A talk given at Dalian Maritime Univ. (Nov. 17, 2012)

Conjectures involving arithmetical sequences

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Nov. 17, 2012

Abstract

A sequence $(a_n)_{n\geqslant 0}$ of natural numbers is said to be *log-concave* (resp. *log-convex*) if $a_{n+1}^2 \geqslant a_n a_{n+2}$ (resp. $a_{n+1}^2 \leqslant a_n a_{n+2}$) for all $n = 0, 1, 2, \ldots$. The log-concavity or log-convexity of combinatorial sequences has been studied extensively by many authors.

During August-September 2012, the speaker formulated many conjectures on monotonicity of $(\sqrt[n]{a_n})_{n\geqslant 1}$ or $(\sqrt[n+1]{a_{n+1}}/\sqrt[n]{a_n})_{n\geqslant 1}$ for various number-theoretic or combinatorial sequences $(a_n)_{n\geqslant 1}$ of positive integers. In this talk we give an introduction to those conjectures and related progress.

Part I. Conjectures on number-theoretic sequences

Firoozbakht's Conjecture

In 1982 Faride Firoozbakht (from Iran) posed the following challenging conjecture while he was studying a proof of the Prime Number Theorem.

Firoozbakht's Conjecture. $\sqrt[n]{p_n} > \sqrt[n+1]{p_{n+1}}$ for all $n = 1, 2, \ldots$, i.e., the sequence $(\sqrt[n]{p_n})_{n \geqslant 1}$ is strictly decreasing

Remark. The conjecture implies that

$$p_{n+1} - p_n < \log^2 p_n - \log p_n + 1$$

for all $n > 4$, which is stronger than Cramer's conjecture that $c := \limsup(p_{n+1} - p_n)/\log^2 p_n$ coincides with 1 (A. Granville thought that c should be $2/e^{\gamma} \approx 1.122918$.)

Verification record. Verified for all primes less than 4×10^{18}.

Comments. Since $\log \sqrt[n]{p_n} \sim (\log n)/n$, the conjecture seems reasonable. I saw it few years ago but soon forgot the reference and the proposer's long name.

A refinement of Firoozbakht's Conjecture

By the Prime Number Theorem, $p_n \sim n \log n$. Note that

$$\log \frac{\sqrt[n+1]{(n+1)\log(n+1)}}{\sqrt[n]{n \log n}} = -\frac{\log n}{n^2} - \frac{\log \log n}{n^2} + O\left(\frac{1}{n^2}\right).$$

This led me to make the following conjecture.

Conjecture (Sun, 2012-09-11) For any integer $n > 4$, we have the inequality

$$\frac{\sqrt[n+1]{p_{n+1}}}{\sqrt[n]{p_n}} < 1 - \frac{\log \log n}{2n^2}.$$

Remark. We have verified the conjecture for all $n \leqslant 3500000$ and all those n with $p_n < 4 \times 10^{18}$ and $p_{n+1} - p_n \neq p_{k+1} - p_k$ for all $1 \leqslant k < n$. If $n = 49749629143526$, then $p_{n+1} - p_n = 1132$, $p_n = 1693182318746371$, and

$$(1 - \sqrt[n+1]{p_{n+1}} / \sqrt[n]{p_n})n^2 / \log \log n \approx 0.5229.$$

Some easy facts

Easy things:

$(\sqrt[n]{n})_{n\geqslant 3}$ is strictly decreasing,

$(\sqrt[n+1]{n+1}/\sqrt[n]{n})_{n\geqslant 4}$ is strictly increasing.

Reason: For the function $f(x) = (\log x)/x$ on the interval $[4.5, +\infty)$, we have

$$f'(x) = \frac{1-\log x}{x^2} < 0 \text{ and } f''(x) = \frac{2\log x - 3}{x^3} > 0,$$

and hence $f(x)$ is strictly decreasing and strictly convex.

For

$$P_n = p_1 p_2 \cdots p_n \quad \text{and} \quad S_n = p_1 + p_2 + \cdots + p_n,$$

$(\sqrt[n]{P_n})_{n\geqslant 1}$ and $(S_n/n)_{n\geqslant 1}$ are strictly increasing (easy to prove).

$\sqrt[n]{P_n}$ —— the *geometric mean* of $p_1, p_2, \ldots, p_n$.

S_n/n —— the *arithmetic mean* of $p_1, p_2, \ldots, p_n$.

Monotonicity related to $S_n = \sum_{k=1}^n p_k$

Theorem (Sun). (i) (July 28-31) $(\sqrt[n]{S_n})_{n\geqslant 2}$ and $(\sqrt[n]{S_n/n})_{n\geqslant 1}$ are strictly decreasing.

(ii) (Discovered on July 29 and proved on August 25)

$$\left(\frac{\sqrt[n+1]{S_{n+1}}}{\sqrt[n]{S_n}}\right)_{n\geqslant 5} \quad \text{and} \quad \left(\frac{\sqrt[n+1]{S_{n+1}/(n+1)}}{\sqrt[n]{S_n/n}}\right)_{n\geqslant 10}$$

are strictly increasing.

A general theorem

For $S_n^{(\alpha)} = \sum_{k=1}^n p_k^\alpha$, we have

Theorem (Sun, Bull. Aust. Math. Soc., in press). Let $\alpha \geqslant 1$.

(i) If $n \geqslant \max\{100, e^{2\times 1.348^\alpha+1}\}$, then

$$\sqrt[n]{\frac{S_n^{(\alpha)}}{n}} > \sqrt[n+1]{\frac{S_{n+1}^{(\alpha)}}{n+1}}$$

and hence

$$\sqrt[n]{S_n^{(\alpha)}} > \sqrt[n+1]{S_{n+1}^{(\alpha)}}.$$

(ii) The sequence

$$\left(\sqrt[n+1]{S_{n+1}^{(\alpha)}/(n+1)} \Big/ \sqrt[n]{S_n^{(\alpha)}/n} \right)_{n \geqslant N(\alpha)}$$

is strictly increasing, where

$$N(\alpha) = \max\left\{350000,\ \lceil e^{((\alpha+1)^2 1.2^{2\alpha+1}+(\alpha+1)1.2^{\alpha+1})/\alpha} \rceil \right\}.$$

The cases $\alpha = 2, 3, 4$

Corollary. The sequences

$$\left(\sqrt[n+1]{S_{n+1}^{(2)}} \Big/ \sqrt[n]{S_n^{(2)}} \right)_{n \geqslant 10},$$

$$\left(\sqrt[n+1]{S_{n+1}^{(3)}} \Big/ \sqrt[n]{S_n^{(3)}} \right)_{n \geqslant 10},$$

$$\left(\sqrt[n+1]{S_{n+1}^{(4)}} \Big/ \sqrt[n]{S_n^{(4)}} \right)_{n \geqslant 17}$$

are all strictly increasing.

On squarefree numbers

A positive integer n is called *squarefree* if $p^2 \nmid n$ for any prime p. Here is the list of all squarefree positive integers not exceeding 30 in alphabetical order:

1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30.

Conjecture (Sun, 2012-08-14) (i) For $n = 1, 2, 3, \ldots$ let s_n be the n-th squarefree positive integer. Then the sequence $(\sqrt[n]{s_n})_{n\geqslant 7}$ is strictly decreasing.

(ii) For $n = 1, 2, 3, \ldots$ let $S(n)$ be the sum of the first n squarefree positive integers. Then the sequence $(\sqrt[n+1]{S(n+1)}/\sqrt[n]{S(n)})_{n\geqslant 7}$ is strictly increasing.

Remark. I have checked both parts of the conjecture via `Mathematica`; for example, $\sqrt[n]{s_n} > \sqrt[n+1]{s_{n+1}}$ for all $n = 7, \ldots, 500000$. Note that $\lim_{n\to\infty} \sqrt[n]{S(n)} = 1$ since $S(n)$ does not exceed the sum of the first n primes.

Conjecture on primitive roots modulo primes

Conjecture (Sun, 2012-08-17). Let $a \in \mathbb{Z}$ be not a perfect power (i.e., there are no integers $m > 1$ and x with $x^m = a$).

(i) Assume that $a > 0$. Then there are infinitely many primes p having a as the *smallest positive* primitive root modulo p. Moreover, if $p_1(a), \ldots, p_n(a)$ are the first n such primes, then the next such prime $p_{n+1}(a)$ is smaller than $p_n(a)^{1+1/n}$.

(ii) Suppose that $a < 0$. Then there are infinitely many primes p having a as the *largest negative* primitive root modulo p. Moreover, if $p_1(a), \ldots, p_n(a)$ are the first n such primes, then the next such prime $p_{n+1}(a)$ is smaller than $p_n(a)^{1+1/n}$ with the only exception $a = -2$ and $n = 13$.

(iii) The sequence $\left(\sqrt[n+1]{P_{n+1}(a)}/\sqrt[n]{P_n(a)}\right)_{n \geqslant 3}$ is strictly increasing with limit 1, where $P_n(a) = \sum_{k=1}^n p_k(a)$.

Remark. The first 5 primes having 24 as the smallest positive primitive root are

$$533821,\ 567631,\ 672181,\ 843781,\ 1035301.$$

Conjecture on twin primes

It is conjectured that there are infinitely many twin primes.

Conjecture (Sun, 2012-08-18) (i) If $\{t_1, t_1+2\}, \ldots, \{t_n, t_n+2\}$ are the first n pairs of twin primes, then the first prime t_{n+1} in the next pair of twin primes is smaller than $t_n^{1+1/n}$, i.e., $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$.

(ii) The sequence $\left(\sqrt[n+1]{T(n+1)}/\sqrt[n]{T(n)}\right)_{n\geqslant 9}$ is strictly increasing with limit 1, where $T(n) = \sum_{k=1}^n t_k$.

Remark. Via `Mathematica` I verified that $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$ for all $n = 1, \ldots, 500000$, and

$$\sqrt[n+1]{T(n+1)}/\sqrt[n]{T(n)} < \sqrt[n+2]{T(n+2)}/\sqrt[n+1]{T(n+1)}$$

for all $n = 9, 10, \ldots, 500000$. Note that $t_{500000} = 115438667$.

After I made the conjecture public, Marek Wolf verified the inequality $\sqrt[n]{t_n} > \sqrt[n+1]{t_{n+1}}$ for all the 44849427 pairs of twin primes below $2^{34} \approx 1.718 \times 10^{10}$.

Conjecture on Sophie Germain primes

A prime p is called a Sophie Germain prime if $2p+1$ is also a prime. It is conjectured that there are infinitely many Sophie Germain primes, but this has not been proved yet.

Conjecture (Sun, 2012-08-18) (i) If $g_1, \ldots, g_n$ are the first n Sophie Germain primes, then the next Sophie Germain prime g_{n+1} is smaller than $g_n^{1+1/n}$ (i.e., $\sqrt[n]{g_n} > \sqrt[n+1]{g_{n+1}}$) with the only exceptions $n = 3, 4$.

(ii) The sequence $(\sqrt[n+1]{G(n+1)}/\sqrt[n]{G(n)})_{n \geqslant 13}$ is strictly increasing with limit 1, where $G(n) = \sum_{k=1}^{n} g_k$.

Remark. Via `Mathematica` I verified that $\sqrt[n]{g_n} > \sqrt[n+1]{g_{n+1}}$ for all $n = 5, \ldots, 200000$, and

$$\sqrt[n+1]{G(n+1)}/\sqrt[n]{G(n)} < \sqrt[n+2]{G(n+2)}/\sqrt[n+1]{G(n+1)}$$

for all $n = 13, 14, \ldots, 200000$. Note that $g_{200000} = 42721961$.

A general conjecture related to Hypothesis H

Schinzel's Hypothesis H. If $f_1(x), \ldots, f_k(x)$ are irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing the product $f_1(q)f_2(q)...f_k(q)$ for all $q \in \mathbb{Z}$, then there are infinitely many $n \in \mathbb{Z}^+$ such that $f_1(n), f_2(n), \ldots, f_k(n)$ are all primes.

General Conjecture (Sun, 2012-09-08) Let $f_1(x), \ldots, f_k(x)$ be irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing $\prod_{j=1}^k f_j(q)$ for all $q \in \mathbb{Z}$. Let $q_1, q_2, \ldots$ be the list (in ascending order) of those $q \in \mathbb{Z}^+$ such that $f_1(q), \ldots, f_k(q)$ are all primes. Then, for all sufficiently large positive integers n, we have

$$q_{n+1} < q_n^{1+1/n}, \text{ i.e., } \sqrt[n]{q_n} > \sqrt[n+1]{q_{n+1}}.$$

Also, there is a positive integer N such that the sequence $\left(\sqrt[n+1]{Q(n+1)}/\sqrt[n]{Q(n)}\right)_{n \geqslant N}$ is strictly increasing with limit 1, where $Q(n) = \sum_{k=1}^n q_k$.

Conjecture on Proth primes

Proth numbers: $k \times 2^n + 1$ with k odd and $0 < k < 2^n$.

F. Proth (1878): A Proth number p is a prime if (and only if) $a^{(p-1)/2} \equiv -1 \pmod{p}$ for some integer a.

A *Proth prime* is a Proth number which is also a prime number; the Fermat primes are a special kind of Proth primes.

Conjecture (Sun, 2012-09-07) (i) The number of Proth primes not exceeding a large integer x is asymptotically equivalent to $c\sqrt{x}/\log x$ for a suitable constant $c \in (3, 4)$.

(ii) If $\mathrm{Pr}(1), \ldots, \mathrm{Pr}(n)$ are the first n Proth primes, then $\mathrm{Pr}(n+1) < \mathrm{Pr}(n)^{1+1/n}$ (i.e., $\sqrt[n]{\mathrm{Pr}(n)} > \sqrt[n+1]{\mathrm{Pr}(n+1)}$) unless $n = 2, 4, 5$. If we set $\mathrm{PR}(n) = \sum_{k=1}^{n} \mathrm{Pr}(k)$, then the sequence $\left(\sqrt[n+1]{\mathrm{PR}(n+1)}/\sqrt[n]{\mathrm{PR}(n)}\right)_{n \geqslant 34}$ is strictly increasing with limit 1.

Remark. I have checked the conjecture for the first 4000 Proth primes.

On irreducible polynomials over finite fields

Let $q > 1$ be a prime power and let $\mathbb{F}_q$ denote the finite field of order q. For $n = 1, 2, 3, \ldots$ let $N_n(q)$ denote the number of monic irreducible polynomials of degree n over $\mathbb{F}_q$.

Theorem (Sun, October 2012)
(i) The sequence $(N_{n+1}(q)/N_n(q))_{n\geqslant 1}$ is strictly increasing if $q \geqslant 9$, and $(N_{n+1}(q)/N_n(q))_{n\geqslant 19}$ is strictly increasing if $q < 9$.

(ii) The sequence $(\sqrt[n]{N_n(q)})_{n>e^{3+7/(q-1)^2}}$ is strictly increasing, and the sequence

$$(\sqrt[n+1]{N_{n+1}(q)}/\sqrt[n]{N_n(q)})_{n\geqslant 5.835\times 10^{14}}$$

is strictly decreasing.

On partitions of integers

A partition of a positive integer n is a way of writing n as a sum of positive integers with the order of addends ignored. Let $p(n)$ denote the number of partitions of n. It is known that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n} \quad \text{(Hardy and Ramanujan)}$$

and hence $\lim_{n\to\infty} \sqrt[n]{p(n)} = 1$.

Conjecture (Sun, 2012-08-02) The sequence $(\sqrt[n]{p(n)})_{n\geqslant 6}$ is strictly decreasing. Moreover, $(\sqrt[n+1]{p(n+1)}/\sqrt[n]{p(n)})_{n\geqslant 26}$ is strictly increasing.

Remark. I have verified the conjecture for n up to 10^5. The log-concavity of $(p(n))_{n\geqslant 25}$ was conjectured by W.Y.C. Chen in August 2010 and proved by J.E. Janoski in his PhD thesis who said that he began the project in the summer of 2010.

On strict partitions of integers

A *strict partition* of $n \in \mathbb{Z}^+$ is a way of writing n as a sum of *distinct* (or odd) positive integers with the order of addends ignored. For $n = 1, 2, 3, \ldots$ we denote by $p_*(n)$ the number of strict partitions of n. It is known that

$$p_*(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4(3n^3)^{1/4}} \quad \text{as } n \to +\infty$$

and hence

$$\lim_{n\to\infty} \sqrt[n]{p_*(n)} = 1.$$

Conjecture (Sun, 2012-08-02) $(p_*(n+1)/p_*(n))_{n\geqslant 32}$ and $(\sqrt[n]{p_*(n)})_{n\geqslant 9}$ are strictly decreasing. Furthermore, the sequence $(\sqrt[n+1]{p_*(n+1)}/\sqrt[n]{p_*(n)})_{n\geqslant 45}$ is strictly increasing.

Remark. I have verified the conjecture for n up to 10^5.

On harmonic numbers of order m

Harmonic numbers: $H_n = \sum_{0<k\leqslant n} 1/k$ $(n = 0, 1, 2, \ldots)$.

Harmonic numbers of order m: $H_n^{(m)} = \sum_{0<k\leqslant n} 1/k^m$.

It is easy to show that $(\sqrt[n]{H_n^{(m)}})_{n\geqslant 2}$ is strictly decreasing for any positive integer m.

Conjecture (Sun, 2012-08-12) For any positive integer m, the sequence

$$\left(\sqrt[n+1]{H_{n+1}^{(m)}} / \sqrt[n]{H_n^{(m)}} \right)_{n\geqslant 3}$$

is strictly increasing.

This conjecture was confirmed in a joint paper with Qing-Hu Hou and Haomin Wen (arXiv:1208.3903).

Bernoulli numbers, Euler numbers and tangent numbers

Bernoulli numbers are given by

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \ (|x| < 2\pi).$$

It is known that $(-1)^{n-1}B_{2n} > 0$ and $B_{2n+1} = 0$ for all $n > 0$.

Euler numbers are given by

$$\frac{2}{e^x + e^{-x}} = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!}.$$

It is known that $(-1)^n E_{2n} > 0$ and $E_{2n+1} = 0$ for all $n = 0, 1, 2, \ldots$.

Tangent numbers are given by

$$\tan x = \sum_{n=1}^{\infty} T(n) \frac{x^{2n-1}}{(2n-1)!}.$$

It is known that $T(n) \in \mathbb{Z}^+$ for all $n = 1, 2, 3, \ldots$.

Bernoulli numbers, Euler numbers and tangent numbers

Conjecture (Sun, 2012-08-02) The sequences

$$(\sqrt[n]{(-1)^{n-1}B_{2n}})_{n\geqslant 1},\ (\sqrt[n]{(-1)^n E_{2n}})_{n\geqslant 1},\ (\sqrt[n]{T(n)})_{n\geqslant 1}$$

are strictly increasing, and the sequences

$$(\sqrt[n+2]{(-1)^n B_{2n+2}}/\sqrt[n]{(-1)^{n-1}B_{2n}})_{n\geqslant 2},$$
$$(\sqrt[n+1]{(-1)^{n+1} E_{2n+2}}/\sqrt[n]{(-1)^n E_{2n}})_{n\geqslant 1},$$
$$(\sqrt[n+1]{T(n+1)}/\sqrt[n]{T(n)})_{n\geqslant 2}$$

are strictly decreasing.

This conjecture was confirmed in a preprint by Florian Luca and Pantelimon Stănică (arXiv:1208.5151).

Part II. Conjectures on combinatorial sequences

On the Fibonacci sequence

The Fibonacci sequence $(F_n)_{n\geqslant 0}$ is given by

$$F_0 = 0,\ F_1 = 1,\ \text{and}\ F_{n+1} = F_n + F_{n-1}\ (n = 1, 2, 3, \ldots).$$

Conjecture (2012-08-11) The sequence $(\sqrt[n]{F_n})_{n\geqslant 2}$ is strictly increasing, and moreover the sequence $(\sqrt[n+1]{F_{n+1}}/\sqrt[n]{F_n})_{n\geqslant 4}$ is strictly decreasing.

This was confirmed in a joint paper with Qing-Hu Hou and Haomin Wen (arXiv:1208.3903). Actually the paper contains a result on general Lucas sequences.

On derangement numbers

The nth derangement number

$$D_n = |\{\sigma \in S_n : \ \sigma(i) \neq i \text{ for all } i = 1, \ldots, k\}|.$$

It is known that $D_n/n! = \sum_{k=0}^{n}(-1)^k/k!$.

Conjecture (Sun, 2012-08-11) The sequence $(\sqrt[n]{D_n})_{n\geqslant 2}$ is strictly increasing, and moreover the sequence $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n\geqslant 3}$ is strictly decreasing.

This was confirmed in a joint paper with Qing-Hu Hou and Haomin Wen (arXiv:1208.3903).

On Bell numbers

The n-th Bell number B_n denotes the number of partitions of $\{1, \ldots, n\}$ into disjoint nonempty subsets. It is known that

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \quad (\text{with } B_0 = 1)$$

and

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}.$$

Conjecture (Sun, 2012-08-11) The sequence $(\sqrt[n]{B_n})_{n\geqslant 1}$ is strictly increasing, and moreover the sequence $(\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n})_{n\geqslant 1}$ is strictly decreasing with limit 1, where B_n is the n-th Bell number.

Remark. In 1994 K. Engel proved the log-convexity of $(B_n)_{n\geqslant 1}$. The above conjecture is still open and seems very challenging!

On Springer numbers

Springer numbers are given by

$$\frac{1}{\cos x - \sin x} = \sum_{n=0}^{\infty} S_n \frac{x^n}{n!}.$$

It is known that S_n equals the numerator of $|E_n(1/4)|$, where $E_n(x)$ is the Euler polynomial of degree n.

Conjecture (Sun, 2012-08-05) The sequence $(S_{n+1}/S_n)_{n\geqslant 0}$ is strictly increasing, and the sequence $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n\geqslant 1}$ is strictly decreasing with limit 1, where S_n is the n-th Springer number.

On central trinomial coefficients

The n-th central trinomial coefficient T_n is the coefficient of x^n in the expansion of $(x^2+x+1)^n$. Here is an explicit expression:

$$T_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$. It is known that

$$(n+1)T_{n+1} = (2n+1)T_n + 3nT_{n-1}.$$

Conjecture (2012-08-11) The sequence $(\sqrt[n]{T_n})_{n\geqslant 1}$ is strictly increasing, and the sequence $(\sqrt[n+1]{T_{n+1}}/\sqrt[n]{T_n})_{n\geqslant 1}$ is strictly decreasing.

F. Luca and P. Stănică confirmed this for large n.

On Motzkin numbers

The n-th Motzkin number

$$M_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} \frac{1}{k+1}$$

is the number of lattice paths from $(0,0)$ to $(n,0)$ which never dip below the line $y = 0$ and are made up only of the allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$. It is known that

$$(n+3)M_{n+1} = (2n+3)M_n + 3nM_{n-1}.$$

Conjecture (Sun, 2012-08-11) The sequence $(\sqrt[n]{M_n})_{n\geqslant 1}$ is strictly increasing, and moreover the sequence $(\sqrt[n+1]{M_{n+1}}/\sqrt[n]{M_n})_{n\geqslant 1}$ is strictly decreasing.

Remark. The log-convexity of the sequence $(M_n)_{n\geqslant 1}$ was first established by M. Aigner in 1998. F. Luca and P. Stănică confirmed the conjecture for large n.

On Schröder numbers

The n-th Schröder number

$$S_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{k+1} = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}\frac{1}{k+1}$$

is the number of lattice paths from the point $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$ that never rise above the line $y=x$

Conjecture (Sun, 2012-08-11) The sequence $(\sqrt[n]{S_n})_{n\geqslant 1}$ is strictly increasing, and moreover the sequence $(\sqrt[n+1]{S_{n+1}}/\sqrt[n]{S_n})_{n\geqslant 1}$ is strictly decreasing, where S_n stands for the n-th Schröder number.

On Domb numbers and Catalan-Larcombe-French numbers

Conjecture (Sun, 2012-08-13) For the Domb numbers

$$D(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} \ (n = 0, 1, 2, \ldots),$$

the sequences $(D(n+1)/D(n))_{n\geqslant 0}$ and $(\sqrt[n]{D(n)})_{n\geqslant 1}$ are strictly increasing. Moreover, the sequence $(\sqrt[n+1]{D(n+1)}/\sqrt[n]{D(n)})_{n\geqslant 1}$ is strictly decreasing.

The Catalan-Larcombe-French numbers $P_0, P_1, P_2, \ldots$ are given by

$$P_n = \sum_{k=0}^{n} \frac{\binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2}{\binom{n}{k}} = 2^n \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k}^2 4^{n-2k},$$

they arose from the theory of elliptic integrals. It is known that $(n+1)P_{n+1} = (24n(n+1)+8)P_n - 128n^2 P_{n-1}$ for all $n \in \mathbb{Z}^+$.

Conjecture (Sun, 2012-08-14) The sequences $(P_{n+1}/P_n)_{n\geqslant 0}$ and $(\sqrt[n]{P_n})_{n\geqslant 1}$ are strictly increasing. Moreover, the sequence $(\sqrt[n+1]{P_{n+1}}/\sqrt[n]{P_n})_{n\geqslant 1}$ is strictly decreasing.

On Franel numbers and Apéry numbers

Franel numbers:

$$f_n := \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots).$$

Apéry numbers:

$$A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \quad (n = 0, 1, 2, \ldots).$$

Conjecture (Sun, 2012-08-11) Both

$$\left(\sqrt[n+1]{f_{n+1}} / \sqrt[n]{f_n} \right)_{n \geqslant 1}$$

and

$$\left(\sqrt[n+1]{A_{n+1}} / \sqrt[n]{A_n} \right)_{n \geqslant 1}$$

are strictly decreasing with limit 1.

F. Luca and P. Stănică confirmed the conjecture for large n.

My above conjectures appeared in the following paper:

Z. W. Sun, *Conjectures involving arithmetical sequences*, in: Arithmetic in Shangri-La (eds., S. Kanemitsu, H. Li and J. Liu), Proc. 6th China-Japan Seminar (Shanghai, August 15-17, 2011), World Sci., Singapore, 2013, pp. 244–258.

You are welcome to solve my conjectures!

Thank you!