Dedicated to Prof. Caiheng Li for his 60th birthday

Seeking for π-series in August 2020

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Abstract

In this talk we tell how the speaker found new series for $1/\pi$ via congruences, and exhibit 23 series for $1/\pi$ found by the speaker in August 2020. We will also give a sketch of our proof of six conjectural series for $1/\pi$ via symbolic computation.

Part I. Ramanujan Series for $\frac{1}{\pi}$ and my Philosophy on Series for $\frac{1}{\pi}$

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

In his famous letter to Hardy, S. Ramanujan mentioned the above series as one of his discoveries.

In 1914 S. Ramanujan published his first paper in England *Modular equations and approximations to π*,
Quart. J. Math. (Oxford), 45(1914), 350–372.

Towards the end of this paper, he wrote "*I shall conclude this paper by giving a few series for $1/\pi$*". Then he listed 17 series for $1/\pi$ and briefly mentioned that the first three series are related to the classical theory of elliptic functions.

S. Ramanujan attributed his mathematical discoveries to inspirations from the God. He once said: **"An equation for me has no meaning, unless it represents a thought of God."**

General forms of Ramanujan-type series

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are 36 rational Ramanujan-type series for $1/\pi$.

Two Examples recorded by Ramanujan:

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}=\frac{4}{\pi}$$

(proved by S. Chowla in 1928),

$$\sum_{k=0}^{\infty}\frac{26390k+1103}{396^{4k}}\binom{4k}{k,k,k,k}=\frac{99^2}{2\pi\sqrt{2}}$$

(proved by J. Borwein and P. Borwein in 1987).

What is needed for proving $\sum_{n=0}^{\infty}(6n+1)\binom{2n}{n}^3/256^n = 4/\pi$

The proofs of Ramanujan series involve lots of things such as modulo forms, elliptic integrals, theta functions, hypergeometric series, modular equations and symbolic computation.

$$P(q) := 1 - 24\sum_{j=1}^{\infty}\frac{jq^j}{1-q^j} \quad \text{(Eisenstein series)},$$

$$\varphi(q) := \sum_{j=-\infty}^{\infty} q^{j^2} \quad \text{(theta function)},$$

$$X = X(q) = q\prod_{j=1}^{\infty}\frac{(1-q^j)^{24}(1-q^{4j})^{24}}{(1-q^{2j})^{48}}.$$

$$\varphi(q)^4 = \sum_{n=0}^{\infty}\binom{2n}{n}X^n,\ P(q^2) = \sqrt{1-64X}\sum_{n=0}^{\infty}(3n+1)\binom{2n}{n}^3 X^n.$$

$$X(e^{-\pi\sqrt{3}}) = \frac{1}{256} \text{ and } P(e^{-2\pi\sqrt{3}}) = \frac{\sqrt{3}}{\pi} + \frac{\sqrt{3}}{4}\varphi(e^{-\pi\sqrt{3}})^4.$$

van Hamme's conjectures

For the two Ramanujan series

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi} \text{ and } \sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi},$$

in 1997 van Hamme conjectured their following p-adic analogues:

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k} \equiv p\left(\frac{-2}{p}\right) \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv 5p\left(\frac{-1}{p}\right) \pmod{p^4},$$

where p is an odd prime.

All the p-adic analogue conjectures of van Hamme were proved before 2017. Following van Hamme's idea, Zudilin [JNT, 2009] proposed more p-adic analogues for Ramanujan-type series.

My Philosophy about Series for $1/\pi$

Part I of the Philosophy (2010). Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k} \equiv 0 \pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1, 2, 2^2, 2^3, \ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d > 1$ and for any prime $p > 3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k} \equiv \begin{cases} (\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p = x^2+dy^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p}) = -1, \end{cases}$$

or $d = 1$, $\gcd(15, m) > 1$, and for any prime $p \equiv 3 \pmod{4}$ with $p \nmid 3m$ we have $\sum_{k=0}^{p-1} a_k/m^k \equiv 0 \pmod{p^2}$.

Philosophy about Series for $1/\pi$ (continued)

Part II of the Philosophy (2011). Let $b, c, m, a_0, a_1, \ldots$ be integers with bm nonzero and the series $\sum_{k=0}^{\infty}(bk+c)a_k/m^k$ convergent. Suppose that there are $d \in \mathbb{Z}^+$, $d' \in \mathbb{Z}$, and rational numbers c_0 and c_1 such that

$$\sum_{k=0}^{p-1}(bk+c)\frac{a_k}{m^k} \equiv p\left(c_0\left(\frac{-d}{p}\right)+c_1\left(\frac{d'}{p}\right)\right) \pmod{p^2}$$

for all sufficiently large primes p. If $d' \geqslant 0$, then

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{C}{\pi}$$

for some C with C^2 rational (and with $C/\sqrt{d}$ rational if $c_0 \neq 0$). If $d' = -d_1 < 0$, then there are rational numbers λ_0 and λ_1 such that

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k} = \frac{\lambda_0\sqrt{d}+\lambda_1\sqrt{d_1}}{\pi}.$$

Remark. Almost all identities of the stated form are *regular*.

An Example Illustrating the Philosophy

Ramanujan Series:

$$\sum_{k=0}^{\infty}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}=\frac{16}{\sqrt{3}\,\pi}.$$

Conjecture (Sun [Sci. China Math. 54(2011)]). For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{12}3)^k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } 12\mid p-1,\ p=x^2+y^2,\ 3\nmid x \text{ and } 3\mid y,\\ -(\frac{xy}{3})4xy \pmod{p^2} & \text{if } 12\mid p-5 \text{ and } p=x^2+y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4,\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3p\left(\frac{p}{3}\right)+\frac{5}{24}p^3B_{p-2}\left(\frac{1}{3}\right) \pmod{p^4}.$$

Another Example Illustrating the Philosophy

I would like to offer \$90 for the first proof of the identity in the following conjecture and \$105 for the first proof of congruences in the conjecture.

Conjecture (Z. W. Sun, 2011). We have

$$\sum_{n=0}^{\infty}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}=\frac{90}{\pi}.$$

For any prime $p>5$, we have

$$\sum_{n=0}^{p-1}\frac{357n+103}{2160^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$
$$\equiv p\left(\frac{-1}{p}\right)\left(54+49\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

Another Example Illustrating the Philosophy (continued)

And

$$\sum_{n=0}^{p-1}\frac{\binom{2n}{n}}{2160^n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-324)^{n-k}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2x^2-2p \pmod{p^2} & \text{if } 2p=x^2+105y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 2p-6x^2 \pmod{p^2} & \text{if } 2p=3x^2+35y^2\ (x,y\in\mathbb{Z}),\\ 20x^2-2p \pmod{p^2} & \text{if } p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 10x^2-2p \pmod{p^2} & \text{if } 2p=5x^2+21y^2\ (x,y\in\mathbb{Z}),\\ 28x^2-2p \pmod{p^2} & \text{if } p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 14x^2-2p \pmod{p^2} & \text{if } 2p=7x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-105}{p})=-1.\end{cases}$$

Remark. The quadratic field $\mathbb{Q}(\sqrt{-105})$ has class number 8.

Part II. Series for $\frac{1}{\pi}$ involving $T_n(b, c)$ (before 2020)

Generalized central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](x^2+x+1)^n \text{ (the coefficient of } x^n \text{ in } (x^2+x+1)^n)$$
$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Note that central binomial coefficients are those

$$\binom{2n}{n} = [x^n](x+1)^{2n} = [x^n](x^2+2x+1)^n \quad (n \in \mathbb{N}).$$

For real numbers b and c, we define the generalized central trinomial coefficient

$$T_n(b,c) := [x^n](x^2+bx+c)^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k.$$

Series for $1/\pi$ involving $T_k(b,c)$ (2011)

I view $T_n(b,c)$ as natural extensions of the central binomial coefficients.

In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a,b,c,d,m integers and $mbcd(b^2-4c)$ nonzero. In August I added 8 new series for $1/\pi$ of type III.

Type I. $\sum_{k=0}^\infty (a+dk)\binom{2k}{k}^2 T_k(b,c)/m^k$.

Type II. $\sum_{k=0}^\infty (a+dk)\binom{2k}{k}\binom{3k}{k} T_k(b,c)/m^k$.

Type III. $\sum_{k=0}^\infty (a+dk)\binom{4k}{2k}\binom{2k}{k} T_k(b,c)/m^k$.

Type IV. $\sum_{k=0}^\infty (a+dk)\binom{2k}{k}^2 T_{2k}(b,c)/m^k$.

Type V. $\sum_{k=0}^\infty (a+dk)\binom{2k}{k}\binom{3k}{k} T_{3k}(b,c)/m^k$.

In October 2011, I found 10 conjectural series for $1/\pi$ of two new types:

Type VI. $\sum_{k=0}^\infty (a+dk) T_k^3(b,c)/m^k$.

Type VII. $\sum_{k=0}^\infty (a+dk)\binom{2k}{k} T_k^2(b,c)/m^k$.

This stimulated several papers by H.-H. Chan, J. Wan, W. Zudilin.

My conjectural series of type VI

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

I would like to offer \$300 as the prize for the person who can provide first rigorous proofs of all the above three identities. The last one was inspired by my following conjecture for primes $p>3$.

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)$$
$$\equiv\frac{p}{19}\left(17563\left(\frac{-14}{p}\right)+4230\left(\frac{-2}{p}\right)\right)\pmod{p^2}.$$

My 2019 conjectural series of type VIII

In November 2019, I introduced a new type series for $1/\pi$.

`Type VIII.` $\sum_{k=0}^\infty (a+dk)T_k(b_1,c_1)T_k(b_2,c_2)^2/m^k = C/\pi$.

Conjecture (Sun, Nov. 2019). We have

$$\sum_{k=0}^\infty \frac{40k+13}{(-50)^k} T_k(4,1)T_k(1,-1)^2 = \frac{55\sqrt{15}}{9\pi}, \qquad \text{(VIII1)}$$

$$\sum_{k=0}^\infty \frac{1435k+113}{3240^k} T_k(7,1)T_k(10,10)^2 = \frac{1452\sqrt{5}}{\pi}, \qquad \text{(VIII2)}$$

$$\sum_{k=0}^\infty \frac{840k+197}{(-2430)^k} T_k(8,1)T_k(5,-5)^2 = \frac{189\sqrt{15}}{2\pi}, \qquad \text{(VIII3)}$$

$$\sum_{k=0}^\infty \frac{39480k+7321}{(-29700)^k} T_k(14,1)T_k(11,-11)^2 = \frac{6795\sqrt{5}}{\pi}. \qquad \text{(VIII4)}$$

Part III. Seeking for π-series in 2020

My 2020 conjectural series of type IX

In August 2019, I wanted to find a new type series for $1/\pi$.

`Type IX.` $\sum_{k=0}^{\infty}(a+dk)\binom{2k}{k}T_k(b_1,c_1)T_k(b_2,c_2)/m^k = C/\pi$.

For $|m| \leqslant 650$ I could not find such a series. To make the search more efficient, later I required that c_1, c_2 are squares if they are positive.

Conjecture (Sun, August 7, 2020). We have

$$\sum_{k=0}^{\infty}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)=\frac{5390}{\pi} \qquad \text{(IX1)}$$

and

$$\sum_{k=0}^{\infty}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)=\frac{98}{3\pi}(10+7\sqrt{5}). \qquad \text{(IX2)}$$

Congruences related to (IX1)

Conjecture (Sun, August 2020). (i) For any integer $n > 1$,

$$n\binom{2n}{n} \ \Big| \ \sum_{k=0}^{n-1}(4290k+367)3136^{n-1-k}\binom{2k}{k}T_k(14,1)T_k(17,16).$$

(ii) Let p be an odd prime with $p \neq 7$. Then

$$\sum_{k=0}^{p-1}\frac{4290k+367}{3136^k}\binom{2k}{k}T_k(14,1)T_k(17,16)$$
$$\equiv \frac{p}{2}\left(1430\left(\frac{-1}{p}\right)+39\left(\frac{3}{p}\right)-735\right) \pmod{p^2}.$$

(iii) For any prime $p > 7$, we have

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3136^k}T_k(14,1)T_k(17,16)$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{-15}{p})=-1.\end{cases}$$

Congruences related to (IX2)

Conjecture (Sun, August 7, 2020). (i) For any prime $p > 7$,

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3136^k}T_k(2,81)T_k(14,81)$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+15y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 20x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=5x^2+6y^2,\\ 2p-12x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-30}{p})=-1,\end{cases}$$

where x and y are integers.

(ii) Let p be an odd prime with $p \neq 7$. Then

$$\sum_{k=0}^{p-1}\frac{540k+137}{3136^k}\binom{2k}{k}T_k(2,81)T_k(14,81)$$

$$\equiv\frac{p}{3}\left(270\left(\frac{-1}{p}\right)-104\left(\frac{-2}{p}\right)+245\left(\frac{-5}{p}\right)\right)\pmod{p^2}.$$

Discovery on August 19, 2020

Motivated by congruences, in 2011 I conjectured that

$$\sum_{k=0}^{\infty} \frac{15k+2}{(-3456)^k} \binom{2k}{k} \binom{3k}{k} T_{3k}(2,-1) = \frac{c}{\pi}$$

for certain algebraic number c. In 2015, on my request H. H. Chan determined that

$$c = \frac{1}{2}\sqrt{72 + 54\sqrt[3]{4} + 12\sqrt[3]{2}}.$$

After reading this story in 2020, Deyi Chen pointed out that one can find the form of an algebraic number by Maple if one knows enough digits of that number.

On Jan. 18, 2012 I guessed that there is an algebraic number C with

$$\sum_{n=0}^{\infty} \frac{6n+1}{(-1728)^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-324)^{n-k} = \frac{C}{\pi}.$$

On August 19, 2020, I conjectured that $C = \frac{24}{25}\sqrt{375 + 120\sqrt{10}}$.

Discovery on August 20, 2020

On June 16, 2011, I observed that

$$\sum_{n=0}^{p-1}\frac{4n+1}{(-160)^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-20)^{n-k}\equiv 0 \pmod{p}$$

for any prime $p>5$, but I was unable to determine the sum modulo p^2.

Conjecture (Sun, August 20, 2020).

$$\sum_{n=0}^{\infty}\frac{4n+1}{(-160)^n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}\binom{n+2k}{2k}\binom{2k}{k}(-20)^{n-k}$$
$$=\frac{\sqrt{30}}{5\pi}\cdot\frac{5+\sqrt[3]{145+30\sqrt{6}}}{\sqrt[6]{145+30\sqrt{6}}}.$$

More such series

Conjecture (Sun, August 21, 2020).

$$\sum_{n=0}^{\infty} \frac{1290n+289}{27648^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-2160)^{n-k} = \frac{96\sqrt{15}}{\pi}.$$

Conjecture (Sun, August 23, 2020).

$$\sum_{n=0}^{\infty} \frac{804n+49}{276480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} 12096^{n-k} = \frac{120\sqrt{15}}{\pi}.$$

Conjecture (Sun, August 27, 2020).

$$\sum_{n=0}^{\infty} \frac{24n+5}{(-20)^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} \left(-\frac{2}{3}\right)^{3k} = \frac{3}{2\pi}(5\sqrt{6}+4\sqrt{15}).$$

Franel numbers

Franel numbers: $f_n = \sum_{k=0}^{n} \binom{n}{k}^3$ $(n \in \mathbb{N})$.

An identity of MacMahon implies that

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+2k}{2k} \binom{2k}{k} (-4)^{n-k} = f_n.$$

Franel numbers of order 4: $f_n^{(4)} = \sum_{k=0}^{n} \binom{n}{k}^4$ $(n \in \mathbb{N})$.

In 2005 Y. Yang used mofular forms of level 10 to discover the following curious identity relating Franel numbers of order four to Ramanujan-type series for $1/\pi$:

$$\sum_{k=0}^{\infty} \frac{4k+1}{36^k} f_k^{(4)} = \frac{18}{\sqrt{15}\,\pi}.$$

More this kind of identities were deduced by S. Cooper in 2012 via modular forms.

$W_n(x)$

For $n \in \mathbb{N}$ the polynomial

$$W_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \binom{2k}{k} \binom{2(n-k)}{n-k} x^k$$
$$= \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}^2 \binom{2(n-k)}{n-k} x^k$$

at $x = -1$ coincides with $(-1)^n f_n^{(4)}$.

In 2011 the author proposed ten identities of the form

$$\sum_{k=0}^{\infty} \frac{ak+b}{m^k} W_k \left(\frac{1}{m} \right) = \frac{C}{\pi},$$

where a, b, m are integers with $am \neq 0$, and C^2 is rational. They were later confirmed by Cooper et al. in 2017.

Six series found in August 2020 have been proved

Theorem (Sun, arXiv:2009.04379).

$$\sum_{k=0}^{\infty}\frac{45k+8}{40^k}W_k\left(\frac{9}{10}\right)=\frac{215\sqrt{15}}{12\pi},$$

$$\sum_{k=0}^{\infty}\frac{1360k+389}{(-60)^k}W_k\left(\frac{16}{15}\right)=\frac{205\sqrt{15}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{735k+124}{200^k}W_k\left(\frac{49}{50}\right)=\frac{10125\sqrt{7}}{56\pi},$$

$$\sum_{k=0}^{\infty}\frac{376380k+69727}{(-320)^k}W_k\left(\frac{81}{80}\right)=\frac{260480\sqrt{5}}{3\pi},$$

$$\sum_{k=0}^{\infty}\frac{348840k+47461}{1300^k}W_k\left(\frac{324}{325}\right)=\frac{1314625\sqrt{2}}{12\pi},$$

$$\sum_{k=0}^{\infty}\frac{41673840k+4777111}{5780^k}W_k\left(\frac{1444}{1445}\right)=\frac{147758475}{\sqrt{95}\,\pi}.$$

Two lemmas

Lemma 1. For $|z| \leqslant 1/30$, we have

$$\sum_{k=0}^{\infty} \frac{z^k}{(1+4z)^{k+1}} W_k\left(\frac{1}{1+4z}\right) = \sum_{n=0}^{\infty} f_n^{(4)} z^n,$$

$$\sum_{k=0}^{\infty} \frac{kz^k}{(1+4z)^{k+1}} W_k\left(\frac{1}{1+4z}\right) = \sum_{n=0}^{\infty} n(f_n^{(4)} + 4s_n) z^n,$$

where

$$s_n := \sum_{0\leqslant j<n} (-1)^{n-1-j} \binom{n-1}{j}\binom{n+j}{j}\binom{2j}{j}\binom{2(n-1-j)}{n-1-j}.$$

Lemma 2. For any $n \in \mathbb{N}$ we have

$$\begin{aligned} &5n(4n+1)((n+2)s_{n+2} - 16ns_n) \\ =&(30n^3+54n^2+7n-2)f_{n+1}^{(4)} + 2(60n^3+58n^2+17n+2)f_n^{(4)}. \end{aligned}$$

An auxiliary theorem

Auxiliary Theorem (Sun, arXiv:2009.04379). Let a, b and x be complex numbers with $|x-1| \geqslant 7.5$. Then

$$\frac{10}{x}(x-1)^2(x-2)\sum_{n=0}^{\infty}\frac{an+b}{(4x)^n}W_n\left(1-\frac{1}{x}\right)$$
$$=\sum_{k=0}^{\infty}(2ax(5x-7)k+a(10x-13)+10b(x-1)(x-2))\frac{f_k^{(4)}}{(4x-4)^k}.$$

Combining this with known identities of the form

$$\sum_{k=0}^{\infty}\frac{ak+b}{m^k}f_k^{(4)}=\frac{c}{\pi}$$

we obtain the six identities in our theorem.

Open series involving $W_n(x)$

Conjecture (Sun, August 2020). We have

$$\sum_{k=0}^{\infty}\frac{4k+1}{6^k}W_k\left(-\frac{1}{8}\right)=\frac{\sqrt{72+42\sqrt{3}}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{392k+65}{(-108)^k}W_k\left(-\frac{49}{12}\right)=\frac{387\sqrt{3}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{168k+23}{112^k}W_k\left(\frac{63}{16}\right)=\frac{1652\sqrt{3}}{9\pi},$$

$$\sum_{k=0}^{\infty}\frac{1512k+257}{(-320)^k}W_k\left(-\frac{405}{64}\right)=\frac{1184\sqrt{35}}{5\pi},$$

$$\sum_{k=0}^{\infty}\frac{56k+9}{324^k}W_k\left(\frac{25}{4}\right)=\frac{1134\sqrt{35}}{125\pi},$$

$$\sum_{k=0}^{\infty}\frac{13000k-1811}{(-1296)^k}W_k\left(-\frac{625}{9}\right)=\frac{49356\sqrt{39}}{5\pi},$$

Open series for $1/\pi$

and

$$\sum_{k=0}^{\infty} \frac{9360k - 1343}{1300^k} W_k\left(\frac{900}{13}\right) = \frac{21515\sqrt{39}}{3\pi},$$

$$\sum_{k=0}^{\infty} \frac{56355k + 2443}{(-5776)^k} W_k\left(-\frac{83521}{361}\right) = \frac{4669535\sqrt{2}}{68\pi},$$

$$\sum_{k=0}^{\infty} \frac{5928k + 253}{5780^k} W_k\left(\frac{1156}{5}\right) = \frac{28951\sqrt{2}}{4\pi}.$$

Conjecture (Sun, August 30).

$$\sum_{n=0}^{\infty} \frac{182n + 31}{576^n} \binom{2n}{n} \sum_{k=0}^{n} \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} \left(-\frac{25}{16}\right)^k = \frac{189}{2\pi}.$$

Main References:

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3. Z.-W. Sun, *New series for powers of* π *and related congruences*, Electron. Res. Arch. **28** (2020), 1273–1342.

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Thank you!