Part I. Representations of positive integers involving primes
Alternating sums of primes

Let $p_n$ be the $n$th prime and define

$$ s_n = p_n - p_{n-1} + \cdots + (-1)^{n-1} p_1. $$

For example,

$$ s_5 = p_5 - p_4 + p_3 - p_2 + p_1 = 11 - 7 + 5 - 3 + 2 = 8. $$

Note that

$$ s_{2n} = \sum_{k=1}^{n} (p_{2k} - p_{2k-1}) > 0, \quad s_{2n+1} = \sum_{k=1}^{n} (p_{2k+1} - p_{2k}) + p_1 > 0. $$

Let $1 \leq k < n$. If $n - k$ is even, then

$$ s_n - s_k = (p_n - p_{n-1}) + \cdots + (p_{k+2} - p_{k+1}) > 0. $$

If $n - k$ is odd, then

$$ s_n - s_k = \sum_{l=k+1}^{n} (-1)^{n-l} p_l - 2 \sum_{j=1}^{k} (-1)^{k-j} p_j \equiv n - k \equiv 1 \pmod{2}. $$

So, $s_1, s_2, s_3, \ldots$ are pairwise distinct.
An amazing recurrence for primes

We may compute the \((n+1)\)-th prime \(p_{n+1}\) in terms of \(p_1, \ldots, p_n\).

**Conjecture** (Z. W. Sun, J. Number Theory 2013). For any positive integer \(n \neq 1, 2, 4, 9\), the \((n+1)\)-th prime \(p_{n+1}\) is the least positive integer \(m\) such that

\[
2s_1^2, \ldots, 2s_n^2
\]

are pairwise distinct modulo \(m\).

**Remark.** I have verified the conjecture for \(n \leq 2 \times 10^5\), and proved that \(2s_1^2, \ldots, 2s_n^2\) are indeed pairwise distinct modulo \(p_{n+1}\).

Let \(1 \leq j < k \leq n\). Then

\[
0 < |s_k - s_j| \leq \max\{s_k, s_j\} \leq \max\{p_k, p_j\} \leq p_n < p_{n+1}.
\]

Also, \(s_k + s_j \leq p_k + p_j < 2p_{n+1}\). If \(2 \nmid k - j\), then

\[
s_k + s_j = p_k - p_{k-1} + \cdots + p_{j+1} \leq p_k < p_{n+1}.
\]

If \(2 \mid k - j\), then \(s_k \equiv s_j \pmod{2}\) and hence \(s_k + s_j \neq p_{n+1}\). Thus

\[
2s_k^2 - 2s_j^2 = 2(s_k - s_j)(s_k + s_j) \not\equiv 0 \pmod{p_{n+1}}.
\]
Conjecture on alternating sums of consecutive primes

**Conjecture** (Z. W. Sun, J. Number Theory, 2013). For any positive integer \( m \), there are consecutive primes \( p_k, \ldots, p_n \) \((k < n)\) not exceeding \( 2m + 2.2\sqrt{m} \) such that

\[
m = p_n - p_{n-1} + \cdots + (-1)^{n-k}p_k.
\]

(Moreover, we may even require that \( m < p_n < m + 4.6\sqrt{m} \) if \( 2 \nmid m \) and \( 2m - 3.6\sqrt{m} + 1 < p_n < 2m + 2.2\sqrt{m} \) if \( 2 \mid m \).)

**Examples.**

\[
10 = 17 - 13 + 11 - 7 + 5 - 3;
\]
\[
20 = 41 - 37 + 31 - 29 + 23 - 19 + 17 - 13 + 11 - 7 + 5 - 3;
\]
\[
303 = p_{76} - p_{75} + \cdots + p_{52},
\]
\[
p_{76} = 383 = \lfloor 303 + 4.6\sqrt{303} \rfloor, \text{ } p_{52} = 239;
\]
\[
2382 = p_{652} - p_{651} + \cdots + p_{44} - p_{43},
\]
\[
p_{652} = 4871 = \lfloor 2 \cdot 2382 + 2.2\sqrt{2382} \rfloor, \text{ } p_{43} = 191.
\]

The conjecture has been verified for \( m \) up to \( 10^7 \).

**Prize.** I would like to offer 1000 US dollars for the first proof.
Goldbach’s conjecture

**Goldbach’s Conjecture** (1742): Every even number \( n > 2 \) can be written in the form \( p + q \) with \( p \) and \( q \) both prime.

**Goldbach’s weak Conjecture** [proved by I. M. Vinogradov (1937) and H. Helfgott (2013)]. Each odd number \( n > 6 \) can be written as a sum of three primes.

Goldbach’s conjecture implies that for any \( n > 2 \) there is a prime \( p \in [n, 2n] \) since \( 2n \neq p + q \) if \( p \) and \( q \) are smaller than \( n \).

**Lemoine’s Conjecture** (1894). Any odd integer \( n > 6 \) can be written as \( p + 2q \), where \( p \) and \( q \) are primes.
Conjectures for twin primes, cousin primes and sexy primes

Conjecture (Sun, 2012-12-22) Any integer $n \geq 12$ can be written as $p + q$ with $p, p + 6, 6q \pm 1$ all prime.

*Remark.* I have verified this for $n$ up to $10^9$.

Conjecture (Sun, 2013-01-03) Let

$$A = \{x \in \mathbb{Z}^+ : 6x - 1 \text{ and } 6x + 1 \text{ are both prime}\},$$

$$B = \{x \in \mathbb{Z}^+ : 6x + 1 \text{ and } 6x + 5 \text{ are both prime}\},$$

$$C = \{x \in \mathbb{Z}^+ : 2x - 3 \text{ and } 2x + 3 \text{ are both prime}\}.$$

Then

$$A + B = \{2, 3, \ldots\}, \quad B + C = \{5, 6, \ldots\}, \quad A + C = \{5, 6, \ldots\} \setminus \{161\}.$$

Also, if we set $2X := X + X$ then

$$2A \supseteq \{702, 703, \ldots\}, \quad 2B \supseteq \{492, 493, \ldots\}, \quad 2C \supseteq \{4006, 4007, \ldots\}.$$
A conjecture refining Bertrand’s Postulate

**Bertrand’s Postulate** (proved by Chebyshev in 1850). For any positive integer $n$, the interval $[n, 2n]$ contains at least a prime.

**Conjecture** (Sun, 2012-12-18) For each positive integer $n$, there is an integer $k \in \{0, \ldots, n\}$ such that $n + k$ and $n + k^2$ are both prime.

**Conjecture** (Sun, 2013-04-15) For any positive integer $n$ there is a positive integer $k \leq 4\sqrt{n+1}$ such that $n^2 + k^2$ is prime.
My conjecture on Heath-Brown primes

**Heath-Brown’s Theorem** (2001). There are infinitely many primes of the form $x^3 + 2y^3$ where $x$ and $y$ are positive integers.

**Conjecture** (Sun, 2012-12-14). Any positive integer $n$ can be written as $x + y$ ($x, y \in \mathbb{N} = \{0, 1, \ldots\}$) with $x^3 + 2y^3$ prime. In general, for each positive odd integer $m$, any sufficiently large integer can be written as $x + y$ ($x, y \in \mathbb{N}$) with $x^m + 2y^m$ prime.

**Conjecture** (Sun, 2013-04-15) For any integer $n > 4$ there is a positive integer $k < n$ such that $2n + k$ and $2n^3 + k^3$ are both prime.
Let’s recall

**Schinzel’s Hypothesis H.** If $f_1(x), \ldots, f_k(x)$ are irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing the product $f_1(q)f_2(q)\ldots f_k(q)$ for all $q \in \mathbb{Z}$, then there are infinitely many $n \in \mathbb{Z}^+$ such that $f_1(n), f_2(n), \ldots, f_k(n)$ are all primes.

The following general hypothesis is somewhat similar to Schinzel’s hypothesis.

**General Conjecture on Representations** (Sun, 2012-12-28) Let $f_1(x, y), \ldots, f_m(x, y)$ be non-constant polynomials with integer coefficients. Suppose that for large $n \in \mathbb{Z}^+$, those $f_1(x, n-x), \ldots, f_m(x, n-x)$ are irreducible, and there is no prime dividing all the products $\prod_{k=1}^m f_k(x, n-x)$ with $x \in \mathbb{Z}$. If $n \in \mathbb{Z}^+$ is large enough, then we can write $n = x + y$ ($x, y \in \mathbb{Z}^+$) such that $|f_1(x, y)|, \ldots, |f_m(x, y)|$ are all prime.
Two curious conjectures on primes

**Conjecture** (Sun, 2013, Prize $200). Any integer \( n > 1 \) can be written as \( x + y \) \((x, y \in \mathbb{Z}^+)\) such that \( x + ny \) and \( x^2 + ny^2 \) are both prime.

For example, \( 20 = 11 + 9 \) with
\[
11 + 20 \times 9 = 191 \text{ and } 11^2 + 20 \times 9^2 = 1741 \text{ both prime.}
\]

**Conjecture** (Sun, 2013, Prize $1000). Any integer \( n > 1 \) can be written as \( k + m \) \((k, m \in \mathbb{Z}^+)\) with \( 2^k + m \) prime.

**Remark.** I have verified this for \( n \leq 10^7 \). For example, \( 9302003 = 311468 + 8990535 \) with \( 2^{311468} + 8990535 \) a prime of 93762 decimal digits.
Unification of Goldbach’s conjecture and the twin prime conjecture

Unification of Goldbach’s Conjecture and the Twin Prime Conjecture (Sun, 2014-01-29). For any integer \( n > 2 \), there is a prime \( q \) with \( 2n - q \) and \( p_{q+2} + 2 \) both prime.

We have verified the conjecture for \( n \) up to \( 10^8 \). Clearly, it is stronger than Goldbach’s conjecture. Now we explain why it implies the twin prime conjecture.

In fact, if all primes \( q \) with \( p_{q+2} + 2 \) prime are smaller than an even number \( N > 2 \), then for any such a prime \( q \) the number \( N! - q \) is composite since

\[
N! - q \equiv 0 \pmod{q} \quad \text{and} \quad N! - q \geq q(q+1) - q > q.
\]

Example. \( 20 = 3 + 17 \) with 3, 17 and \( p_{3+2} + 2 = 11 + 2 = 13 \) all prime.
Super Twin Prime Conjecture

If $p, p + 2$ and $\pi(p)$ are all prime, then we call $\{p, p + 2\}$ a super twin prime pair.

**Super Twin Prime Conjecture** (Sun, 2014-02-05). Any integer $n > 2$ can be written as $k + m$ with $k$ and $m$ positive integers such that $p_k + 2$ and $p_{p_m} + 2$ are both prime.

**Example.** $22 = 20 + 2$ with $p_{20} + 2 = 71 + 2 = 73$ and $p_{p_2} + 2 = p_3 + 2 = 5 + 2 = 7$ both prime.

**Remark.** If all those positive integer $m$ with $p_{p_m} + 2$ prime are smaller than an integer $N > 2$, then by the conjecture, for each $j = 1, 2, 3, \ldots$, there are positive integers $k(j)$ and $m(j)$ with $k(j) + m(j) = jN$ such that $p_{k(j)} + 2$ and $p_{p_{m(j)}} + 2$ are both prime, and hence $k(j) \in ((j - 1)N, jN)$ since $m(j) < N$; thus

$$\sum_{j=1}^{\infty} \frac{1}{p_{k(j)}} \geq \sum_{j=1}^{\infty} \frac{1}{p_{jN}},$$

which is impossible since the series on the right-hand side diverges while the series on the left-hand side converges by Brun’s theorem.
Practical numbers

A positive integer \( n \) is called a *practical* number if every \( m = 1, \ldots, n \) can be written as a sum of some distinct divisors of \( n \), i.e., there are distinct divisors \( d_1, \ldots, d_k \) of \( n \) such that

\[
\frac{m}{n} = \sum_{i=1}^{k} \frac{1}{d_i}.
\]

For example, 6 is practical since 1, 2, 3, 6 divides 6, and also \( 4 = 1 + 3 \) and \( 5 = 2 + 3 \). As any positive integer has a unique representation in base 2 with digits in \( \{0, 1\} \), powers of 2 are all practical. 1 is the only odd practical number.

Practical numbers below 50:
1, 2, 4, 6, 8, 12, 16, 18, 20, 24, 28, 30, 32, 36, 40, 42, 48.

I view prime numbers 2, 3, 5, 7, 11, \ldots, as *men* and practical numbers 1, 2, 4, 6, 8, 12, \ldots as *women*. If one of \( n \) and \( n + 1 \) is prime and the other is practical, then I call \( \{n, n + 1\} \) a *couple*. 
Goldbach-type results for practical numbers

**Theorem** (Stewart [Amer. J. Math., 76(1954)]). If \(p_1 < \cdots < p_r\) are distinct primes and \(a_1, \ldots, a_r\) are positive integers then \(m = p_1^{a_1} \cdots p_r^{a_r}\) is practical if and only if \(p_1 = 2\) and

\[
p_{s+1} - 1 \leq \sigma(p_1^{a_1} \cdots p_s^{a_s}) \quad \text{for all} \quad 0 < s < r,
\]

where \(\sigma(n)\) stands for the sum of all divisors of \(n\).

The behavior of practical numbers is quite similar to that of primes. G. Melfi proved the following Goldbach-type conjecture of M. Margenstern.

**Theorem** (G. Melfi [J. Number Theory 56(1996)]). Each positive even integer is a sum of two practical numbers, and there are infinitely many practical numbers \(m\) with \(m - 2\) and \(m + 2\) also practical.

**Conjecture** (Sun, 2013). Any even integer \(2n > 4\) can be written as \(p + q = (p + 1) + (q - 1)\), where \(p\) and \(q\) are primes with \(p + 1\) and \(q - 1\) both practical.
In 2013, I introduced two kinds of sandwiches.

**First kind of sandwiches:** \( \{ p - 1, p, p + 1 \} \) with \( p \) prime and \( p \pm 1 \) practical.

**Second kind of sandwiches:** \( \{ q - 1, q, q + 1 \} \) with \( q \) practical and \( q \pm 1 \) prime.

**Conjecture** (Sun, 2013).

(i) Each \( n = 4, 5, \ldots \) can be written as \( p + q \), where \( \{ p - 1, p, p + 1 \} \) is a sandwich of the first kind, and \( q \) is either prime or practical.

(ii) Each even number \( n > 8 \) can be written as \( p + q + r \), where \( \{ p - 1, p, p + 1 \} \) and \( \{ q - 1, q, q + 1 \} \) are sandwiches of the first kind, and \( \{ r - 1, r, r + 1 \} \) is a sandwich of the second kind.
Two conjectures on Euler’s totient function

For any positive integer \( m \), define

\[
\varphi(m) = |\{1 \leq a \leq m : (a, m) = 1\}|.
\]

The function \( \varphi \) is called *Euler’s totient function*. If \( m = p_1^{a_1} \cdots p_k^{a_k} \) with \( p_1, \ldots, p_k \) distinct primes and \( a_1, \ldots, a_k \) positive integers, then

\[
\varphi(m) = m \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right) = \prod_{i=1}^{k} p_i^{a_i-1}(p_i - 1).
\]

**Conjecture 1** (Sun, 2013). Any integer \( n > 5 \) can be written as \( k + m \) with \( k \) and \( m \) positive integers such that \((\varphi(k) + \varphi(m))/2\) is prime.

**Conjecture 2** (Sun, 2014). Any integer \( n > 8 \) can be written as \( k + m \) with \( k \) and \( m \) distinct positive integers such that \( \varphi(k)\varphi(m) \) is a square.
Part II. Representations involving powers and polygonal numbers
Sums of four squares

Lagrange’s Four-square Theorem (1770). Each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as the sum of four squares.

S. Ramanujan’s Observation (confirmed by L.E. Dickson in 1927). There are totally 54 quadruples $(a, b, c, d) \in (\mathbb{Z}^+)^4$ with $a \leq b \leq c \leq d$ such that each $n \in \mathbb{N}$ can be written as $aw^2 + bx^2 + cy^2 + dz^2$ with $w, x, y, z \in \mathbb{Z}$. The 54 quadruples are

$(1, 1, 1, 1), (1, 1, 1, 2), (1, 1, 2, 2), (1, 2, 2, 2), (1, 1, 1, 3), (1, 1, 2, 3), (1, 2, 2, 3), (1, 1, 3, 3), (1, 2, 3, 3), (1, 1, 4, 4), (1, 1, 2, 4), (1, 2, 2, 4), (1, 1, 3, 4), (1, 2, 3, 4), (1, 2, 4, 4), (1, 1, 5, 5), (1, 1, 2, 5), (1, 2, 2, 5), (1, 1, 3, 5), (1, 2, 3, 5), (1, 2, 4, 5), (1, 1, 6, 6), (1, 1, 2, 6), (1, 2, 2, 6), (1, 1, 3, 6), (1, 2, 3, 6), (1, 2, 4, 6), (1, 2, 5, 6), (1, 1, 7, 7), (1, 1, 2, 7), (1, 2, 2, 7), (1, 2, 3, 7), (1, 2, 4, 7), (1, 2, 5, 7), (1, 1, 8, 8), (1, 2, 3, 8), (1, 2, 4, 8), (1, 2, 5, 8), (1, 1, 9, 9), (1, 2, 3, 9), (1, 2, 4, 9), (1, 1, 10, 10), (1, 2, 3, 10), (1, 2, 4, 10), (1, 2, 5, 10), (1, 1, 11, 11), (1, 2, 4, 11), (1, 1, 12, 12), (1, 2, 4, 12), (1, 1, 13, 13), (1, 2, 4, 13), (1, 1, 14, 14), (1, 2, 4, 14).
Universal sums of four mixed powers

If any \( n \in \mathbb{N} \) can be written as \( f(x_1, \ldots, x_n) \) with \( x_1, \ldots, x_n \) in \( \mathbb{N} \) (or \( \mathbb{Z} \)), then we say that \( f \) is universal over \( \mathbb{N} \) (or \( \mathbb{Z} \)).

**Theorem** (Z.-W. Sun [JNT 175(2017)]) For any \( a \in \{1, 4\} \) and \( k \in \{4, 5, 6\} \), \( aw^k + x^2 + y^2 + z^2 \) is universal over \( \mathbb{N} \).

**Theorem** (Z.-W. Sun [Nanjing Univ. J. Math. Biquarterly 34(2017)]) Let \( a, b, c, d \in \mathbb{Z}^+ \) with \( a \leq b \leq c \leq d \), and let \( h, i, j, k \in \{2, 3, \ldots\} \) with at most one of \( h, i, j, k \) equal to two. Suppose that \( h \leq i \) if \( a = b \), \( i \leq j \) if \( b = c \), and \( j \leq k \) if \( c = d \). If \( f(w, x, y, z) = aw^h + bx^i + cy^j + dz^k \) is universal over \( \mathbb{N} \), then \( f(w, x, y, z) \) must be among the following 9 polynomials:

\[
\begin{align*}
&w^2 + x^3 + y^4 + 2z^3, \ w^2 + x^3 + y^4 + 2z^4, \ w^2 + x^3 + 2y^3 + 3z^3, \\
&w^2 + x^3 + 2y^3 + 3z^4, \ w^2 + x^3 + 2y^3 + 4z^3, \ w^2 + x^3 + 2y^3 + 5z^3, \\
&w^2 + x^3 + 2y^3 + 6z^3, \ w^2 + x^3 + 2y^3 + 6z^4, \ w^3 + x^4 + 2y^2 + 4z^3.
\end{align*}
\]

**Conjecture** (Sun [Nanjing Univ. J. Math. Biquarterly 34(2017)]) All the 9 polynomials are universal over \( \mathbb{N} \).
Triangular numbers

Triangular numbers are those

\[ T_n = \sum_{r=0}^{n} r = \frac{n(n + 1)}{2} \quad (n \in \mathbb{N}). \]

Note that

\[ T_{-n-1} = \frac{(-n-1)(-n)}{2} = T_n \quad \text{for all} \quad n \in \mathbb{N}. \]

**Theorem** (conjectured by Fermat and proved by Gauss). Each \( n \in \mathbb{N} \) can be written as \( T_x + T_y + T_z \) with \( x, y, z \in \mathbb{N} \).

**Liouville’s Theorem** (Liouville, 1862). Let \( a, b, c \in \mathbb{Z}^+ \) and \( a \leq b \leq c \). Then any \( n \in \mathbb{N} \) can be written in the form \( aT_x + bT_y + cT_z \) if and only if \( (a, b, c) \) is among

\( (1, 1, 1), (1, 1, 2), (1, 1, 4), (1, 1, 5), (1, 2, 2), (1, 2, 3), (1, 2, 4). \)
Mixed Sums of Squares and Triangular Numbers

Euler’s Observation:

\[ 8n + 1 = (2x)^2 + (2y)^2 + (2z + 1)^2 \]

\[ \Rightarrow n = \frac{x^2 + y^2}{2} + T_z = \left(\frac{x + y}{2}\right)^2 + \left(\frac{x - y}{2}\right)^2 + T_z. \]

Lionnet’s Assertion (proved by Lebesgue & Réalis in 1872). Any \( n \in \mathbb{N} \) is the sum of two triangular numbers and a square.

B. W. Jone and G. Pall [Acta Math. 1939]. Every \( n \in \mathbb{N} \) is the sum of a square, an even square and a triangular number.

Theorem. (i) (Z. W. Sun, Acta Arith. 2007) Any \( n \in \mathbb{N} \) is the sum of an even square and two triangular numbers.

(ii) (Conjectured by Z. W. Sun and proved by B. K. Oh and Sun [JNT, 2009]) Any positive integer \( n \) can be written as the sum of a square, an odd square and a triangular number.
Mixed Sums of Squares and Triangular Numbers

In 2005 Z. W. Sun [Acta Arith. 2007] investigated what kind of mixed sums $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$ (with $a, b, c \in \mathbb{Z}^+$) are universal (i.e., all natural numbers can be so represented). This project was completed via three papers: Z. W. Sun [Acta Arith. 2007], S. Guo, H. Pan & Z. W. Sun [Integers, 2007], and B. K. Oh & Sun [JNT, 2009].

List of all universal $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$:

$T_x + T_y + z^2$, $T_x + T_y + 2z^2$, $T_x + T_y + 4z^2$, $T_x + 2T_y + z^2$,
$T_x + 2T_y + 2z^2$, $T_x + 2T_y + 3z^2$, $T_x + 2T_y + 4z^2$, $2T_x + T_y + z^2$,
$2T_x + 4T_y + z^2$, $2T_x + 5T_y + z^2$, $T_x + 3T_y + z^2$, $T_x + 4T_y + z^2$,
$T_x + 4T_y + 2z^2$, $T_x + 6T_y + z^2$, $T_x + 8T_y + z^2$, $T_x + y^2 + z^2$,
$T_x + y^2 + 2z^2$, $T_x + y^2 + 3z^2$, $T_x + y^2 + 4z^2$, $T_x + y^2 + 8z^2$,
$T_x + 2y^2 + 2z^2$, $T_x + 2y^2 + 4z^2$, $2T_x + y^2 + z^2$, $2T_x + y^2 + 2y^2$,
$4T_x + y^2 + 2z^2$. 
Polygonal Numbers

Polygonal numbers are nonnegative integers constructed geometrically from the regular polygons. For $m = 3, 4, 5, \ldots$, the $m$-gonal numbers are given by

$$p_m(n) = (m - 2)\binom{n}{2} + n \ (n = 0, 1, 2, \ldots).$$

Clearly

$$p_3(n) = T_n, \ p_4(n) = n^2, \ p_5(n) = \frac{3n^2 - n}{2}, \ p_6(n) = 2n^2 - n = T_{2n-1}.$$ 

The larger $m$ is, the more sparse $m$-gonal numbers are.

**Fermat’s Assertion** (1638). Any natural number $n$ can be written as the sum of $m$ $m$-gonal numbers.

**Confirmation:** $m = 4$ (Lagrange 1770), $m = 3$ (Gauss 1796), $m \geq 5$ (Cauchy 1813).
Mixed Sums of Three Polygonal Numbers

**Conjecture** [Z. W. Sun, 2009]. Let $3 \leq i \leq j \leq k$ and $k \geq 5$. Then each $n \in \mathbb{N}$ can be written as the sum of an $i$-gonal number, a $j$-gonal number and a $k$-gonal number, if and only if $(i, j, k)$ is among the following 31 triples:

$(3, 3, 5), (3, 3, 6), (3, 3, 7), (3, 3, 8), (3, 3, 10), (3, 3, 12), (3, 3, 17), (3, 4, 5), (3, 4, 6), (3, 4, 7), (3, 4, 8), (3, 4, 9), (3, 4, 10), (3, 4, 11), (3, 4, 12), (3, 4, 13), (3, 4, 15), (3, 4, 17), (3, 4, 18), (3, 4, 27), (3, 5, 5), (3, 5, 6), (3, 5, 7), (3, 5, 8), (3, 5, 9), (3, 5, 11), (3, 5, 13), (3, 7, 8), (3, 7, 10), (4, 4, 5), (4, 5, 6).

**Remark.** Sun proved the ‘only if’ part. The ‘if’ part is difficult!

Sun [Sci. China Math. 58(2015)] also showed that there are only 95 candidates for universal sums over $\mathbb{N}$ of the form $ap_i(x) + bp_j(y) + cp_k(z)$. 
On \( x(ax + b) + y(ay + c) + z(az + d) \) with \( x, y, z \in \mathbb{Z} \)

If any \( n \in \mathbb{N} \) can be written as \( f(x_1, \ldots, x_n) \) with \( x_1, \ldots, x_n \in \mathbb{Z} \), then we say that \( f \) is universal over \( \mathbb{Z} \).

As \( T_n = T_{-n-1} \) for all \( n \in \mathbb{N} \), we see that

\[
\{ T_n : n \in \mathbb{N} \} = \{ T_{2x} = x(2x + 1) : x \in \mathbb{Z} \}
\]

and hence \( x(2x + 1) + y(2y + 1) + z(2z + 1) \) is universal over \( \mathbb{Z} \).

**Theorem** (Z.-W. Sun [JNT 171(2017)]) Let \( a, b, c, d \in \mathbb{N} \) with \( a > 2 \) and \( b \leq c \leq d \leq a \). Then \( x(ax + b) + y(ay + c) + z(az + d) \) is universal over \( \mathbb{Z} \) if and only if the quadruple \((a, b, c, d)\) is among

\[
(3, 0, 1, 2), (3, 1, 1, 2), (3, 1, 2, 2), (3, 1, 2, 3), (4, 1, 2, 3).
\]
On $x(ax + 1) + y(by + 1) + z(cz + 1)$ with $x, y, z \in \mathbb{Z}$

**Theorem** (Z.-W. Sun [JNT 171(2017)]) (i) Let $a, b, c \in \mathbb{Z}^+$ with $a \leq b \leq c$. If $f_{a,b,c}(x, y, z) := x(ax + 1) + y(by + 1) + z(cz + 1)$ is universal over $\mathbb{Z}$, then $(a, b, c)$ is among the following 17 triples:

(1, 1, 2), (1, 2, 2), (1, 2, 3), (1, 2, 4), (1, 2, 5),
(2, 2, 2), (2, 2, 3), (2, 2, 4), (2, 2, 5), (2, 2, 6),
(2, 3, 3), (2, 3, 4), (2, 3, 5), (2, 3, 7), (2, 3, 8), (2, 3, 9), (2, 3, 10).

(ii) $f_{a,b,c}(x, y, z)$ is universal over $\mathbb{Z}$ if $(a, b, c)$ is among

(1, 2, 3), (1, 2, 4), (1, 2, 5), (2, 2, 4), (2, 2, 5), (2, 3, 3), (2, 3, 4).

**Conjecture** (Sun). $f_{a,b,c}(x, y, z)$ is universal over $\mathbb{Z}$ if $(a, b, c)$ is among (2, 2, 6), (2, 3, 5), (2, 3, 7), (2, 3, 8), (2, 3, 9), (2, 3, 10).

In 2017, Ju and Oh [arXiv:1701.02974] proved that

$f_{2,2,6}(x, y, z)$ and $f_{2,3,c}(x, y, z)$ ($c = 5, 7$)

are universal over $\mathbb{Z}$. The universality of $f_{2,3,c}(x, y, z)$ over $\mathbb{Z}$ for $c = 8, 9, 10$ remains open.
Write  $n = a^2 + b^2 + 3^c + 5^d$ 

**Conjecture** (Z.-W. Sun, April 28, 2018). Any integer $n > 1$ can be written as $a^2 + b^2 + 3^c + 5^d$ with $a, b, c, d \in \mathbb{N} = \{0, 1, 2, \ldots \}$.

*Remark.* I have verified this for $n$ up to $2 \times 10^{10}$, and I’d like to offer 3500 US dollars as the prize for the first proof of this conjecture. I also conjecture that $5^d$ in the conjecture can be replaced by $2^d$.

**Example.**

$2 = 0^2 + 0^2 + 3^0 + 5^0$, $5 = 0^2 + 1^2 + 3^1 + 5^0$, $25 = 1^2 + 4^2 + 3^1 + 5^1$.

**Conjecture** (Z.-W. Sun, April 26, 2018). Any integer $n > 1$ can be written as the sum of two squares and two central binomial coefficients.

*Remark.* I have verified this for $n$ up to $10^{10}$.

**Example.**

\[
2435 = 32^2 + 33^2 + \binom{2 \times 4}{4} + \binom{2 \times 5}{5}.
\]
Sums of two triangular numbers and two powers of 5

Recall that those $T_n = n(n+1)/2$ with $n \in \mathbb{N}$ are called triangular numbers. As claimed by Fermat and proved by Gauss, each $n \in \mathbb{N}$ is the sum of three triangular numbers.

**Conjecture** (Z.-W. Sun, April 23, 2018). Any integer $n > 1$ can be written as $T_a + T_b + 5^c + 5^d$ with $a, b, c, d \in \mathbb{N}$.

*Remark.* I have verified this for $n$ up to $10^{10}$.

**Conjecture** (Z.-W. Sun, April 23, 2018). Any integer $n > 1$ can be written as the sum of $p_5(a) + p_5(b) + 3^c + 3^d$ with $a, b, c, d \in \mathbb{N}$, where $p_5(k)$ denotes the pentagonal number $k(3k - 1)/2$.

*Remark.* I have verified this for $n$ up to $7 \times 10^6$.

**Example.**

$$285 = p_5(1) + p_5(11) + 3^3 + 3^4, \quad 13372 = p_5(17) + p_5(65) + 3^4 + 3^8.$$
Representations involving tetrahedral numbers

Those numbers

\[ t_n := \sum_{k=0}^{n} T_n = \sum_{k=0}^{n} \frac{k^2 + k}{2} = \frac{n(n+1)(n+2)}{6} = \binom{n+2}{3} \quad (n \in \mathbb{N}) \]

are called tetrahedral numbers.

**Pollock’s Conjecture** (Pollock, 1850). Each \( n \in \mathbb{N} \) is the sum of five tetrahedral numbers.

**Conjecture** (Z.-W. Sun, Feb. 2019). (i) Any \( n \in \mathbb{N} \) can be written as \( 2 \binom{w}{3} + \binom{x}{3} + \binom{y}{3} + \binom{z}{3} \) with \( w, x, y, z \in \mathbb{N} \).

(ii) Each \( n \in \mathbb{N} \) can be written as \( w^3 + \binom{x}{3} + \binom{y}{3} + \binom{z}{3} \) with \( w, x, y, z \in \mathbb{N} \).

**Remark.** I verified parts (i) and (ii) for \( n \) up to \( 5 \times 10^5 \) and \( 2 \times 10^6 \). Later, a student of mine continued the verification for \( n \) up to \( 10^8 \).

**Example.** \( 1284 = 10^3 + \binom{7}{3} + \binom{9}{3} + \binom{11}{3} \).
2-4-6-8 Conjecture

2-4-6-8 Conjecture (Z.-W. Sun, Feb. 18, 2019). Any positive integer \( n \) can be written as

\[
\binom{w}{2} + \binom{x}{4} + \binom{y}{6} + \binom{z}{8}
\]

with \( w, x, y, z \in \{2, 3, \ldots\} \).

Remark. I verified this for \( n \) up to \( 3 \times 10^7 \). Later, Max Alekseyev and Yaakov Baruch extended the verification for \( n \) up to \( 2 \times 10^{11} \) and \( 2 \times 10^{12} \) respectively.

Example.

\[
23343989 = \binom{365}{2} + \binom{76}{4} + \binom{40}{6} + \binom{34}{8}.
\]

The 2-4-6-8 conjecture is very strong since

\[
\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} = \frac{25}{24} \approx 1.04.
\]

Prize. 2468 US dollars for a proof, and 2468 RMB for a concrete counterexample.
Part III. On universal sums

\[ \frac{x(ax + b)}{2} + \frac{y(cy + d)}{2} + \frac{z(ez + f)}{2} \text{ over } \mathbb{N} \text{ or } \mathbb{Z} \]
Universal tuples \((a, b, c, d, e, f)\) over \(\mathbb{N}\)

Instead of \(c p_m(x)\), we may consider more general polynomials

\[
\psi_{a,b}(x) := \frac{x(ax + b)}{2} \quad \text{with} \quad a \in \mathbb{Z}^+, \ b \in \mathbb{Z}, \ b > -a \quad \text{and} \quad a \equiv b \pmod{2}.
\]

Clearly, \(\psi_{a,b}(\mathbb{N}) \subseteq \mathbb{N}\), \(\psi_{a,a}(x) = aT_x\) and \(\psi_{2a,0}(x) = ax^2\).

For positive integers \(a, c, e\) and integers \(b > -a, \ d > -c, \ f > -e\) with \(a + b, c + d, e + f\) all even, if

\[
\psi_{a,b}(x) + \psi_{c,d}(y) + \psi_{e,f}(z) = \frac{x(ax + b)}{2} + \frac{y(cy + d)}{2} + \frac{z(ez + f)}{2}
\]

is universal over \(\mathbb{N}\) then we simply call the ordered tuple \((a, b, c, d, e, f)\) universal over \(\mathbb{N}\).

Note that those universal tuples \((a, b, c, d, e, f)\) with \(b(a - b) = d(c - d) = f(e - f) = 0\) have been determined.
Universal tuples \((a, b, c, d, e, f)\) with \(a \mid b, c \mid d\) and \(e \mid f\)

**Sun** (arXiv:1502.03056): The following tuples are universal over \(\mathbb{N}\).

\[
\begin{align*}
(1, 3, 1, 1, 1, 1), & \quad (1, 3, 1, 3, 1, 1), & \quad (1, 5, 1, 1, 1, 1), & \quad (1, 5, 1, 3, 1, 1), \\
(1, 7, 1, 1, 1, 1), & \quad (1, 7, 1, 3, 1, 1), & \quad (1, 9, 1, 1, 1, 1), & \quad (2, 0, 1, 3, 1, 1), \\
(2, 0, 1, 3, 1, 3), & \quad (2, 0, 1, 5, 1, 1), & \quad (2, 0, 1, 5, 1, 3), & \quad (2, 0, 1, 7, 1, 1), \\
(2, 0, 1, 7, 1, 3), & \quad (2, 0, 1, 9, 1, 1), & \quad (2, 0, 1, 9, 1, 3), & \quad (2, 0, 1, 11, 1, 1), \\
(2, 0, 1, 11, 1, 3), & \quad (2, 0, 1, 13, 1, 1), & \quad (2, 0, 1, 13, 1, 3), & \quad (2, 0, 1, 15, 1, 1), \\
(2, 0, 2, 0, 1, 3), & \quad (2, 2, 1, 3, 1, 1), & \quad (2, 2, 1, 5, 1, 1), & \quad (2, 2, 1, 7, 1, 1), \\
(2, 2, 2, 0, 1, 3), & \quad (2, 2, 2, 0, 1, 5), & \quad (2, 2, 2, 0, 1, 7), & \quad (2, 2, 2, 0, 1, 9), \\
(2, 4, 1, 1, 1, 1), & \quad (2, 4, 2, 0, 1, 1), & \quad (2, 4, 2, 0, 1, 3), & \quad (2, 4, 2, 2, 1, 1), \\
(2, 4, 2, 2, 0, 0), & \quad (2, 6, 1, 1, 1, 1), & \quad (2, 6, 1, 3, 1, 1), & \quad (2, 6, 2, 0, 1, 1), \\
(2, 6, 2, 0, 1, 3), & \quad (2, 6, 2, 2, 1, 1), & \quad (2, 6, 2, 2, 2, 0), & \quad (2, 8, 1, 1, 1, 1), \\
(2, 8, 2, 0, 1, 1), & \quad (2, 8, 2, 0, 1, 3), & \quad (2, 8, 2, 2, 2, 0), & \quad (2, 10, 2, 0, 1, 1), \\
(2, 10, 2, 0, 1, 3), & \quad (2, 12, 2, 0, 1, 1), & \quad (2, 12, 2, 0, 1, 3), & \quad (2, 14, 2, 0, 1, 1), \\
(3, 3, 2, 0, 1, 3), & \quad (3, 9, 2, 0, 1, 1), & \quad (3, 9, 2, 0, 1, 3), & \quad (4, 0, 1, 3, 1, 1), \\
(4, 0, 1, 5, 1, 1), & \quad (4, 0, 1, 7, 1, 1), & \quad (4, 4, 1, 3, 1, 1), & \quad (8, 0, 1, 3, 1, 1).
\end{align*}
\]
Universal tuples \((a, b, c, d, e, f)\) with \(a \mid b,\ c \mid d\) and \(e \mid f\)

We have the following conjecture concerning other possible universal tuples \((a, b, c, d, e, f)\) over \(\mathbb{N}\) with \(a \mid b,\ c \mid d\) and \(e \mid f\).

**Conjecture** (Sun, arXiv:1502.03056) The following 10 tuples

\[(4, 0, 2, 0, 1, 3),\ (4, 0, 2, 0, 1, 5),\ (4, 0, 2, 6, 1, 1),\ (4, 0, 2, 6, 2, 0),\]
\[(4, 4, 2, 0, 1, 3),\ (4, 8, 2, 0, 1, 1),\ (4, 8, 2, 0, 1, 3),\ (4, 12, 2, 0, 1, 1),\]
\[(6, 0, 2, 0, 1, 3),\ (6, 6, 2, 0, 1, 3)\]

are universal over \(\mathbb{N}\).

**Theorem** (Sun, arXiv:1502.03056) Let \(a, c, e\) be positive integers and let \(b > -a,\ d > -c\) and \(f > -e\) be integers with \(a + b,\ c + d,\ e + f\) all even. Suppose that \(a \geq c \geq e\), and \(b \geq d\) if \(a = c\), and \(d \geq f\) if \(c = e\), and that the ordered tuple \((a, b, c, d, e, f)\) is universal over \(\mathbb{N}\). If \(a \mid b,\ c \mid d\) and \(e \mid f\), but \(b(a - b),\ d(c - d),\ f(e - f)\) are not all zero, then \((a, b, c, d, e, f)\) must be among the 56 tuples in the above result and the 10 tuples in the above conjecture.
Universal tuples \((a, b, c, d, e, f)\) with \(a \nmid b\) or \(c \nmid d\) or \(e \nmid f\)

**Theorem** (Sun, arXiv:1502.03056) Let \(a, c, e\) be positive integers and let \(b > -a, d > -c\) and \(f > -e\) be integers with \(a + b, c + d, e + f\) all even, and \(a \nmid b\) or \(c \nmid d\) or \(e \nmid f\). Suppose that \(a \geq c \geq e\), and \(b \geq d\) if \(a = c\), and \(d \geq f\) if \(c = e\), and that the ordered tuple \((a, b, c, d, e, f)\) is universal over \(\mathbb{N}\). Then \((a, b, c, d, e, f)\) must be among our list of 407 tuples.

In our list, those tuples \((a, b, c, d, e, f)\) with \(a \geq 16\) are as follows.

\[
(16, -14, 2, 0, 1, 1), (16, -10, 2, 0, 1, 1), (16, -10, 2, 0, 1, 3),
(16, -8, 3, -1, 1, 1), (16, -4, 2, 0, 1, 1), (17, -15, 3, 1, 1, 1),
(17, -15, 3, 1, 1, 3), (18, -10, 2, 0, 1, 1), (20, -16, 2, 2, 1, 1),
(20, -16, 2, 6, 1, 1), (20, -12, 3, -1, 1, 1), (20, -4, 2, 0, 1, 1),
(21, -19, 2, 2, 1, 1), (21, -9, 2, 0, 1, 1), (21, -5, 2, 0, 1, 1),
(25, -23, 2, 0, 1, 1).
\]
Conjecture (Sun, arXiv:1502.03056). All the 407 tuples $(a, b, c, d, e, f)$ are universal over $\mathbb{N}$.

I’m unable to prove none of the 407 tuples is universal over $\mathbb{N}$ but many of the tuples can be proved to be universal over $\mathbb{Z}$.
Universal tuples \((a, b, c, d, e, f)\) over \(\mathbb{Z}\)

For \(a \in \mathbb{Z}^+\), clearly \(\psi_{a,-b}(\mathbb{Z}) = \psi_{a,b}(\mathbb{Z})\) for all \(b = 0, \ldots, a\) with \(b \equiv a \pmod{2}\), and \(\psi_{a,a}(\mathbb{Z}) = \psi_{4a,2a}(\mathbb{Z})\) since \(\{T_x : x \in \mathbb{Z}\} = \{x(2x+1) : x \in \mathbb{Z}\}\). Thus we are led to find all the sums

\[
\psi_{a,b}(x) + \psi_{c,d}(y) + \psi_{e,f}(z) = \frac{x(ax + b)}{2} + \frac{y(cy + d)}{2} + \frac{z(ez + f)}{2}
\]

which are universal over \(\mathbb{Z}\), where \(a, c, e \in \mathbb{Z}^+, \ b, d, f \in \mathbb{N}, \ b < a\) and \(a \equiv b \pmod{2}\), \(d < c\) and \(c \equiv d \pmod{2}\), and \(f < e\) and \(e \equiv f \pmod{2}\). If the sum in (*) is universal over \(\mathbb{Z}\), then we also say that the ordered tuple \((a, b, c, d, e, f)\) is universal over \(\mathbb{Z}\).

**Theorem** (Sun, arXiv:1502.03056). Let \(a, b, c, d, e, f \in \mathbb{N}\) with \(a > b, c > d, e > f, a \equiv b \pmod{2}, c \equiv d \pmod{2}, e \equiv f \pmod{2}\), \(a \geq c \geq e \geq 2\), and \(b \geq d\) if \(a = c\), and \(d \geq f\) if \(c = e\). Suppose that the ordered tuple \((a, b, c, d, e, f)\) is universal over \(\mathbb{Z}\). Then \((a, b, c, d, e, f)\) must be among the 12082 tuples listed at http://oeis.org/A286944.
Universal tuples \((a, b, c, d, e, f)\) over \(\mathbb{Z}\)

**Conjecture** (Sun, arXiv:1502.03056). All the 12082 tuples listed at http://oeis.org/A286944 are universal over \(\mathbb{Z}\).

On the list of 12082 tuples, those tuples \((a, b, c, d, e, f)\) with \(a > 100\) are as follows:

\[ (a, a - 22, 6, 2, 5, 3) \ (a = 102, 105, 109, 110, 112, 116, 117, 121, 128), \]
\[ (a, a - 4, 7, 5, 3, 1) \ (a = 101, 103, 104, 105, 107, 111, 112, 114, 116, \]
\[ \quad 117, 118, 119, 121, 123, 124, 127, 129, 130, 131), \]
\[ (a, a - 4, 3, 1, 2, 0) \ (a = 101, 102, 104, 105, 107, 111, 112, 114, 116, \]
\[ \quad 120, 122, 123, 126, 128, 129, 130, 132, 133), \]
\[ (a, a - 38, 7, 1, 3, 1) \ (a = 102, 103, 104, 105, 106, 108, \]
\[ \quad 111, 115, 117, 118, 119), \]
\[ (a, a - 28, 7, 1, 3, 1) \ (a = 101, 103, 104, 105, 107, 108, 109, 110, 112, \]
\[ \quad 114, 116, 117, 118, 119, 120, 122, 125, 126, 127, 130, \]
\[ \quad 133, 134, 137, 139, 140, 142, 143, 145, 146, 151, 153, \]
\[ \quad 155, 158, 160, 161, 163, 164, 165, 170, 171). \]
Conjectural universal sums $ax^2 + by^2 + z(cz + d)/2$ over $\mathbb{Z}$

**Conjecture** (Sun, arXiv:1502.03056). The following polynomials are universal over $\mathbb{Z}$.

- $x^2 + y^2 + \frac{z(5z + 1)}{2}$
- $x^2 + y^2 + \frac{z(5z + 3)}{2}$
- $x^2 + y^2 + \frac{z(9z + 5)}{2}$
- $x^2 + y^2 + z(5z + 3)$
- $x^2 + 2y^2 + \frac{z(5z + 1)}{2}$
- $x^2 + 2y^2 + \frac{z(5z + 3)}{2}$
- $x^2 + 2y^2 + \frac{z(7z + 1)}{2}$
- $x^2 + 2y^2 + \frac{z(7z + 3)}{2}$
- $x^2 + 2y^2 + \frac{z(9z + 1)}{2}$
- $x^2 + 2y^2 + z(4z + 3)$
- $x^2 + 2y^2 + z(5z + 1)$
- $x^2 + 2y^2 + z(5z + 2)$
- $x^2 + 2y^2 + z(5z + 4)$
- $x^2 + 2y^2 + \frac{z(13z + 11)}{2}$
- $x^2 + 2y^2 + z(7z + 3)$
- $x^2 + 2y^2 + \frac{z(15z + 7)}{2}$
- $2x^2 + 2y^2 + \frac{z(5z + 3)}{2}$
- $x^2 + 3y^2 + z(3z + 1)$
- $2x^2 + 3y^2 + \frac{z(3z + 1)}{2}$
- $x^2 + 4y^2 + \frac{z(3z + 1)}{2}$
- $x^2 + 4y^2 + z(5z + 3)$
- $2x^2 + 4y^2 + \frac{z(3z + 1)}{2}$
- $3x^2 + 4y^2 + \frac{z(3z + 1)}{2}$
- $x^2 + 5y^2 + \frac{z(3z + 1)}{2}$

$40/55$
Conjectural universal sums $ax^2 + by^2 + z(cz + d)/2$ over $\mathbb{Z}$

\[
x^2 + 5y^2 + \frac{z(5z + 1)}{2}, \quad x^2 + 6y^2 + z(3z + 1), \quad x^2 + 6y^2 + \frac{z(3z + 1)}{2},
\]
\[
2x^2 + 6y^2 + \frac{z(3z + 1)}{2}, \quad x^2 + 7y^2 + \frac{z(3z + 1)}{2}, \quad x^2 + 7y^2 + z(3z + 1),
\]
\[
x^2 + 7y^2 + \frac{z(7z + 3)}{2}, \quad x^2 + 8y^2 + \frac{z(3z + 1)}{2}, \quad x^2 + 10y^2 + \frac{z(3z + 1)}{2},
\]
\[
x^2 + 11y^2 + \frac{z(3z + 1)}{2}, \quad x^2 + 15y^2 + \frac{z(3z + 1)}{2}.
\]

Remark. I have shown that the following sums are universal over $\mathbb{Z}$.

\[
x^2 + y^2 + \frac{z(3z + 1)}{2}, \quad x^2 + y^2 + z(3z + r) \quad (r = 1, 2),
\]
\[
x^2 + y^2 + z(4z + r) \quad (r = 1, 3), \quad x^2 + y^2 + 2z(3z + 2),
\]
\[
x^2 + 2y^2 + c\frac{z(3z + 1)}{2} \quad (c = 1, 2, 4), \quad x^2 + 3y^2 + \frac{z(3z + 1)}{2},
\]
\[
x^2 + 3y^2 + z(3z + 2), \quad 2x^2 + 3y^2 + z(3z + 2), \quad 2x^2 + 6y^2 + \frac{z(3z + 1)}{2}.
\]
Progress on the little 1-3-5 conjecture

**Little 1-3-5 Conjecture** (Sun, arXiv:1502.03056; Prize $135). The tuple $(5, 1, 3, 1, 1, 1)$ is universal over $\mathbb{N}$, i.e., any $n \in \mathbb{N}$ can be written as

$$\frac{x(x+1)}{2} + \frac{y(3y+1)}{2} + \frac{z(5z+1)}{2}$$

with $x, y, z \in \mathbb{N}$.

*Remark.* This is different from the following 1-3-5 conjecture whose integral version was proved Hai-Liang Wu and Sun (arXiv:1710.08763) for sufficiently large integers $n \not\equiv 0 \pmod{16}$.

**1-3-5 Conjecture** (Sun, JNT 175(2017); Prize $1350). Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.

Though we cannot solve the little 1-3-5 conjecture, we are able to show its integral version.

**Theorem** (Sun, arXiv:1502.03056). Any $n \in \mathbb{N}$ can be written as

$$x(x + 1)/2 + y(3y + 1)/2 + z(5z + 1)/2$$

with $x, y, z \in \mathbb{Z}$. 
Part IV. Some representations of positive rational numbers
Egyptian fractions

Unit fractions have the form $1/n$ with $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.

A sum of finitely many distinct unit fractions is called a *Egyptian fraction* as it was first studied by the ancient Egyptians around 1650 B.C.

As

$$\frac{1}{n} = \frac{1}{n+1} + \frac{1}{n(n+1)},$$

any positive rational number $r = m/n$ with $m, n \in \mathbb{Z}^+$ is an Egyptian fraction.

This easy fact was first proved by Fibonacci in 1202 and it implies that the series $\sum_{n=1}^{\infty} 1/n$ diverges.

For example,

$$\frac{2}{3} = \frac{1}{3} + \frac{1}{3} = \frac{1}{3} + \left(\frac{1}{3+1} + \frac{1}{3 \times 4}\right) = \frac{1}{3} + \frac{1}{4} + \frac{1}{12}.$$
On Egyptian fractions involving primes

**Euler:** $\sum_p 1/p$ diverges, where $p$ runs over all primes.

**Dirichlet’s Theorem:** If $a \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ are relatively prime, then there are infinitely many primes $p \equiv a \pmod{m}$.

For any $m \in \mathbb{Z}^+$ there are infinitely many primes $p$ with $p - 1$ (or $p + 1$) a multiple of $m$.

**Conjecture 1.3 with $500 Prize (S., Sept. 9-10, 2015).** For any positive rational number $r$, there is a finite set $P_r^-$ of primes such that

$$\sum_{p \in P_r^-} \frac{1}{p - 1} = r,$$

also there is a finite set $P_r^+$ of primes such that

$$\sum_{p \in P_r^+} \frac{1}{p + 1} = r.$$

**Verification:** Qing-Hu Hou at Tianjin Univ. has verified this for all rational numbers $r \in (0, 1]$ with denominators not exceeding 100.
Examples:

\[ \frac{1}{2-1} = \frac{1}{3-1} + \frac{1}{5-1} + \frac{1}{7-1} + \frac{1}{13-1}, \]
\[ \frac{1}{2+1} = \frac{1}{3+1} + \frac{1}{5+1} + \frac{1}{7+1} + \frac{1}{11+1} + \frac{1}{23+1}, \]
\[ \frac{1}{19} = \frac{1}{37-1} + \frac{1}{137-1} + \frac{1}{191-1} + \frac{1}{229-1} + \frac{1}{331-1} + \frac{1}{397-1} + \frac{1}{761-1} + \frac{1}{1021-1} + \frac{1}{331+1} + \frac{1}{359+1} + \frac{1}{701+1} + \frac{1}{911+1} (\text{Sun}). \]

In 2018, Prof. Guo-Niu Han found 2065 distinct primes \( p_1 < \cdots < p_{2065} \) with \( p_{2065} \approx 4.7 \times 10^{218} \) such that

\[ \frac{1}{p_1+1} + \cdots + \frac{1}{p_{2065}+1} = 2. \]
A similar conjecture involving practical numbers

\( n \in \mathbb{Z}^+ \) is *practical* if each \( m = 1, \ldots, n \) can be written as the sum of some distinct (positive) divisors of \( n \). 1 is the only odd practical number, and all powers of two are practical numbers.

For \( x > 0 \) let \( P(x) = |\{q \leq x : q \text{ is practical}\}|. \) Then

\[
P(x) \sim c \frac{x}{\log x}
\]

for some constant \( c > 0 \),

(conjectured by M. Margenstern in 1991 and proved by A. Weingartner in 2014).

**Conjecture 1.4** (Sun, Sept. 12, 2015). Any positive rational number \( r \) can be written as \( \sum_{j=1}^{k} 1/q_j \), where \( q_1, \ldots, q_k \) are distinct practical numbers.

*Example.*

\[
\frac{10}{11} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{48} + \frac{1}{132} + \frac{1}{176}
\]

with 2, 4, 8, 48, 132, 176 all practical numbers.
A similar conjecture involving practical numbers

\( n \in \mathbb{Z}^+ \) is \textit{practical} if each \( m = 1, \ldots, n \) can be written as the sum of some distinct (positive) divisors of \( n \). 1 is the only odd practical number, and all powers of two are practical numbers.

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\textit{Example}.

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In Nov. 2016, David Eppstein proved the above conjecture.
Motivations from Pell’s equations

If \( d \in \mathbb{Z}^+ \) is not a square, then the Pell equation
\[
x^2 - dy^2 = 1
\]
has infinitely many integral solutions.

Let \( r = a/b \) with \( a, b \in \mathbb{Z}^+ \) and \( \gcd(a, b) = 1 \). If \( r \) is not a square of rational numbers (i.e., \( ab \) is not a square), then there is a positive integer \( k \) such that \((ka)(kb) + 1 = abk^2 + 1\) is a square, i.e., we can write \( r = m/n \) with \( m, n \in \mathbb{Z}^+ \) such that \( mn + 1 \) is a square.

For any \( r = a/b \) with \( a, b \in \mathbb{Z}^+ \) and \( \gcd(a, b) = 1 \), by Schinzel’s Hypothesis there is a positive integer \( x \) such that \( p = ax + 1 \) and \( q = bx + 1 \) are both prime and hence
\[
\frac{ax}{bx} = \frac{p - 1}{q - 1}.
\]

Motivated by the above, the speaker posed some conjectures on writing each positive rational number as a special ratio.
**m/n with \( p_m + p_n \) a square**

**Conjecture** (i) (Sun, 2015-07-03) The set

\[
\left\{ \frac{m}{n} : \, m, n \in \mathbb{Z}^+ \text{ and } p_m + p_n \text{ is a square} \right\}
\]

contains any positive rational number \( r \).

(ii) (Sun, 2015-08-20) Any positive rational number \( r \neq 1 \) can be written as \( m/n \) with \( m, n \in \mathbb{Z}^+ \) such that \( p_{p_m} + p_{p_n} \) is a square.

We have verified part (i) of this conjecture for all those \( r = a/b \) with \( a, b \in \{1, \ldots, 200\} \), and part (ii) for all those \( r = a/b \neq 1 \) with \( a, b \in \{1, \ldots, 60\} \). For example, \( 2 = 20/10 \) with

\[
p_{20} + p_{10} = 71 + 29 = 10^2,
\]

and \( 2 = 92/46 \) with

\[
p_{p_{92}} + p_{p_{46}} = p_{479} + p_{199} = 3407 + 1217 = 68^2.
\]
\( m/n \) with \( \varphi(m) \) and \( \sigma(n) \) both squares

**Conjecture** (Sun, 2015-07-08). Any positive rational number \( r \) can be written as \( m/n \) with \( m, n \in \mathbb{Z}^+ \) such that both \( \varphi(m) \) and \( \sigma(n) \) are both squares.

We have verified this for all \( r = a/b \) with \( a, b \in \{1, \ldots, 150\} \).

**Examples:**

\[
\frac{4}{5} = \frac{136}{170} \quad \text{with} \quad \varphi(136) = 8^2 \quad \text{and} \quad \sigma(170) = 18^2,
\]

and

\[
\frac{5}{4} = \frac{1365}{1092} \quad \text{with} \quad \varphi(1365) = 24^2 \quad \text{and} \quad \sigma(1092) = 56^2.
\]
\[ m/n \text{ with } \pi(m)\pi(n) \text{ a positive square} \]

**Conjecture** (Sun, 2015-07-05). Any positive rational number \( r \) can be written as \( m/n \) with \( m, n \in \mathbb{Z}^+ \) such that \( \pi(m)\pi(n) \) is a positive square, where \( \pi(x) = |\{p \leq x : p \text{ is prime}\}|. \)

(ii) (Sun, 2015-07-06) Any positive rational number \( r \) can be written as \( m/n \) with \( m, n \in \mathbb{Z}^+ \) such that \( \pi(m) \) and \( \pi(\pi(n)) \) are positive squares.

**Remark.** We have verified part (i) of this conjecture for all those rational numbers \( r = a/b \) with \( a, b \in \{1, \ldots, 60\} \). For example,

\[
\frac{49}{58} = \frac{1076068567}{1273713814}
\]

with

\[
\pi(1076068567)\pi(1273713814) = 54511776 \cdot 63975626 = 59054424^2.
\]
Conjecture (Sun, 2015-07-02). Any positive rational number $r$ can be written as $m/n$ with $m, n \in \mathbb{Z}^+$ such that $p(m)^2 + p(n)^2$ is prime, where $p(\cdot)$ is the partition function.

Remark. This conjecture implies that there are infinitely many primes of the form $p(m)^2 + p(n)^2$ with $m, n \in \mathbb{Z}^+$. We have verified it for all those $r = a/b$ with $a, b \in \{1, \ldots, 100\}$.

Example. $4/5 = 124/155$ with

$$p(124)^2 + p(155)^2 = 2841940500^2 + 66493182097^2$$

$$= 4429419891190341567409$$

prime.
Conjecture related to twin primes

Conjecture. (i) (Sun, 2015-06-30) Any rational number \( r > 0 \) can be written as \( m/n \) with \( m, n \in U \), where

\[
U = \{ n \in \mathbb{Z}^+ : n \pm 1 \text{ and } p_n + 2 \text{ are all prime} \}.
\]

(ii) (Sun, 2015-06-28) Any rational number \( r > 0 \) can be written as \( m/n \) with \( m, n \in V \), where

\[
V = \{ n \in \mathbb{Z}^+ : p_n + 2 \text{ and } p_{p_n} + 2 \text{ are both prime} \}.
\]

(iii) Any rational number \( r > 0 \) can be written as \( m/n \), where \( m \) and \( n \) belong to the set

\[
W = \{ k \in \mathbb{Z}^+ : p_k + 2 \text{ is prime and } p_{p_k+2} - p_{p_k} = 6 \}.
\]

Remark. We have verified parts (i)-(ii) for those \( r = a/b \) with \( a, b \in \mathbb{Z}^+ \) not exceeding 100 and 400 respectively. For example, \( 4/5 = 11673840/14592300 \) with \( 11673840, 14592300 \in U \), and \( 2 = 1782/891 \) with \( 891, 1782 \in W \). Part (iii) implies that there are infinitely many twin prime pairs \( \{ q, q + 2 \} \) with \( p_{q+2} - p_q = 6 \).
References

For main sources of my conjectures mentioned here, you may look at:


Thank you!