Given at the Hangzhou Conf. on q-Series and Combin. (April 28–May 1, 2023)
and the 16th Int. Conf. Number Theory & Related Prob. (May 19-23, 2023)

Series and Congruences involving Harmonic Numbers

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May 20, 2023

Abstract

In this talk we introduce various series involving harmonic numbers. In particular, we focus on how the speaker found many conjectural series with summands involving harmonic numbers.

Rogers & Straub [Int. JNT 9(2013)] proved the 520-series

A solution of Sun's $520 challenge concerning $\frac{520}{\pi}$

SIAM Annual Meeting, San Diego
Symbolic Computation and Special Functions

Armin Straub

July 10, 2013

University of Illinois at Urbana–Champaign & Max-Planck-Institut für Mathematik, Bonn

Based on joint work with:

Mathew Rogers
University of Montreal

$520 prize for the 520-series

Sun's challenge

CONJ

$$\frac{520}{\pi} = \sum_{n=0}^{\infty} \frac{1054n+233}{480^n} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} (-1)^k 8^{2k-n}$$

- roughly, each two terms of the outer sum give one correct digit

> *I would like to offer $520 (520 US dollars) for the person who could give the first correct proof of (*) in 2012 because May 20 is the day for Nanjing University.*
>
> **Zhi-Wei Sun** (2011)

Harmonic numbers

Harmonic numbers:

$$H_n := \sum_{0<k\leqslant n} \frac{1}{k} \quad (n = 0, 1, 2, \ldots).$$

Harmonic numbers of order m:

$$H_n^{(m)} := \sum_{0<k\leqslant n} \frac{1}{k^m} \quad (n = 0, 1, 2, \ldots).$$

J. Wolstenholme (1862): For any prime $p > 3$, we have

$$H_{p-1} \equiv 0 \pmod{p^2},\ H_{p-1}^{(2)} \equiv 0 \pmod{p},\ \binom{2p-1}{p-1} \equiv 1 \pmod{p^3}.$$

J.W.L. Glaisher (1900): Let $p > 3$ be a prime. Then

$$H_{p-1}^{(m)} \equiv \begin{cases} \frac{pm}{m+1} B_{p-1-m} \pmod{p^2} & \text{if } m \in \{2, 4, \ldots, p-3\}, \\ -\frac{p^2 m(m+1)}{2(m+2)} B_{p-2-m} \pmod{p^3} & \text{if } m \in \{1, 3, \ldots, p-4\}, \end{cases}$$

where $B_0, B_1, B_2, \ldots$ are the Bernoulli numbers.

Basic series involving harmonic numbers

Basic series involving harmonic numbers:

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3) \text{ (Euler)},$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k^3} = \frac{\pi^4}{72} \text{ (Goldbach, 1742)},$$

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17}{360}\pi^4 \text{ (D. Borwein and J.M. Borwein, 1995)},$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12} \text{ (S.W. Coffman, 1987)},$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k2^k} = \frac{5}{8}\zeta(3) \text{ (B. Cloitre, 2004)}.$$

Arithmetic theory of harmonic numbers

For any prime p, those $H_k = \sum_{0<j\leqslant k} 1/j$ $(k = 1, \ldots, p-1)$ are p-adic integers.

Z.-W. Sun [Proc. AMS 140(2012), 415-428]: Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{p-1} H_k^2 \equiv 2p-2 \pmod{p^2},\ \ \sum_{k=1}^{p-1} H_k^3 \equiv 6 \pmod{p},$$

and

$$\sum_{k=1}^{p-1} k^2 H_k^2 \equiv -\frac{4}{9} \pmod{p}),\ \ \sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p}.$$

When $p > 5$, we have

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}.$$

Arithmetic theory of harmonic numbers (continued)

Z.-W. Sun and L.-L. Zhao [Colloq. Math. 130(2013), 67-78]: For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \frac{7}{24} pB_{p-3} \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p},$$

where the first congruence was originally conjectured by Sun [Proc. AMS 140(2012)].

Another Congruence (conjectured by Sun [Proc. AMS 140(2012)] and confirmed by R. Meštrović [Int. J. Number Theory 9(2012), 1081-1085]):

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv \frac{4}{5} pB_{p-5} \pmod{p^2} \text{ for any prime } p > 3.$$

Conjectural series involving harmonic numbers (2014)

Conjecture (Z.-W. Sun [Nanjing Univ. J. Math. Biquarterly 32(2015)] (i) We have

$$\sum_{k=1}^{\infty}\frac{H_{2k}+2/(3k)}{k^2\binom{2k}{k}}=\zeta(3),$$
$$\sum_{k=1}^{\infty}\frac{H_{2k}+2H_k}{k^2\binom{2k}{k}}=\frac{5}{3}\zeta(3),$$
$$\sum_{k=1}^{\infty}\frac{H_{2k}+17H_k}{k^2\binom{2k}{k}}=\frac{5}{2}\sqrt{3}\,\pi K,$$

where $K := L(2,(\frac{-3}{\cdot})=\sum_{k=1}^{\infty}(\frac{k}{3})/k^2$.
(ii) Let $p>3$ be a prime. Then

$$\sum_{k=1}^{(p-1)/2}\frac{3H_{2k}+2/k}{k^2\binom{2k}{k}}\equiv B_{p-3}\pmod{p},\ \cdots\cdots$$

Remark. Part (i) was confirmed by J. Ablinger [Experiment. Math. 26(2017)].

More conjectures made in 2014

It is known that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^3 \binom{2k}{k}} = \frac{2}{5}\zeta(3),$$

which plays an important role in Apéry's proof of the irrationality of $\zeta(3)$.

Conjecture (Z.-W. Sun, 2014).

$$\sum_{k=1}^{\infty} \frac{H_{2k} - H_k + 2/k}{k^4 \binom{2k}{k}} = \frac{11}{9}\zeta(5).$$

Remark. This was confirmed by J. Ablinger [Experiment. Math. 26(2017)] by symbolic computation via the software `Sigma`.

More conjectures made in 2014

Conjecture (Z.-W. Sun, 2014).

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{10H_k-3/k}{k^3\binom{2k}{k}}=\frac{\pi^4}{30}$$

and

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{H_{2k}+4H_k}{k^3\binom{2k}{k}}=\frac{2}{75}\pi^4.$$

This was confirmed by W. Chu [Contrib. Discrete. Math. 15(2020)] and also K. C. Au [arXiv:2201.01676].

Conjecture (Z.-W. Sun, 2014).

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^3\binom{2k}{k}}\left(H_k^{(3)}+\frac{1}{5k^3}\right)=\frac{2}{5}\zeta(3)^2.$$

This was confirmed by W. Chu [Contrib. Discrete. Math. 15(2020)].

Ramanujan-type series for $1/\pi$

General forms of Classical Ramanujan-type Series for $1/\pi$:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$
$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are totally 36 known Ramanujan-type series for $1/\pi$ with a, b, m rational. I prefer their forms in terms of binomial coefficients rather than hypergeometric series.

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Remark. This yielded the record for the calculation of π during 1989-1994.

Long's conjecture

Motivated by the Ramanujan series

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi}$$

L. Long [Pacific J. Math. 249(2011)] conjectured the congruence

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\sum_{j=1}^{k}\left(\frac{1}{(2j-1)^2}-\frac{1}{16j^2}\right)\equiv 0 \pmod{p}$$

for any odd prime p, which was confirmed by H. Swisher in 2015. Note that the congruence can be rewritten as

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(2)}-\frac{5}{16}H_k^{(2)}\right)\equiv 0 \pmod{p}.$$

Guo and Lian's conjecture

In 2022 I conjectured further that for any prime $p > 3$ we have

$$\sum_{k=0}^{(p-1)/2} (6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(2)} - \frac{5}{16}H_k^{(2)}\right) \equiv \frac{p}{4}\left(\frac{2}{p}\right)E_{p-3} \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(2)} - \frac{5}{16}H_k^{(2)}\right) \equiv \frac{p}{16}E_{p-3}\left(\frac{1}{4}\right) \pmod{p^2}.$$

In 2022 C. Wei [Ramanujan J.] deduced the two identities

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(2)} - \frac{5}{16}H_k^{(2)}\right) = -\frac{\sqrt{2}}{48}\pi$$

and

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}\left(H_{2k}^{(2)} - \frac{5}{16}H_k^{(2)}\right) = \frac{\pi}{12}$$

conjectured by Guo and Lian [J. Difference Equ. Appl. 27(2021)], as well as their q-analogues.

Wei and Ruan's work

Motivated by Bauer's series

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}$$

and Ramanujan's series

$$\sum_{k=0}^{\infty}(8k+1)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}=\frac{2\sqrt{3}}{\pi},$$

Wei and G. Ruan [arXiv:2210.01331] proved the two new identities:

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\left(H_{2k}^{(2)}-\frac{1}{2}H_k^{(2)}\right)=-\frac{\pi}{12},$$

$$\sum_{k=0}^{\infty}(8k+1)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\left(H_{2k}^{(2)}-\frac{5}{18}H_k^{(2)}\right)=\frac{\sqrt{3}\,\pi}{54}.$$

(Just like Guo and Lian, Wei and Ruan did not use second-order harmonic numbers.)

A series discoveries in Oct. 2022

Conjecture 1 (Z.-W. Sun, arXiv:2210.07238). We have

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\left(H_{2k}^{(2)}-\frac{25}{92}H_k^{(2)}\right)=\frac{2\pi}{69},$$

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\left(H_{2k}^{(3)}-\frac{43}{352}H_k^{(3)}\right)=\frac{555}{77}\cdot\frac{\zeta(3)}{\pi}-\frac{32}{11}G,$$

where $G=L(2,(\frac{-4}{\cdot}))=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}$ is the Catalan constant.

Remark. The first identity was later confirmed by C. Wei [arXiv:2211.1148].

Conjecture 2 (Z.-W. Sun, arXiv:2210.07238). We have

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\left(H_{2k}^{(3)}-\frac{7}{64}H_k^{(3)}\right)=\frac{57}{16}\cdot\frac{\zeta(3)}{\sqrt{2}\,\pi}-L,$$

where

$$L=L\left(2,\left(\frac{-8}{\cdot}\right)\right)=\sum_{n=1}^{\infty}\frac{(\frac{-8}{n})}{n^2}=\sum_{k=0}^{\infty}\frac{(-1)^{k(k-1)/2}}{(2k+1)^2}.$$

Zeilberger-type series

In 1993, D. Zeilberger used the Wilf-Zeilberger method to obtain the new identity

$$\sum_{k=1}^{\infty} \frac{21k-8}{k^3 \binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}.$$

Define

$$F(n,k) = \frac{1}{\binom{2n}{n}(n+1)^2 \binom{2n+k+1}{n+1}^2}$$

and

$$G(n,k) = \frac{n!^4 (n+k)!^2}{2(2n+1)!(2n+k+2)!^2} P(n,k),$$

where $P(n,k)$ denotes

$$(n+1)^2(21n+13) + 2k^3 + k^2(13n+11) + k(28n^2+48n+20).$$

Then $\langle F, G \rangle$ is a **WZ pair** in the sense that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k).$$

Other Zeilberger-type series

J. Guillera [Ramanujan J. 15(2008)] used the WZ method to give three new Zeilberger-type series:

$$\sum_{k=1}^{\infty} \frac{(4k-1)(-64)^k}{k^3\binom{2k}{k}^3} = -16G,$$

$$\sum_{k=1}^{\infty} \frac{(3k-1)(-8)^k}{k^3\binom{2k}{k}^3} = -2G,$$

$$\sum_{k=1}^{\infty} \frac{(3k-1)16^k}{k^3\binom{2k}{k}^3} = \frac{\pi^2}{2},$$

where G denotes the Catalan constant $\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2}$.

Q.-H. Hou, C. Krattenthaler and Z.-W. Sun [Proc. Amer. Math. Soc. 147(2019)] provided a q-analogue of the last identity:

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} \frac{1-q^{3n+2}}{1-q} \cdot \frac{(q;q)_n^3(-q;q)_n}{(q^3;q^2)_n^3} = (1-q)^2 \frac{(q^2;q^2)_\infty^4}{(q;q^2)_\infty^4},$$

where $|q| < 1$, $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, $(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k)$.

My serial discoveries in Oct. 2022

Conjecture 3 (Z.-W. Sun, arXiv:2210.07238). (i) We have

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(2)}-\frac{25}{8}H_{k-1}^{(2)}\right)=\frac{47\pi^4}{2880},$$

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(3)}+\frac{43}{8}H_{k-1}^{(3)}\right)=\frac{711}{28}\zeta(5)-\frac{29}{14}\pi^2\zeta(3).$$

(ii) We have

$$\sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(2)}-\frac{5}{4}H_{k-1}^{(2)}\right)=\frac{\pi^4}{24},$$

$$\sum_{k=1}^{\infty}\frac{(3k-1)16^k}{k^3\binom{2k}{k}^3}\left(H_{2k-1}^{(3)}+\frac{7}{8}H_{k-1}^{(3)}\right)=\frac{\pi^2}{2}\zeta(3).$$

Remark. The first identity in part (ii) was confirmed by C. Wei [arXiv:2211.1148] and also K. C. Au [arXiv:2212.02986]. The first identity in part (i) was confirmed by K. C. Au [arXiv:2212.02986].

Au's method

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

K. C. Au [arXiv:2212.02986] used the WZ method to obtain the identity with a, b, c, d near 0:

$$\sum_{k=0}^{\infty} \frac{(a+1)_k(b+1)_k}{(c+k+1)(d+k+1)(c+1)_k(d+1)_k}$$
$$= \sum_{n=1}^{\infty} \frac{(a+1)_n(b+1)_n(c-a+1)_n(d-a+1)_n(c-b+1)_n(d-b+1)_n P(n)}{(c+1)_{2n}(d+1)_{2n}(c+d-a-b+1)_{2n} Q(n)},$$

where

$$Q(n) = (a+n)(b+n)(a-c-n)(a-d-n)(c-b+n)(d-b+n),$$

and $P(n)$ is a very complicated polynomial in a, b, c, d, n.

Au's method

Expanding both sides at $(a, b, c, d) = (0, 0, 0, 0)$, Au recovered Zeilberger's series

$$\sum_{n=1}^{\infty} \frac{(21n-8)(1)_n^6}{n^3(1)_{2n}^3} = \zeta(2).$$

Let $[a^i b^j c^k d^l]$ denote the coefficient of $a^i b^j c^k d^l$ of the identity obtained by Au (on the last page). Via computing $\frac{11}{4}[a^2] + [ac] + \frac{5}{8}[ab]$, he confirmed the identity

$$\sum_{n=1}^{\infty} \frac{21n-8}{n^3\binom{2n}{n}^3} \left(H_{2n-1}^{(2)} - \frac{25}{8} H_{n-1}^{(2)} \right) = \frac{47\pi^4}{2880}$$

conjectured by the speaker.

New series with summands involving harmonic numbers

Via a similar method, K. C. Au [arXiv:2212.02986] also proved that

$$\sum_{k=1}^{\infty}\frac{(1)_k^6}{(1)_{2k}^3}\left(\frac{21k-8}{k^3}(H_{2k}-H_k)+\frac{7-4k}{k^4}\right)=\zeta(3),$$
$$\sum_{k=1}^{\infty}\frac{4^{2k}(1)_k^6}{(1)_{2k}^3}\left(\frac{3k-1}{k^3}(H_{2k}-H_k)+\frac{2k-1}{2k^4}\right)=\frac{\pi^2}{3}\log 2+\frac{7}{6}\zeta(3).$$

On Dec. 4, 2022, I rewrote these two identities in better form. For example, the first one has the equivalent form:

$$\sum_{k=1}^{\infty}\frac{(21k-8)(H_{2k-1}-H_{k-1})-7/2}{k^3\binom{2k}{k}^3}=\zeta(3).$$

This form inspired me to discover many new conjectural series involving harmonic numbers.

Series with binomial coefficients in the denominators

In 2010 Z.-W. Sun conjectured that

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},$$

which was confirmed by J. Guillera and M. Rogers in 2014.

Conjecture (Sun, 2022-12-05). We have

$$\sum_{k=1}^{\infty}\frac{8^k((10k-3)(H_{2k-1}-H_{k-1})-1)}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{7}{2}\zeta(3)$$

and

$$\sum_{k=1}^{\infty}\frac{8^k((10k-3)(H_{3k-1}-H_{k-1})-8/3)}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{2\pi^2\log 2+7\zeta(3)}{4}.$$

Series with binomial coefficients in the denominators

In 2010 Z.-W. Sun conjectured that

$$\sum_{k=1}^{\infty} \frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}} = 8\pi^2,$$

which was later confirmed by J. Guillera.

Conjecture (Sun, 2022-12-05). We have

$$\sum_{k=1}^{\infty} \frac{64^{k-1}((11k-3)(2H_{2k-1}+H_{k-1})-4)}{k^3\binom{2k}{k}^2\binom{3k}{k}} = \frac{7}{2}\zeta(3)$$

and

$$\sum_{k=1}^{\infty} \frac{64^{k-1}((11k-3)(3H_{3k-1}-6H_{k-1})-7)}{k^3\binom{2k}{k}^2\binom{3k}{k}} = \frac{6\pi^2\log 2-21\zeta(3)}{8}.$$

Series with binomial coefficients in the denominators

In 2010 Z.-W. Sun conjectured that

$$\sum_{k=1}^{\infty} \frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}} = 12\pi^2,$$

which was confirmed by J. Guillera and M. Rogers in 2014.

Conjecture (Sun, 2022-12-09). We have

$$\sum_{k=1}^{\infty} \frac{81^k((35k-8)(H_{4k-1}-H_{k-1})-35/4)}{k^3\binom{2k}{k}^2\binom{4k}{2k}} = 12\pi^2 \log 3 + 39\zeta(3).$$

A General Conjecture

Part (i) of the General Conjecture (Z.-W. Sun, Dec. 2022). If we have an identity

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k}=\frac{c\sqrt{d}}{\pi}$$

with $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and d is a positive squarefree integer, then

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3}{m^k}(6(ak+b)(H_{2k}-H_k)+a)=c\sqrt{d}\frac{\log|m|}{\pi},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{m^k}(6(ak+b)(H_{2k}-H_k)+a)$$
$$\equiv\left(\frac{-d}{p}\right)(a+b(m^{p-1}-1))\pmod{p^2}$$

for any prime $p\nmid dm$.

Part (ii) of the General Conjecture

Part (ii) of the General Conjecture (Z.-W. Sun, Dec. 2022). If we have an identity

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}=\frac{c\sqrt{d}}{\pi}$$

with $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and d is a positive squarefree integer, then

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}((ak+b)(3H_{3k}+2H_{2k}-5H_k)+a)=c\sqrt{d}\frac{\log|m|}{\pi},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k}\left((ak+b)(3H_{3k}+2H_{2k}-5H_k)+a\right)$$
$$\equiv\left(\frac{-d}{p}\right)(a+b(m^{p-1}-1))\pmod{p^2}$$

for any odd prime $p\nmid dm$.

Parts (iii) of the General Conjecture

Part (ii) of the General Conjecture (Z.-W. Sun, Dec. 2022). If we have an identity

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}=\frac{c\sqrt{d}}{\pi}$$

with $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and d is a positive squarefree integer, then

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}(4(ak+b)(H_{4k}-H_k)+a)=c\sqrt{d}\frac{\log|m|}{\pi},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}(4(ak+b)(H_{4k}-H_k)+a)$$
$$\equiv\left(\frac{-d}{p}\right)\left(a+b(m^{p-1}-1)\right)\pmod{p^2}$$

for any odd prime $p\nmid dm$.

Parts (iv) of the General Conjecture

Part (iv) of the General Conjecture (Z.-W. Sun, Dec. 2022). If we have an identity

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}=\frac{c\sqrt{d}}{\pi}$$

with $a,b,m\in\mathbb{Z}$, $am\neq 0$, $c\in\mathbb{Q}\setminus\{0\}$, and d is a positive squarefree integer, then

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}(3(ak+b)(2H_{6k}-H_{3k}-H_k)+a)=c\sqrt{d}\frac{\log|m|}{\pi},$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}(3(ak+b)(2H_{6k}-H_{3k}-H_k)+a)$$
$$\equiv\left(\frac{-d}{p}\right)\left(a+b(m^{p-1}-1)\right)\pmod{p^2}$$

for any odd prime $p\nmid dm$.

Remark. Having seen this conjecture posted to MathOverflow, K. C. Au provided a rough idea for proving those identities.

More conjectural series

Conjecture (Z.-W. Sun, arXiv:2210.07238). We have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}\left((6k+1)(H_{2k}-2H_k)+3\right)=\frac{9\sqrt{3}\log 3}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}(6k+1)(3H_{3k}-H_k)=\frac{9\sqrt{3}\log 2}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\left((8k+1)(3H_{2k}-4H_k)+6\right)=\frac{16\sqrt{3}\log 2}{\pi}.$$

Remark. This is motivated by the Ramanujan series

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}=\frac{3\sqrt{3}}{\pi}\ \text{and}\ \sum_{k=0}^{\infty}(8k+1)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}=\frac{2\sqrt{3}}{\pi}.$$

Powers of $\arcsin x$

By taking derivatives of both sides of the identity

$$\left(\arcsin\frac{x}{2}\right)^3 = 3\sum_{k=0}^{\infty}\frac{\binom{2k}{k}x^{2k+1}}{(2k+1)16^k}\sum_{0\leqslant j<k}\frac{1}{(2j+1)^2}\quad (|x|<2),$$

we get

$$3\left(\arcsin\frac{x}{2}\right)^2\times\frac{1/2}{\sqrt{1-(x/2)^2}} = 3\sum_{k=0}^{\infty}\frac{\binom{2k}{k}x^{2k}}{16^k}\sum_{0\leqslant j<k}\frac{1}{(2j+1)^2}$$

and hence

$$\frac{(\arcsin(x/2))^2}{\sqrt{4-x^2}} = \sum_{k=1}^{\infty}\frac{\binom{2k}{k}x^k}{16^k}\sum_{j=1}^{k}\frac{1}{(2j-1)^2}.$$

Thus we have

$$\frac{(\arcsin(x/2))^2}{\sqrt{4-x^2}} = \sum_{k=1}^{\infty}\frac{\binom{2k}{k}x^{2k}}{16^k}\left(H_{2k}^{(2)}-\frac{1}{4}H_k^{(2)}\right).$$

Series with summands involving only one binomial coefficient

Conjecture (Sun, 2022-11-14) We have the identity

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k}\left(H_{2k}^{(3)}-\frac{1}{8}H_k^{(3)}\right)=\frac{35\sqrt{2}}{64}\zeta(3)-\frac{\sqrt{2}}{8}\pi G.$$

Remark. In contrast, we have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k}\left(H_{2k}^{(2)}-\frac{1}{4}H_k^{(2)}\right)=\frac{\pi^2}{16\sqrt{2}}.$$

`Mathematica` yields that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k}H_k=-\sqrt{2}\log(12-8\sqrt{2})$$

and

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{8^k}H_{2k}=\frac{\log(3/2+\sqrt{2})}{\sqrt{2}}.$$

Series with summands involving only one binomial coefficient

Conjecture (Sun, 2022-11-14) We have the identity

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} \left(H_{2k}^{(3)} - \frac{1}{8} H_k^{(3)} \right) = \frac{2\zeta(3)}{3\sqrt{3}} - \frac{\pi K}{8},$$

where

$$K := L\left(2, \left(\frac{-3}{\cdot}\right)\right) = \sum_{k=1}^{\infty} \frac{\left(\frac{k}{3}\right)}{k^2}.$$

Remark In contrast, we have

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} \left(H_{2k}^{(2)} - \frac{1}{4} H_k^{(2)} \right) = \frac{\pi^2}{36\sqrt{3}}.$$

`Mathematica` yields that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} H_k = -\frac{2}{\sqrt{3}} \log(84-48\sqrt{3}) \text{ and } \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{16^k} H_{2k} = \frac{\log((7+4\sqrt{3})/9)}{\sqrt{3}}.$$

Series with summands involving two binomial coefficients

Conjecture (Sun, 2022-12-30). We have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}(3H_{3k}-H_k)=\left(\log\frac{8}{9}\right)\sum_{k=0}^{\infty}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}.$$

Remark For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\pmod{p^2}$$

by Sun [Finite Fields Appl., 2013], and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\begin{cases}\binom{(2p-2)/3}{(p-1)/3}\pmod{p^2} & \text{if } p\equiv 1\pmod 3,\\ p/\binom{(2p+2)/3}{(p+1)/3}\pmod{p^2} & \text{if } p\equiv 2\pmod 3,\end{cases}$$

as conjectured by Z.-W. Sun [Sci. China Math., 2011] and proved by C. Wang and Sun [J. Math. Anal. Appl., 2022].

The speaker actually has made several similar conjectures.

Conjectural series for $\zeta(4)$ and $\zeta(5)$

In 2010, via p-adic congruences the speaker conjectured that

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

This was confirmed by K. C. Au in 2022.

Conjecture (Sun, 2022-12-09) (i) We have

$$\sum_{k=1}^{\infty}\frac{(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}\left((28k^2-18k+3)(4H_{2k-1}-3H_{k-1})-20k+6\right)=\frac{\pi^4}{2}$$

and

$$\sum_{k=1}^{\infty}\frac{(-64)^k((28k^2-18k+3)(2H_{2k-1}^{(2)}-3H_{k-1}^{(2)})-2)}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-31\zeta(5).$$

Remark. We also have corresponding conjectural p-adic congruences.

Conjectural series for $(\log 24)/\pi^2$

The following conjecture was motivated by the known series

$$\sum_{k=0}^{\infty}(252k^2+63k+5)\frac{\binom{2k}{k}^3\binom{3k}{k}\binom{4k}{2k}}{(-24^4)^k}=\frac{48}{\pi^2}.$$

Conjecture (Sun, 2022-12-09) (i) We have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^3\binom{3k}{k}\binom{4k}{2k}}{(-24^4)^k}\left((252k^2+63k+5)(4H_{4k}+3H_{3k}-7H_k)+504k+63\right)$$
$$=\frac{192\log 24}{\pi^2}.$$

(ii) For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3\binom{3k}{k}\binom{4k}{2k}}{(-24^4)^k}\left((252k^2+63k+5)(4H_{4k}+3H_{3k}-7H_k)+504k+63\right)$$
$$\equiv 63p+5p^2q_p(24^4)-\frac{5}{2}p^3q_p(24^4)^2 \pmod{p^4},$$

where $q_p(m)$ denotes the Fermat quotient $(m^{p-1}-1)/p$.

Conjectural series for $(\log 10)/\pi^2$

The following conjecture was motivated by the conjectural identity

$$\sum_{k=0}^{\infty}(532k^2+126k+9)\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{10^{6k}}=\frac{375}{4\pi^2}.$$

Conjecture (Sun, 2023-01-16) (i) We have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{10^{6k}}\left(3(532k^2+126k+9)(H_{6k}-H_k)+532k+63\right)$$
$$=\frac{1125\log 10}{4\pi^2}.$$

(ii) For any odd prime $p\neq 5$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}^2\binom{6k}{3k}}{10^{6k}}\left(3(532k^2+126k+9)(H_{6k}-H_k)+532k+63\right)$$
$$\equiv 63p+\frac{9}{2}p^2q_p(10^6)-\frac{9}{4}p^3q_p(10^6)^2 \pmod{p^4}.$$

More such conjectural series

Conjecture (Z.-W. Sun, 2023-01-17). (i) For $k \in \mathbb{N}$, set

$$H(k) := 6H_{6k} + 4H_{4k} - 3H_{3k} - 2H_{2k} - 5H_k.$$

Then

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2\binom{3k}{k}\binom{4k}{2k}\binom{6k}{3k}}{(-2^{22}3^3)^k}\left((1640k^2+278k+15)H(k)+3280k+278\right)$$
$$= \frac{256}{\sqrt{3}\pi^2}\log(2^2 23^3).$$

(ii) For $k \in \mathbb{N}$, set

$$\mathcal{H}(k) := 4H_{8k} - 2H_{4k} + H_{2k} - 3H_k.$$

Then

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^3\binom{4k}{2k}\binom{8k}{4k}}{(2^{18}7^4)^k}\left((1920k^2+304k+15)\mathcal{H}(k)+1920k+152\right)$$
$$= \frac{56\sqrt{7}}{\pi^2}(9\log 2 + 2\log 7).$$

A conjectural series for $(\log 2)/\pi^3$

The following conjecture was motivated by the identity

$$\sum_{k=0}^{\infty}(168k^3+76k^2+14k+1)\frac{\binom{2k}{k}^7}{2^{20k}}=\frac{32}{\pi^3}$$

conjectured by B. Gourevich.

Conjecture (Sun, 2022-12-09) We have

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7}{2^{20k}}\left(7(168k^3+76k^2+14k+1)(H_{2k}-H_k)+252k^2+76k+7\right)$$
$$=\frac{320\log 2}{\pi^3}.$$

Also,

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^7}{2^{20k}}\Big((168k^3+76k^2+14k+1)(16H_{2k}^{(2)}-5H_k^{(2)})+8(6k+1)\Big)=\frac{80}{3\pi}.$$

A conjectural series for π^6

The following conjecture is motivated by the identity

$$\sum_{k=1}^{\infty}\frac{(21k^3-22k^2+8k-1)256^k}{k^7\binom{2k}{k}^7}=\frac{\pi^4}{8}$$

conjectured by Guillera in 2003.

Conjecture (Sun, 2022-12-09) (i) We have

$$\sum_{k=1}^{\infty}\frac{256^k}{k^7\binom{2k}{k}^7}\Big((21k^3-22k^2+8k-1)(4H_{2k-1}^{(2)}-5H_{k-1}^{(2)})-6k+2\Big)=\frac{\pi^6}{24}.$$

(ii) For any odd prime p, we have

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^7}{256^k}\Big((21k^3+22k^2+8k+1)(4H_{2k}^{(2)}-5H_k^{(2)})+6k+2\Big)\equiv 2p \pmod{p^5}.$$

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Thank you!