A talk given at Dalian Univ. of Technology (Nov. 16, 2012) and Nankai University (Dec. 1, 2012)

The Riddle of Primes

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Dec. 1, 2012

Abstract

In November 2012 the speaker posed many new conjectures about prime numbers. In this talk we will mainly introduce the speaker's various refinements of some famous conjectures on primes including the twin prime conjecture, Goldbach's conjecture, Lemoine's conjecture, Legendre's conjecture and Oppermann's conjecture. We will also talk about some new-type conjectures made by the speaker.

Part I. Introduction to Goldbach's conjecture, Lemoine's conjecture and the twin prime conjecture

Goldbach's conjecture

Goldbach's Conjecture (1742): Every even number $n > 2$ can be written in the form $p + q$ with p and q both prime.

Goldbach's weak Conjecture. Each odd number $n > 6$ can be written as a sum of three primes.

Goldbach's conjecture implies Goldbach's weak conjecture, for, if $n > 6$ is odd then $n - 3 > 2$ is even and hence $n - 3 = p + q$ for some primes p and q.

Goldbach's conjecture implies that for any $n > 2$ there is a prime $p \in [n, 2n]$ since $2n \neq p + q$ if p and q are smaller than n.

Vinogradov's Theorem

Vinogradov's Theorem (1937). Each sufficiently large odd number can be written as a sum of three primes.

Ming-Chit Liu & Tian-Ze Wang (2002) Any odd integer $n > e^{3100} \approx 2 \times 10^{1346}$ is a sum of three primes.

Deshouillers, Effinger, Zinovier (1997). The Generalized Riemann Hypothesis implies Goldbach's weak conjecture.

Lemoine's conjecture

Lemoine's Conjecture (1894). Any odd integer $n > 6$ can be written as $p + 2q$, where p and q are primes.

Examples: $7 = 3 + 2 \times 2$, $11 = 5 + 3 \times 3$, $19 = 5 + 2 \times 7$.

Lemoine's conjecture is stronger than Goldbach's weak conjecture, and it is essentially as difficult as the Goldbach conjecture for even numbers.

Twin primes

If p and $p+2$ are both prime, then $\{p, p+2\}$ is called a *twin prime* pair.

Examples: $\{5, 7\}$, $\{11, 13\}$, $\{17, 19\}$, $\{29, 31\}$, $\{41, 43\}$.

Twin Prime Conjecture. There are infinitely many twin prime pairs.

Polignac's Conjecture (1849). For any positive even integer n, there are infinitely many cases of two consecutive primes with difference n.

If p and $p+4$ are both prime, then $\{p, p+4\}$ is called a *cousin prime* pair. If p and $p+6$ are both prime, then $\{p, p+6\}$ is called a *sexy prime* pair.

Hardy and Littlewood's Conjecture

Let n be any positive even number. Define

$$C_n = C_2 \prod_{p|n,\ p>2} \frac{p-1}{p-2},$$

where

$$C_2 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right) \approx 0.6601618.$$

Note that $C_2 = C_4 = C_6/2$.

Hardy-Littlewood Conjecture. We have

$$|\{\{p,q\} : \ p \text{ and } q < p \text{ are primes with } p+q=n\}| \sim 2C_n \frac{n}{\log^2 n}.$$

If $\pi_n(x)$ denotes the number of prime gaps of size n below x, then

$$\pi_n(x) \sim 2C_n \frac{x}{\log^2 x}.$$

Chen's Theorem

The linear equations $p + q = 2n$ and $p - q = 2$ with p and q prime variables are of the same level of difficulty.

Chen's Theorem (1973). All large even numbers can be written as $p + q$ where p is a prime and q has at most two prime factors.

A prime p is called a *Chen prime* if $p + 2$ is a product of at most two primes.

For example, 19 is a Chen prime since $21 = 3 \times 7$.

Chen Jing-run (1966): There are infinitely many Chen primes.

Part II. The speaker's refinements of known conjectures

Ordowski's conjecture and Ming-Zhi Zhang's problem

As conjectured by Fermat and proved by Euler, any prime $p \equiv 1 \pmod 4$ can be written uniquely as a sum of two squares.

Tomasz Ordowski's Conjecture (Oct. 3-4, 2012) For any prime $p \equiv 1 \pmod 4$ write $p = a_p^2 + b_p^2$ with $a_p > b_p > 0$. Then

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv 1 \pmod 4} a_p}{\sum_{p\leqslant N,\, p\equiv 1 \pmod 4} b_p} = 1 + \sqrt{2}$$

and

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv 1 \pmod 4} a_p^2}{\sum_{p\leqslant N,\, p\equiv 1 \pmod 4} b_p^2} = \frac{9}{2}.$$

Ming-Zhi Zhang's Problem (1990s) Whether any odd integer $n > 1$ can be written as $a + b$ with $a, b \in \mathbb{Z}^+$ and $a^2 + b^2$ prime ?
(Cf. http://oeis.org/A036468)

My conjecture involving $x^2 + xy + y^2$

It is known that any prime $p \equiv 1 \pmod 3$ can be written uniquely in the form $x^2 + xy + y^2$ with $x > y > 0$.

Conjecture (Sun, Nov. 3, 2012). (i) For any prime $p \equiv 1 \pmod 3$ write $p = x_p^2 + x_p y_p + y_p^2$ with $x_p > y_p > 0$. Then

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv 1 \pmod 3} x_p}{\sum_{p\leqslant N,\, p\equiv 1 \pmod 3} y_p} = 1 + \sqrt{3}$$

and

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv 1 \pmod 3} x_p^2}{\sum_{p\leqslant N,\, p\equiv 1 \pmod 3} y_p^2} = \frac{52}{9}.$$

(ii) Any integer $n > 1$ with $n \neq 8$ can be written as $x + y$, with $x, y \in \mathbb{Z}^+$ and $x^2 + xy + y^2$ prime.

My conjectures involving $x^2+3xy+y^2$

Conjecture (Sun, Nov. 4, 2012). (i) For any prime $p \equiv \pm 1 \pmod 5$ write $p = x_p^2 + 3x_py_p + y_p^2$ with $x_p > y_p > 0$. Then

$$\lim_{N\to\infty} \frac{\sum_{p\leqslant N,\, p\equiv \pm 1 \pmod 5} x_p}{\sum_{p\leqslant N,\, p\equiv \pm 1 \pmod 5} y_p} = 1+\sqrt{5}.$$

(ii) Any integer $n>1$ can be written as $x+y$, with $x,y\in\mathbb{Z}^+$ and $x^2+3xy+y^2$ prime.

Conjecture (Sun, Nov. 5, 2012). For any integer $n \geqslant 1188$, there are primes p and q with $p^2+3pq+q^2$ prime such that $p+(1+\{n\}_2)q = n$, where $\{n\}_m$ denotes the least nonnegative residue of n mod m.

A refinement of Lemoine's conjecture

Conjecture (Sun, Nov. 7-8, 2012) (i) Each odd integer $n > 15$ can be written as $p + 2q$ with $p, q, p^2 + 60q^2$ all prime.

(ii) Any odd integer $n > 1424$ can be written as $p + 2q$ with $p, q, p^4 + (2q)^4$ all prime.

(iii) In general, for any $a, b \in \mathbb{Z}^+$ all sufficiently large odd numbers can be written as $p + 2q$, where p and q are primes with $p^{2^a} + (2^b - 1)(2q)^{2^a}$ prime.

Remark. It is not yet proven that there are infinitely many primes in the form $x^4 + y^4$ with $x, y \in \mathbb{Z}$.

Another conjecture

Conjecture (Sun, Nov. 7-8, 2012) (i) For any $a = 0, 1, 3, 4, \ldots$, any sufficiently large integer n can be written as $p + (1 + \{n\}_2)q$, where p and q are primes with $(2^a + 2)(p + q)^2 + pq$ prime. In particular, any integer $n \geqslant 105$ can be written as $p + (1 + \{n\}_2)q$ with $p, q, (2^a + 2)(p + q)^2 + pq$ all prime.

(ii) For any integer $n \geqslant 9608$ there are primes p and q with $(p + q)^4 + (pq)^2$ prime such that $p + q = 2n$.

Remark. In 1998 Friedlander and Iwaniec [Ann. of Math.] proved that there are infinitely many primes in the form $x^4 + y^2$ with $x, y \in \mathbb{Z}^+$.

A conjecture involving $p^2 + q^2 - 1$

Conjecture (Sun, Nov. 11, 2012). (i) Any integer $n \geqslant 785$ can be written as $p + (1 + \{n\}_2)q$ with $p, q, p^2 + q^2 - 1$ all prime.

(ii) Let d be any integer with $d \not\equiv 1 \pmod 3$. Then large even numbers can be written as $p + q$ with $p, q, p^2 + q^2 + d$ all prime. If $5 \nmid d$, then all large odd numbers can be represented as $p + 2q$ with $p, q, p^2 + q^2 + d$ all prime.

Remark. I have verified part (i) for n up to 1.4×10^8.

Conjecture (Sun, Nov. 11, 2012). For any integer $n \geqslant 7830$ there are primes p and $q < p$ with $p^4 + q^4 - 1$ prime such that $p + (1 + \{n\}_2)q = n$.

A conjecture involving $pq \pm 6$

Conjecture (2012-11-11) (i) For any integer $n > 14491$ there are primes p and $q < p$ with $pq + 6$ prime such that $p + (1 + \{n\}_2)q = n$. For any integer $n > 22093$ there are primes p and $q < p$ with $pq - 6$ prime such that $p + (1 + \{n\}_2)q = n$.

(ii) Let d be any nonzero multiple of 6. Then any sufficiently large integer n can be written as $p + (1 + \{n\}_2)q$ with $p > q$ and $p, q, pq + d$ all prime.

Remark. I have verified part (i) for n up to 5×10^7.

Conjecture on sexy primes

Conjecture (Sun, 2012-11-11). (i) For any even $n > 8012$ and odd $n > 15727$, there are primes p and $q < p$ with $p - 6$ and $q + 6$ also prime such that $p + (1 + \{n\}_2)q = n$.

(ii) If d_1 and d_2 are integers divisible by 6, then any sufficiently large integer n can be written as $p = (1 + \{n\}_2)q$ with $p > q$ and $p, q, p - d_1, q + d_2$ all prime.

Remark. I have verified part (i) for n up to 10^8.

Conjecture on twin primes and cousin primes

Conjecture (Sun, Nov. 12-13, 2012). (i) Any integer $n > 62371$ with $n \not\equiv 2 \pmod 6$ can be written as $p + (1 + \{-n\}_6)q$, where p and $q < p$ are primes with $p - 2$ and $q + 2$ also prime. Also, any integer $n > 6896$ with $n \equiv 2 \pmod 6$ can be written as $p - q$ with $q < n/2$ and $p, q, p - 2, q + 2$ all prime.

(ii) Any integer $n > 66503$ with $n \not\equiv 4 \pmod 6$ can be written as $p + (1 + \{n\}_6)q$, where p and $q < p$ are primes with $p - 4$ and $q + 4$ also prime. Also, any integer $n > 7222$ with $n \equiv 4 \pmod 6$ can be written as $p - q$ with $q < n/2$ and $p, q, p - 4, q + 4$ all prime.

Remark. I have verified part (i) for n up to 10^8.

A conjecture implying the twin prime conjecture

Conjecture (Sun, Nov. 13, 2012). For any odd $n > 4676$ and even $n > 30986$, there are primes p and $q < p$ such that $\{3(p-q) \pm 1\}$ is a twin prime pair and $p + (1 + \{n\}_2)q = n$.

Remark. I have verified this for n up to 10^8.

Note that if $p + q = 2n$ and $p - q = 2d$ then the interval $[n - s, n + d]$ of length $2d$ contains p and q. As the interval $[m! + 2, m! + m]$ od length $m - 2$ contains no prime for each $m = 2, 3, \ldots$, the above conjecture implies that there are infinitely many twin primes.

A conjecture implying Oppermann's conjecture

Legendre's Conjecture For any integer $n > 1$ there is a prime between n^2 and $(n+1)^2$.

Oppermann's Conjecture. For any integer $n > 1$, both of the intervals $(n^2 - n, n^2)$ and $(n^2, n^2 + n)$ contain primes.

Conjecture (Sun, Nov. 10, 2012). (i) For any integer $n \geqslant 2733$, there is a prime $p < n$ such that $n^2 - n + p$ and $n^2 + n - p$ are both prime.

(ii) For any integer $n \geqslant 3513$, there is a prime $p \in (n, 2n)$ such that $n^2 - n + p$ and $n^2 + n - p$ are both prime.

Clearly this conjecture implies Oppermann's Conjecture.

Part III. Some conjectures of new types

A conjecture on $6n \pm p$

Conjecture (2012-11-10). (i) For any integer $n > 5$ there is a prime $p < n$ such that both $6n - p$ and $6n + p$ are prime.

(ii) For any given non-constant integer-valued polynomial $P(x)$ with positive leading coefficient, if $n \in \mathbb{Z}^+$ is large enough then there is a prime $p < n$ such that $6P(n) \pm p$ are both prime.

Remark. I have verified part (i) for n up to 10^8. If we take $P(x) = x(x+1)/2, x^2, x^3$ in part (ii), then it suffices to require that n is greater than $1933, 2426, 6772$ respectively.

A conjecture joint with Olivier Gerard

Conjecture (O. Gerard and Z. W. Sun, 2012-11-18). Any integer $n \geqslant 400$ different from 757, 1069, 1238 can be written as $p + (1 + \{n\}_2)q$, where p and q are odd primes with

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = 1,$$

where $(-)$ denotes the Legendre symbol.

Remark. We have verified the conjecture for n up to 10^8. It was announced via a message to `Number Theory Mailing List` on Nov. 19, 2012 with the title

Refining Goldbach's conjecture by using quadratic residues.

More on Legendre symbols

Conjecture (Sun, 2012-11-22). Let m be any integer. Then, for any sufficiently large integer n there are primes $p > q > 2$ such that

$$p + q = n \quad \text{and} \quad \left(\frac{p-m}{q}\right) = \left(\frac{q+m}{p}\right) = 1 \quad \text{if } 2 \mid n,$$

and

$$p + 2q = n \quad \text{and} \quad \left(\frac{p-2m}{q}\right) = \left(\frac{q+m}{p}\right) = 1 \quad \text{if } 2 \nmid n.$$

Remark. Note that

$$(p-m) + (q+m) = p + q \quad \text{and} \quad (p-2m) + 2(q+m) = p + 2q.$$

A conjecture involving prime differences

Conjecture (Sun, 2012-11-27) (i) Any integer $n > 3120$ can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ and $\{xy - 1, xy + 1\}$ a twin prime pair.
(ii) Let m be any positive integer. Then any sufficiently large integer n with $(m-1)n$ even can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $xy - m$ and $xy + m$ are both prime.

Remark. (a) We have verified part (i) for n up to 2×10^8.

(b) Amarnath Murthy (2005) conjectured that any integer $n > 3$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $xy - 1$ prime. (Cf. http://oeis.org/A109909)

(c) Clearly part (ii) implies that for any positive even integer d there are infinitely many primes p and q with difference d.

A general conjecture

Conjecture (Sun, 2012-11-27) (i) Any $n \in \mathbb{Z}^+$ not among $1, 8, 10, 18, 20, 41, 46, 58, 78, 116, 440$ can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $n^2 - xy = x^2 + xy + y^2$ and $n^2 + xy = x^2 + 3xy + y^2$ are both prime. For any integer $n \geqslant 4687$ there are $x, y \in \mathbb{Z}^+$ such that $n^4 + xy$ and $n^4 - xy$ are both prime.
(ii) In general, for any $a = 0, 1, 2, 4, 5, 6, \ldots$ and positive odd integer m, each sufficiently large integer n can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $|mn^a - xy|$ and $mn^a + xy$ are both prime.

More conjectures involving xy

Conjecture (Sun, 2012-11-28) (i) Any $n \in \mathbb{Z}^+$ different from 1, 6, 16, 24 can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $(xy)^2 + 1$ prime.

(ii) In general, for each $a = 0, 1, 2, \ldots$, any sufficiently large integer n can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $(xy)^{2^a} + 1$ prime.

Remark. Part (i) implies a well known conjecture which states that there are infinitely many primes of the form $x^2 + 1$.

Conjecture (Sun, 2012-11-29) (i) Any integer $n > 1$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $(xy)^2 + xy + 1$ prime.

(ii) In general, for each prime p, any sufficiently large integer n can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $((xy)^p - 1)/(xy - 1)$ prime.

A conjecture requiring $2xy + 1$ prime

Conjecture (Sun, 2012-11-29) (i) Every $n = 2, 3, 4, \ldots$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $2xy + 1$ prime. Moreover, any integer $n > 357$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $2xy \pm 1$ twin primes.

(ii) In general, for each positive odd integer m, any sufficiently large integer n can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $2xy - m$ and $2xy + m$ are both prime.

Remark. I have verified part (i) for n up to 10^8.

A conjecture of new type

Conjecture (Sun, 2012-11-30) (i) Any integer $n > 7$ can be written as $p + q$ ($q \in \mathbb{Z}^+$) with p and $2pq + 1$ prime.

(ii) For each odd integer $n \not\equiv 5 \pmod 6$, sufficiently large integer n can be written as $p + q$ ($q \in \mathbb{Z}^+$) with p and $2pq + m$ both prime.

Remark. I have verified part (i) for n up to 4×10^8. Concerning part (ii), for

$$m = 3, -3, 7, 9, -9, -11, 13, 15$$

it suffices to require that n is greater than

$$1, 29, 16, 224, 29, 5, 10, 52$$

respectively.

More conjectures of the new type

Conjecture (Sun, 2012-12-01) (1) Any integer $n > 2572$ with $n \neq 6892$ can be written as $p + q$ with p and $pq \pm n$ all prime.

(2) Any integer $n > 2162$ can be written as $p + q$ with $n^2 - pq = p^2 + pq + q^2$ and $n^2 + pq = p^2 + 3pq + q^2$ both prime.

(3) Any integer $n > 2$ not among 13, 16, 46, 95, 157 can be written as $p + q$ with $(pq)^2 + pq + 1$ prime.

(4) Any integer $n > 2$ different from 5, 10, 34, 68 can be written as $p + q$ with p and $(2pq)^4 + 1$ prime.

I have many other conjectures similar to the above ones.

For sources of my conjectures, you may look at my preprint

Conjectures involving primes and quadratic forms

available from

`http://arxiv.org/abs/1211.1588`

You are welcome to solve my conjectures!

Thank you!