 < j \leq k} 1/j$ and $\gamma = 0.5772 \ldots$ is the Euler constant.

**Conjecture** (Z. W. Sun). (i) (June 30, 2013) Any integer $n > 4$ can be written in the form $H(j) + H(k)/2$ with $j, k \in \{2, 3, \ldots\}$.
(ii) (July 2, 2013) Each positive integer $n \neq 23$ can be written in the form $x^2 + H(k)$, where $x$ and $k > 1$ are integers.
(iii) (July 3, 2013) Every integer $n > 4$ can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $xy = H(k)$ for some $k > 1$. 
For sources of my conjectures, you may visit my homepage http://math.nju.edu.cn/~zwsun

You are welcome to solve my conjectures!

Thank you!