A talk given at the 5th National Conference on Number Theory

Correspondence between Series and Congruences

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Preface

In November 2009 I posted to arxiv the article

Zhi-Wei Sun, **Open Conjectures on Congruences**,
http://arxiv.org/abs/0911.5665.
which now contains my 100 unsolved conjectures!

Conj. A30. Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^3} \equiv -\frac{2}{p^2} \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^2}.$$

(It is known that $p^{-2}\sum_{k=1}^{p-1} 1/k \equiv -B_{p-3}/3 \pmod p$.)

Conj. A97. For any odd prime p and positive integer n,

$$\frac{1}{n\binom{2n}{n}} \sum_{k=0}^{n-1} \binom{(p-1)k}{k, \ldots, k} \quad \text{is always a } p\text{-adic integer.}$$

Conj. A13. We have

$$\sum_{k=1}^{\infty} \frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}} = \sum_{k=1}^{\infty} \frac{(\frac{k}{3})}{k^2} = 0.7813024\ldots.$$

More examples of conjectured congruences

Conj. A66. Let A_n be the Apéry number $\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$. For any odd prime p, we have

$$\sum_{k=0}^{p-1}A_k\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z},\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8);\end{cases}$$

also,

$$\sum_{k=0}^{p-1}(-1)^kA_k\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+3y^2 \text{ with } x,y\in\mathbb{Z},\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3).\end{cases}$$

Conj. A78. Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\left(\frac{[x^k](x^2+2x+9)^k}{(-4)^k}\right)^3$$

$$\equiv\begin{cases}4x^2-2p\ \pmod{p^2} & \text{if } p\equiv 1,7\ \pmod{24}\ \&\ p=x^2+6y^2,\\ 2p-8x^2\ \pmod{p^2} & \text{if } p\equiv 5,11\ \pmod{24}\ \&\ p=2x^2+3y^2,\\ 0\ \pmod{p^2} & \text{if } \left(\frac{-6}{p}\right)=-1.\end{cases}$$

The purpose of this talk

How could I find the 100 open conjectures on congruences ? In fact, I have formulated some **philosophical viewpoints** and they led me find those conjectures.

In this talk I'll talk about one philosophy of mine about series and congruences.

We observe that there is a correspondence between a wide range of series (for power of π or the zeta function) and congruences modulo powers of a prime p involving the Bernoulli number B_{p-3} or the Euler number E_{p-3}.

In this talk we will illustrate the above philosophy by many examples and give a survey of results and conjectures in the field.

Part A. Review of some known series for powers of π or the Riemann zeta function

Classical series for π

Leibniz:

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}$$

$$\frac{\pi}{4} = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = L\left(1, \left(\frac{-1}{\cdot}\right)\right).$$

Euler:

$$\zeta(2) = \sum_{k=0}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}, \quad \zeta(4) = \sum_{k=0}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}.$$

Two more series:

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5}\zeta(3) \text{ (Apéry)}, \quad \sum_{k=1}^{\infty} \frac{1}{k^4 \binom{2k}{k}} = \frac{17}{36}\zeta(4).$$

Inverse sine function and related π-series

$$\arcsin\frac{x}{2} = \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)4^k}\left(\frac{x}{2}\right)^{2k+1},$$

$$2\arcsin^2\frac{x}{2} = \sum_{k=1}^{\infty}\frac{x^{2k}}{k^2\binom{2k}{k}}.$$

Thus

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)4^k} = \frac{\pi}{2},\quad \sum_{k=1}^{\infty}\frac{4^k}{k^2\binom{2k}{k}} = \frac{\pi^2}{2};$$

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k} = \frac{\pi}{3},\quad \sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}} = \frac{\pi^2}{18};$$

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)8^k} = \frac{\pi}{4}\sqrt{2},\quad \sum_{k=1}^{\infty}\frac{2^k}{k^2\binom{2k}{k}} = \frac{\pi^2}{8};$$

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)}\left(\frac{3}{16}\right)^k = \frac{2\pi}{9}\sqrt{3},\quad \sum_{k=1}^{\infty}\frac{3^k}{k^2\binom{2k}{k}} = \frac{2}{9}\pi^2.$$

Series involving harmonic numbers

Harmonic numbers:

$$H_n := \sum_{k=1}^{n} \frac{1}{k}, \quad H_n^{(2)} := \sum_{k=1}^{n} \frac{1}{k^2} \ (n = 0, 1, 2, \ldots).$$

Series related to the zeta function:

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3) \text{ (Euler)}, \quad \sum_{k=1}^{\infty} \frac{H_k}{k^3} = \frac{\pi^4}{72} \text{ (Goldbach, 1742)},$$

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17}{360}\pi^4 \text{ (D. Borwein and J.M. Borwein, 1995)},$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12} \text{ (S.W. Coffman, 1987)},$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k2^k} = \frac{5}{8}\zeta(3) \text{ (B. Cloitre, 2004)}.$$

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$ and $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$.

Series for $1/\pi$

G. Bauer (1859):

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} = \frac{2}{\pi}.$$

Ramanujan (1914) gave 14 series for $1/\pi$, for example,

$$\sum_{k=0}^{\infty}\frac{6k+1}{4^k}\cdot\frac{(1/2)_k^3}{(1)_k^3} = \sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k} = \frac{4}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{26390k+1103}{99^{4k}}\cdot\frac{(1/2)_k(1/4)_k(3/4)_k}{(1)_k^3}$$

$$= \sum_{k=0}^{\infty}\frac{26390k+1103}{396^{4k}}\binom{4k}{k,k,k,k} = \frac{99^2}{2\pi\sqrt{2}}.$$

Note that

$$\binom{4k}{k,k,k,k} = \frac{(4k)!}{(k!)^4} = \binom{2k}{k}^2\binom{4k}{2k}.$$

Ramanujan-type series for $1/\pi$

General form:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Remark. This yielded the record for the calculation of π during 1989-1994.

Part B. Congruences involving central binomial coefficients or squares of central binomial coefficients

Central binomial coefficients

Central binomial coefficients:

$$\binom{2n}{n} \quad (n = 0, 1, 2, \ldots).$$

Catalan numbers:

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \ (n = 0, 1, 2, \ldots).$$

Asymptotic formula:

$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{n\pi}}.$$

This follows from the Stirling formula

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n.$$

Classical congruences for central binomial coefficients

If p is an odd prime, then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p} \text{ for every } k = \frac{p+1}{2}, \ldots, p-1.$$

Wolstenholme's Congruence (1862). For any prime $p > 3$ we have

$$H_{p-1} = \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}$$

and

$$\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}.$$

Morley's Congruence (1895). For any prime $p > 3$ we have

$$\binom{p-1}{(p-1)/2} \equiv \left(\frac{-1}{p}\right) 4^{p-1} \pmod{p^3}.$$

Gauss' congruence

Gauss' Congruence. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\binom{(p-1)/2}{(p-1)/4} \equiv 2x \pmod p.$$

Further Refinement of Gauss' Result (Chowla, Dwork and Evans, 1986):

$$\binom{(p-1)/2}{(p-1)/4} \equiv \frac{2^{p-1}+1}{2}\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

It follows that

$$\binom{(p-1)/2}{(p-1)/4}^2 \equiv 2^{p-1}(4x^2 - 2p) \pmod{p^2}.$$

Three conjectured congruences of Rodriguez-Villegas

Congruences related to $\mathbb{F}_p$-points of some Calabi-Yau manifolds:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{108^k} \equiv b(p) \pmod{p^2}, \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{4k}{2k}}{256^k} \equiv c(p) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}}{12^{3k}} \equiv \begin{cases} -a(p) \pmod{p^2} & \text{if } p \equiv 5 \pmod{12}, \\ a(p) \pmod{p^2} & \text{otherwise,} \end{cases},$$

$$a(p) = [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6 = \eta(4z)^6,$$

$$b(p) = [q^p]q\prod_{n=1}^{\infty}(1-q^{6n})^3(1-q^{2n})^3 = \eta^3(6z)\eta^3(2z),$$

$$c(p) = [q^p]q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2$$

$$= \eta^2(8z)\eta(4z)\eta(2z)\eta^2(z),$$

On $\eta(z)$, $a(p)$, $b(p)$ and $c(p)$

The Dedekind η-function:

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \quad (\mathrm{Im}(z) > 0 \text{ and } q = e^{2\pi iz}).$$

F. Klein and R. Fricke (1892):

$$a(p) = \begin{cases} 4x^2 - 2p & \text{if } p \equiv 1 \pmod 4 \text{ \& } p = x^2 + y^2 \text{ with } 2 \nmid x \text{ \& } 2 \mid y, \\ 0 & \text{if } p \equiv 3 \pmod 4). \end{cases}$$

J. Stienstra and F. Beukers (1985):

$$b(p) = \begin{cases} 4x^2 - 2p & \text{if } p \equiv 1 \pmod 3 \text{ \& } p = x^2 + 3y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 & \text{if } p \equiv 2 \pmod 3; \end{cases}$$

$$c(p) = \begin{cases} 4x^2 - 2p & \text{if } p \equiv 1, 3 \pmod 8 \text{ \& } p = x^2 + 2y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 & \text{if } p \equiv 5, 7 \pmod 8. \end{cases}$$

Progress on the three congruences conjectured by Rodriguez-Villegas

Via an advanced approach involving the p-adic Gamma function and Gauss and Jacobi sums, E. Mortenson [2005] managed to provide a partial solution to the three congruences, with the following things open:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv b(p)=0 \pmod{p^2}\quad \text{if } p\equiv 5 \pmod 6,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv c(p) \pmod{p^2}\quad \text{if } p\equiv 3 \pmod 4,$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv -a(p) \pmod{p^2}\quad \text{if } p\equiv 5 \pmod 6.$$

Quite recently the author solved all the remaining open cases!

p-adic Gamma function

Mortenson's approach is similar to S. Ahlgren and Ken Ono's proof of the Beukers conjecture on Apéry numbers; the p-adic Gamma function plays an important role in such an approach.

p-**adic Gamma function**:

$$\Gamma_p(n) := (-1)^n \prod_{\substack{1<k<n \\ p\nmid n}} k \quad (n = 1, 2, 3, \ldots)$$

and

$$\Gamma_p(x) = \lim_{n \to x} \Gamma_p(n) \ \text{ for any } p-\text{adic integer } x.$$

A Connection of Γ_p to Binomial Coefficients:

If $p = 2n + 1$ is an odd prime, then for $k = 0, \ldots, n$ we have

$$(-1)^{(p+1)/2} \frac{\Gamma_p(k+1/2)^2}{\Gamma_p(k+1)^2} \equiv \binom{n}{k}\binom{n+k}{k}(-1)^k \equiv \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}.$$

My conjectures involving products of two binomial coefficients

Conj. B2. Let $p \equiv 1 \pmod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4} \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k}$$
$$\equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

Remark. I proved the congruences modulo p and later Zhi-Hong Sun [Proc. AMS] confirmed the conjecture.

I also have conjectures on $\sum_{k=0}^{p-1} \binom{2k}{k}\binom{4k}{2k}/m^k$ mod p^2 with $m = 48, 63, 72, 128, -192$. For example,

Conj. A50. Let $p > 3$ be a prime. If $(\frac{p}{7}) = 1$ and $p = x^2 + 7y^2$ with $(\frac{x}{7}) = 1$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{63^k} \equiv \left(\frac{p}{3}\right)\left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

Two more conjectures involving products of two binomial coefficients

Conj. A46. Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{24^k} \equiv \begin{cases} \binom{2(p-1)/3}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod 3, \\ 0 \pmod p & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Conj. A44. Let $p \equiv 1 \mod 4$ be a prime and write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{864^k} \equiv \begin{cases} (-1)^{\lfloor x/6 \rfloor}(2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ (\frac{xy}{3})(2y - \frac{p}{2y}) \pmod{p^2} & \text{if } p \equiv 5 \pmod{12}. \end{cases}$$

Remark. I have proved that $\sum_{k=0}^{p-1}\binom{6k}{3k}\binom{3k}{k}/864^k \equiv 0 \pmod{p^2}$ for any prime $p \equiv 3 \pmod 4$ with $p > 3$. Recently Zhi-Hong Sun proved the conjectured congruence modulo p.

Connections to elliptic curves over $\mathbb{F}_p$ and character sums

Let $p = 2n+1 > 3$ be a prime and let $\lambda \not\equiv 0, 1 \pmod p$ be a rational p-adic integer. Consider the elliptic curve $E_p(\lambda) : y^2 = x(x-1)(x-\lambda)$ of Legendre form over $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$. It is known (cf. Loh and Rhoades [IJNT, 2006]) that

$$\#E_p(\lambda) - p - 1 = \sum_{x=0}^{p-1} \left(\frac{x(x-1)(x-\lambda)}{p}\right)$$

$$\equiv (-1)^{(p+1)/2}(p+1) \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-\lambda)^{kp}\left(1 + 2kp\sum_{j=1}^{n}\frac{1}{k+j}\right) \pmod{p^2}$$

$$\equiv (-1)^{(p+1)/2} \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-\lambda)^{k} \pmod p$$

$$\equiv (-1)^{(p+1)/2} \sum_{k=0}^{p-1} \binom{2k}{k}^2 \left(\frac{\lambda}{16}\right)^{k} \pmod p.$$

Part C. Series VS Congruences

Γ-function VS the p-adic Γ-function

Known facts about the Γ-function:

$$\Gamma(x)\Gamma(1-x) = \frac{x}{\sin \pi x},\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},\ \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\pi.$$

***p*-adic analogues**:

$$\Gamma_p(x)\Gamma_p(1-x) = (-1)^{R_p(x)} \quad (p \neq 2)$$

where $R_p(x)$ is the least positive positive with $x \equiv R_p(x) \pmod p$.

$$\Gamma_p^2\left(\frac{1}{2}\right) = (-1)^{(p+1)/2} = -\left(\frac{-1}{p}\right).$$

(Van Hammer) If $p = x^2 + y^2$ with x odd then

$$-\Gamma_p\left(\frac{3}{4}\right)^{-4} \equiv 4x^2 - 2p \pmod{p^2}.$$

Hamme's Observations

L. Van Hamme [1997] conjectured some p-adic analogues of Ramanujan-type series.

Examples: Ramanujan series

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi},\quad \sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{64^k}=\frac{\pi}{\Gamma(3/4)^4}.$$

Hammer's p-adic analogues:

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\left(\frac{-1}{p}\right)p \pmod{p^3}$$

(Conjectured by Hammer and confirmed by Mortenson [2008]);

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+y^2\ (2\nmid x),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod{4}.\end{cases}$$

(proved by van Hammer).

Some Joint Work

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod p\ \ (d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod p\ \ \text{for } p>3.$$

Sun & R. Tauraso [Adv. in Appl. Math. 45(2010)].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\ \ \text{for } p>3,$$

My result on $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ mod p^2

Sun [Sci. China Math. 53(2010)]: Let p be an odd prime and let $m\in\mathbb{Z}$ with $p\nmid m$. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{m^k}\equiv\left(\frac{m^2-4m}{p}\right)+u_{p-(\frac{m^2-4m}{p})}\pmod{p^2},$$

where $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0=0,\ u_1=1,\ \text{and}\ u_{n+1}=(m-2)u_n-u_{n-1}\ (n=1,2,3,\ldots).$$

In particular,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv(-1)^{(p-1)/2}\pmod{p^2}.$$

Remark. I only found two values of p such that the last congruence holds mod p^3: $p=149,\ 241$.

My unexpected discovery in Jan. 2010

Let p be an odd prime. I wanted to know $\sum_{k=0}^{(p-1)/2}\binom{2k}{k}/k \bmod p^2$ and I found that $\sum_{k=1}^{(p-1)/2}\binom{2k}{k}/k \equiv 0 \pmod{p^3}$ for $p = 149, 241$.

A conjecture of Rodriguez-Villegas proved by Mortenson [JNT, 2003] states that $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k \equiv (\frac{-1}{p}) \pmod{p^2}$. I found that it holds mod p^3 for $p = 149, 241$.

A conjecture of Hammer proved by Mortenson [PAMS, 2008] asserts that

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{-1}{p}\right) p \pmod{p^3}.$$

I found that it holds mod p^4 for $p = 149, 241$.

Connections to Euler numbers

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1,\ \sum_{2|k} \binom{n}{k} E_{n-k} = 0\ (n = 1, 2, 3, \ldots).$$

It is known that $E_1 = E_3 = E_5 = \cdots = 0$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!}\ \left(|x| < \frac{\pi}{2}\right).$$

Z. W. Sun [arXiv:1001.4453]:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2}\ (p > 3),$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3},$$

Connections between series and congruences involving E_{p-3}

Series:

$$\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}}=\frac{\pi^2}{18},\ \sum_{k=1}^{\infty}\frac{4}{k^2\binom{2k}{k}}=\frac{\pi^2}{2},\ \sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}.$$

Congruences:

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)\frac{4}{3}E_{p-3}\pmod p\ (p>3),$$

$$\sum_{k=1}^{(p-1)/2}\frac{4^k}{k^2\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)4E_{p-3}\pmod p,$$

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\ (\mathrm{mod}\ p^4).$$

My philosophy about series involving π or the ζ-function

In a message to `Number Theory List` on March 15, 2010, I expressed the following viewpoint:

Almost every series with summation related to $\pi = 3.14\ldots$ or the Riemann zeta function corresponds to a congruence for Euler numbers or Bernoulli numbers. Conversely, many congruences for E_{p-3} or B_{p-3} modulo a prime p yield corresponding series related to π or the zeta function.

Example: It is known that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} = \frac{\pi^2}{10}.$$

I conjectured that for any prime $p > 5$ we have

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv \frac{H_{p-1}}{5p} \pmod{p^3}.$$

Find new series for π^3

There are very few interesting series for π^3. The only well-known series for π^3 is the following one:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=\frac{\pi^3}{32}.$$

Conj. A32. For any prime $p>5$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}\equiv(-1)^{(p-1)/2}\left(\frac{H_{p-1}}{4p^2}+\frac{p^2}{36}B_{p-5}\right)\pmod{p^3}.$$

Motivated by this conjecture, I guessed that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}=\frac{7}{216}\pi^3.$$

After I announced this conjecture, Olivier Gerard pointed out there is a computer proof via `Mathematica` (version 7).

Find new series for π^3

Let p be an odd prime. I proved that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv(-1)^{(p-1)/2}-p^2E_{p-3}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-2)^k}\equiv(-1)^{(p-1)/2}2^{p-1}\pmod{p^3}.$$

For $k=0,\ldots,p-1$, it is easy to see that

$$\binom{p-1}{k}(-1)^k\equiv 1+pH_k+\frac{p^2}{2}(H_k^2-H_k^{(2)})\pmod{p^3}.$$

So, it is natural to investigate $\sum_{k=0}^{p-1}\binom{2k}{k}H_k^{(2)}/2^k \bmod p$.

Theorem (Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}H_k^{(2)}\equiv -E_{p-3}\pmod{p}.$$

Note that $\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{2^k}H_k^{(2)}$ is divergent!

Find new series for π^3

A useful observation: Let p be an odd prime. Then, for any $k = 1, \ldots, p-1$ we have

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2}.$$

Note also that

$$H^{(2)}_{p-k} = H^{(2)}_{p-1} - \frac{1}{(p-k+1)^2} - \cdots - \frac{1}{(p-1)^2} \equiv -H^{(2)}_{k-1} \pmod{p}.$$

Thus via the transformation $k \to p-k$ we should investigate $\sum_k 2^k H^{(2)}_{k-1}/(k\binom{2k}{k})$ which cannot be evaluated via `Mathematica`.

Theorem (Sun, Sept. 2010). We have

$$\sum_{k=1}^{\infty} \frac{2^k H^{(2)}_{k-1}}{k\binom{2k}{k}} = \frac{\pi^3}{48}.$$

A sketch of the proof

Using the fact that

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{for any } a, b > 0,$$

and the dilogarithm function $\mathrm{Li}_2(x)$ given by

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad (|x| < 1),$$

I deduced that

$$\sum_{k=1}^{\infty} \frac{2^{k-1} H_{k-1}^{(2)}}{k\binom{2k}{k}} = \int_{-1}^{1} \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt$$
$$= \frac{\pi^3}{96} \quad \text{(by Mathematica 7)}.$$

Remark. The indefinite integral

$$\int \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt$$

is **very very** complicated. It occurs more than two pages!

Six conjectured series for π^2 and other constants

Conjecture (Z. W. Sun, 2010): We have

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},$$

$$\sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2,$$

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=-27\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2},$$

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

Conjecture involving x^2+7y^2

Conj. A1. Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac p7)=1\ \&\ p=x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac p7)=-1.\end{cases}$$

Remark. With helps of many sophisticated combinatorial identities, I recently proved that

$$\sum_{k=0}^{(p-1)/2}(21k+8)\binom{2k}{k}^3\equiv 8p+\left(\frac{-1}{p}\right)32p^3E_{p-3}\pmod{p^4}$$

which has the equivalent form

$$\sum_{k=1}^{(p-1)/2}\frac{21k-8}{k^3\binom{2k}{k}^3}\equiv(-1)^{(p+1)/2}4E_{p-3}\pmod{p}.$$

In 1993 via the WZ method D. Zeilberger obtained that

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}=\zeta(2)=\frac{\pi^2}{6}.$$

Conjecture involving $x^2 + 11y^2$

Conj. A4. Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k}$$

$$\equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{11}) = 1 \ \&\ 4p = x^2 + 11y^2\ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{p}{11}) = -1, \text{ i.e., } p \equiv 2, 6, 7, 8, 10 \pmod{11}. \end{cases}$$

Furthermore,

$$\sum_{k=0}^{p-1} (11k + 3) \frac{\binom{2k}{k}^2 \binom{3k}{k}}{64^k} \equiv 3p + \frac{7}{2} p^4 B_{p-3} \pmod{p^5},$$

$$p \sum_{k=1}^{(p-1)/2} \frac{(11k - 3)64^k}{k^3 \binom{2k}{k}^2 \binom{3k}{k}} \equiv 32 \frac{2^{p-1} - 1}{p} - \frac{64}{3} p^2 B_{p-3} \pmod{p^3}.$$

Remark. It is well-known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $(\frac{p}{11}) = 1$ we can write $4p = x^2 + 11y^2$ with $x, y \in \mathbb{Z}$.

Conjecture involving $x^2 + 163y^2$

Conj. A12. Let $p > 5$ be a prime with $p \neq 23, 29$.

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k,k,k}}{(-640320)^{3k}} \equiv \begin{cases} (\frac{-10005}{p})(x^2 - 2p) \pmod{p^2} & \text{if } (\frac{p}{163}) = 1 \ \&\ 4p = x^2 + 163y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{163}) = -1. \end{cases}$$

Remark. It is well known that the only imaginary quadratic fields with class number one are those $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 7, 11, 19, 43, 67, 163$. For each of the 9 values of d we have corresponding conjectures similar to the above one.

Conjecture for $\mathbb{Q}(\sqrt{-d})$ with class number two

Let $d > 0$ be a squarefree integer. It is known that $\mathbb{Q}(\sqrt{-d})$ has class number two if and only if d is among

$5, 6, 10, 13, 15, 22, 35, 37, 51, 58, 91, 115, 123, 187, 235, 267, 403, 427.$

Except for $d = 35, 91, 115, 187, 235, 403, 427$ we have found explicit conjectures involving $x^2 + dy^2$.

Conj. A13 (for $\mathbb{Q}(\sqrt{-15})$). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 4 \pmod{15} \ \&\ p = x^2 + 15y^2, \\ 20x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2, 8 \pmod{15} \ \&\ p = 5x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1. \end{cases}$$

Also, for any $a \in \mathbb{Z}^+$ we have

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} \frac{15k+4}{(-27)^k} \binom{2k}{k}^2 \binom{3k}{k} \equiv 4\left(\frac{p^a}{3}\right) \pmod{p^2}.$$

More Conjectures on Congruences

For more conjectures of mine on congruences, see

Z. W. Sun, *Open Conjectures on Congruences*,

arXiv:0911.5665.

You are welcome to solve my conjectures!

Thank you!