Restricted Sums of Three or Four Squares

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Abstract

The classical Gauss-Legendre theorem states that each \( n = 0, 1, 2, \ldots \) not of the form \( 4^k(8m + 1) \) \((k, m = 0, 1, 2, \ldots)\) can be written as the sum of three squares. Lagrange’s four-square theorem asserts that any natural number can be written as the sum of four squares. In this talk we introduce various conjectures refining these two theorems; for example, our 24-conjecture states that any nonnegative integers can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \) nonnegative integers such that both \( x \) and \( x + 24y \) are squares. We will also mention some recent results as well as related techniques in this direction.
Part I. The Birth of the 1-3-5-Conjecture
Lagrange’s theorem

**Lagrange’s Theorem.** Each \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \) can be written as the sum of four squares.

*Examples.* \( 3 = 1^2 + 1^2 + 1^2 + 0^2 \) and \( 7 = 2^2 + 1^2 + 1^2 + 1^2 \).

A. Diophantus (AD 299-215, or AD 285-201) was aware of this theorem as indicated by examples given in his book *Arithmetica*.

In 1621 Bachet translated Diophantus’ book into Latin and stated the theorem in the notes of his translation.

In 1748 L. Euler found the four-square identity

\[
(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = (x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2 \\
+ (x_1y_3 - x_3y_1 + x_2y_4 - x_4y_2)^2 + (x_1y_4 - x_4y_1 - x_2y_3 + x_3y_2)^2.
\]

and hence reduced the theorem to the case with \( n \) prime.

The theorem was first proved by J. L. Lagrange in 1770.
The representation function $r_4(n)$

It is known that only the following numbers have a unique representation as the sum of four unordered squares:

$$1, 3, 5, 7, 11, 15, 23$$

and

$$2^{2k+1}m \ (k = 0, 1, 2, \ldots \text{ and } m = 1, 3, 7).$$

For example, $4^k \times 14 = (2^k3)^2 + (2^{k+1})^2 + (2^k)^2 + 0^2$.

Jacobi considered the fourth power of the theta function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and this led him to show that

$$r_4(n) = 8 \sum_{d|n \text{ and } 4|d} d \quad \text{for all } n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\},$$

where

$$r_4(n) := |\{(w, x, y, z) \in \mathbb{Z}^4 : w^2 + x^2 + y^2 + z^2 = n\}|.$$
Natural numbers as sums of polygonal numbers

For $m = 3, 4, 5, \ldots$, the polygonal numbers of order $m$ (or $m$-gonal numbers) are given by

$$p_m(n) := (m - 2) \binom{n}{2} + n \ (n = 0, 1, 2, \ldots).$$

Clearly, $p_4(n) = n^2$, $p_5(n) = n(3n - 1)/2$ and $p_6(n) = n(2n - 1)$.

Fermat’s Claim. Let $m \geq 3$ be an integer. Then any $n \in \mathbb{N}$ can be written as the sum of $m$ polygonal numbers of order $m$.

This was proved by Lagrange in the case $m = 4$, by Gauss in the case $m = 3$, and by Cauchy in the case $m \geq 5$.

Conjecture (Z.-W. Sun, March 14, 2015). Each $n \in \mathbb{N}$ can be written as

$$p_5(x_1) + p_5(x_2) + p_5(x_3) + 2p_5(x_4) \ (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

Theorem (conjectured by the speaker and proved by X.-Z. Meng and Z.-W. Sun [Acta Arith. 180(2017)]). $n \in \mathbb{N}$ can be written as

$$p_6(x_1) + p_6(x_2) + 2p_6(x_3) + 4p_6(x_4) \ (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$
Upgrade Waring’s problem

In 1770 E. Waring proposed the following famous problem.

**Waring’s Problem.** Whether for each integer \( k > 1 \) there is a positive integer \( g(k) = r \) (as small as possible) such that every \( n \in \mathbb{N} \) can be written as

\[
x_1^k + x_2^k + \ldots + x_r^k \quad \text{with} \quad x_1, \ldots, x_r \in \mathbb{N}.
\]

In 1909 D. Hilbert proved that \( g(k) \) always exists. It is conjectured that

\[
g(k) = 2^k + \left\lceil \left( \frac{3}{2} \right)^k \right\rceil - 2.
\]

**New Problem** (Z.-W. Sun, March 30-31, 2016). Determine \( t(k) \) for any integer \( k > 1 \), where \( t(k) \) is the smallest positive integer \( t \) such that

\[
\{ a_1x_1^k + a_2x_2^k + \ldots + a_tx_t^k : x_1, \ldots, x_t \in \mathbb{N} \} = \mathbb{N}
\]

for some \( a_1, \ldots, a_t \in \mathbb{Z}^+ \) with \( a_1 + a_2 + \ldots + a_t = g(k) \).
A conjecture

**Conjecture** (Z.-W. Sun, March 30-31, 2016). (i) $t(3) = 5$. In fact,

$$\{ u^3 + v^3 + 2x^3 + 2y^3 + 3z^3 : u, v, x, y, z \in \mathbb{N} \} = \mathbb{N}.$$

(ii) $t(4) = 7$. In fact, we have

$$\{ x_1^4 + x_2^4 + 2x_3^4 + 2x_4^4 + 3x_5^4 + 3x_6^4 + 7x_7^4 : x_1, \ldots, x_7 \in \mathbb{N} \} = \mathbb{N},$$
$$\{ x_1^4 + x_2^4 + 2x_3^4 + 2x_4^4 + 3x_5^4 + 4x_6^4 + 6x_7^4 : x_1, \ldots, x_7 \in \mathbb{N} \} = \mathbb{N}.$$

(iii) $t(5) = 8$. In fact,

$$\{ x_1^5 + x_2^5 + 2x_3^5 + 3x_4^5 + 4x_5^5 + 5x_6^5 + 7x_7^5 + 14x_8^5 : x_1, \ldots, x_8 \in \mathbb{N} \} = \mathbb{N},$$
$$\{ x_1^5 + x_2^5 + 2x_3^5 + 3x_4^5 + 4x_5^5 + 6x_6^5 + 8x_7^5 + 12x_8^5 : x_1, \ldots, x_8 \in \mathbb{N} \} = \mathbb{N}.$$

(iv) $t(6) = 10$. In fact,

$$\{ x_1^6 + x_2^6 + x_3^6 + 2x_4^6 + 3x_5^6 + 5x_6^6 + 6x_7^6 + 10x_8^6 + 18x_9^6 + 26x_{10}^6 : x_i \in \mathbb{N} \} = \mathbb{N}.$$

(v) In general, $t(k) \leq 2k - 1$ for any integer $k > 2$. 
Discoveries on April 8, 2016

Motivated by my conjecture that any $n \in \mathbb{N}$ can be written as

$$x_1^3 + x_2^3 + 2x_3^3 + 2x_4^3 + 3x_5^3 \ (x_1, x_2, x_3, x_4, x_5 \in \mathbb{N}),$$

on April 8, 2016 I considered to write $n \in \mathbb{N}$ as $\sum_{i=1}^{5} a_i x_i^2$ ($x_i \in \mathbb{N}$) with certain restrictions on $x_1, \ldots, x_5$.

**Conjecture (Z.-W. Sun)** Let $n > 1$ be an integer.

(i) $n$ can be written as

$$x_1^2 + x_2^2 + x_3^2 + 2x_4^2 + 2x_5^2 = x_1^2 + x_2^2 + x_3^2 + (x_4 + x_5)^2 + (x_4 - x_5)^2 \ (x_i \in \mathbb{N})$$

with $x_1 + x_2 + x_3 + x_4 + x_5$ prime.

(ii) We can write $n$ as

$$x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 5x_5^2 \ (x_1, x_2, x_3, x_4, x_5 \in \mathbb{N})$$

with $x_1 + x_2 + x_3 + x_4$ a square.

**Remark.** Squares are sparser than prime numbers.
1-3-5-Conjecture (1350 US dollars for the first solution)

1-3-5-Conjecture (Z.-W. Sun, April 9, 2016): Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.

Examples.

$$7 = 1^2 + 1^2 + 1^2 + 2^2 \text{ with } 1 + 3 \times 1 + 5 \times 1 = 3^2,$$
$$8 = 0^2 + 2^2 + 2^2 + 0^2 \text{ with } 0 + 3 \times 2 + 5 \times 2 = 4^2,$$
$$31 = 5^2 + 2^2 + 1^2 + 1^2 \text{ with } 5 + 3 \times 2 + 5 \times 1 = 4^2,$$
$$43 = 1^2 + 5^2 + 4^2 + 1^2 \text{ with } 1 + 3 \times 5 + 5 \times 4 = 6^2.$$

The conjecture has been verified by Qing-Hu Hou for all $n \leq 10^{10}$.

We guess that if $a, b, c$ are positive integers with $\gcd(a, b, c)$ squarefree such that each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $ax + by + cz$ a square then we must have $\{a, b, c\} = \{1, 3, 5\}$. 
Graph for the number of such representations of $n$
无　　解

digital 时有，
把酒问青天。
一二三四五，
自然藏玄机。

四个平方和，
遍历自然数。
奇妙一三五，
更上一层楼。

苍天捉弄人，
数论妙无穷。
吾辈虽努力，
难解一三五！

时势唤英雄，
攻关需豪杰。
人间若无解，
天神会证否？
Related conjectures

**Conjecture** (Z.-W. Sun, 2016): (i) Each \( n \in \mathbb{N} \setminus \{7, 15, 23, 71, 97\} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) \((x, y, z, w \in \mathbb{N})\) with \( x + 3y + 5z \) twice a square. Also, any \( n \in \mathbb{N} \setminus \{7, 43, 79\} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) \((x, y, z, w \in \mathbb{N})\) with \( 3x + 5y + 6z \) a square, and any \( n \in \mathbb{N} \setminus \{5, 7, 15\} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) \((x, y, z, w \in \mathbb{N})\) with \( 3x + 5y + 6z \) twice a square.

(ii) Let \( a, b, c \in \mathbb{Z}^+ \) with \( \gcd(a, b, c) \) squarefree. If there are only finitely many positive integers which cannot be written as \( x^2 + y^2 + z^2 + w^2 \) \((x, y, z, w \in \mathbb{N})\) with \( ax + by + cz \) a square, then \( \{a, b, c\} \) must be among

\[
\{1, 3, 5\}, \ \{2, 6, 10\}, \ \{3, 5, 6\}, \ \{6, 10, 12\}.\]

**Remark.** Qing-Hu Hou at Tianjin Univ. has verified part (i) for \( n \) up to \( 10^9 \).
1-2-3-Conjecture (Companion of 1-3-5-Conjecture)

1-2-3-Conjecture (Z.-W. Sun, July 24, 2016). Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + 2w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that \( x + 2y + 3z \) is a square.

Examples:

- \[ 14 = 1^2 + 1^2 + 2^2 + 2 \times 2^2 \] with \( 1 + 2 \times 1 + 3 \times 2 = 3^2 \),
- \[ 30 = 3^2 + 2^2 + 3^2 + 2 \times 2^2 \] with \( 3 + 2 \times 2 + 3 \times 3 = 4^2 \),
- \[ 33 = 1^2 + 0^2 + 0^2 + 2 \times 4^2 \] with \( 1 + 2 \times 0 + 3 \times 0 = 1^2 \),
- \[ 84 = 4^2 + 6^2 + 0^2 + 2 \times 4^2 \] with \( 4 + 2 \times 6 + 3 \times 0 = 4^2 \),
- \[ 169 = 10^2 + 6^2 + 1^2 + 2 \times 4^2 \] with \( 10 + 2 \times 6 + 3 \times 1 = 5^2 \),
- \[ 225 = 10^2 + 6^2 + 9^2 + 2 \times 2^2 \] with \( 10 + 2 \times 6 + 3 \times 9 = 7^2 \).

Another Conjecture (Sun, 2017). Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + 2w^2 \) with \( x, y, z, w \in \mathbb{Z} \) and \( x + 2y + 3z = 1 \).

Remark. This was proved by Hai-Liang Wu and Z.-W. Sun in 2017 for sufficient large integers \( n \).
Graph for the number of such representations of $n$
Part II. Sums of Four Squares with Linear Restrictions
Diagonal ternary quadratic forms

For $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots \}$, we define

$$E(a, b, c) := \{ n \in \mathbb{N} : n \neq ax^2 + by^2 + cz^2 \text{ for any } x, y, z \in \mathbb{N} \}.$$  

It is known that $E(a, b, c)$ is an infinite set.

**Gauss-Legendre Theorem.** $E(1, 1, 1) = \{ 4^k(8l + 7) : k, l \in \mathbb{N} \}$.

There are totally 102 diagonal ternary quadratic forms $ax^2 + by^2 + cz^2$ with $a, b, c \in \mathbb{Z}^+$ and $\gcd(a, b, c) = 1$ for which the structure of $E(a, b, c)$ is known explicitly. For example,

$$E(1, 1, 2) = \{ 4^k(16l + 14) : k, l \in \mathbb{N} \},$$

$$E(1, 1, 5) = \{ 4^k(8l + 3) : k, l \in \mathbb{N} \},$$

$$E(1, 2, 3) = \{ 4^k(16l + 10) : k, l \in \mathbb{N} \},$$

$$E(1, 2, 6) = \{ 4^k(8l + 5) : k, l \in \mathbb{N} \}.$$
\[ n = x^2 + y^2 + z^2 + w^2 \] with \( P(x, y, z) = 0 \)

**Theorem (Z.-W. Sun, 2016)** (i) Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that \( P(x, y, z) = 0 \), whenever \( P(x, y, z) \) is among the polynomials

\[
    x(x - y), \ x(x - 2y), \ (x - y)(x - 2y), \ (x - y)(x - 3y), \\
    x(x + y - z), \ (x - y)(x + y - z), \ (x - 2y)(x + y - z).
\]

(ii) Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that \( (2x - 3y)(x + y - z) = 0 \), provided that

\[
\{ x^2 + y^2 + 13z^2 : x, y, z \in \mathbb{N} \} \supseteq \{ 8q + 5 : q \in \mathbb{N} \}. \quad (\ast)
\]

**Remark.** We may require \( x(x - y) = 0 \) since \( E(1, 1, 1) \cap E(1, 1, 2) = \emptyset \). Also, we may require that \( xy \) (or \( 2xy \), or \( (x^2 + y^2)(x^2 + z^2) \)) is a square. It seems that \( (\ast) \) does hold.

**Lemma.** Let \( n \in \mathbb{N} \). Then \( n \not\in E(1, 2, 6) \) if and only if \( n = x^2 + y^2 + z^2 + w^2 \) for some \( x, y, z, w \in \mathbb{N} \) with \( x + y = z \). Also, \( n \not\in E(1, 2, 3) \) if and only if \( n = x^2 + y^2 + z^2 + w^2 \) for some \( x, y, z, w \in \mathbb{Z} \) with \( x + y = 2z \).
\[ n = x^2 + y^2 + z^2 + w^2 \text{ with } P(x, y, z, w) = 0 \]

**Conjecture** (Z.-W. Sun, 2016) Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that \( P(x, y, z, w) = 0 \), whenever \( P(x, y, z, w) \) is among the polynomials

\[
\begin{align*}
(x - y)(x + y - 3z), & \quad (x - y)(x + 2y - z), & \quad (x - y)(x + 2y - 2z), \\
(x - y)(x + 2y - 7z), & \quad (x - y)(x + 3y - 3z), & \quad (x - y)(x + 4y - 6z), \\
(x - y)(x + 5y - 2z), & \quad (x - 2y)(x + 2y - z), & \quad (x - 2y)(x + 2y - 2z), \\
(x - 2y)(x + 3y - 3z), & \quad (x + y - z)(x + 2y - 2z), \\
(x - y)(x + y + 3z - 3w), & \quad (x - y)(x - y + 3z - 5w), \\
(x - y)(3x + 3y + 3z - 5w), & \quad (x - y)(3x - 3y + 5z - 7w), \\
(x - y)(3x - 3y + 7z - 9w).
\end{align*}
\]
Sums of a fourth power and three squares

**Theorem** (Z.-W. Sun, March 27, 2016). Each \( n \in \mathbb{N} \) can be written as \( w^4 + x^2 + y^2 + z^2 \) with \( w, x, y, z \in \mathbb{N} \).

**Proof.** For \( n = 0, 1, 2, \ldots, 15 \), the result can be verified directly. Now let \( n \geq 16 \) be an integer and assume that the result holds for smaller values of \( n \).

**Case 1.** \( 16 | n \).

By the induction hypothesis, we can write

\[
\frac{n}{16} = x^4 + y^2 + z^2 + w^2 \quad \text{with } x, y, z, w \in \mathbb{N}.
\]

It follows that \( n = (2x)^4 + (4y)^2 + (4z)^2 + (4w)^2 \).

**Case 2.** \( n = 4^k q \) with \( k \in \{0, 1\} \) and \( q \equiv 7 \pmod{8} \).

In this case, \( n - 1 \not\in E(1, 1, 1) \), and hence \( n = 1^4 + y^2 + z^2 + w^2 \) for some \( y, z, w \in \mathbb{N} \).

**Case 3.** \( 16 \nmid n \) and \( n \neq 4^k (8l + 7) \) for any \( k \in \{0, 1\} \) and \( l \in \mathbb{N} \).

In this case, \( n \not\in E(1, 1, 1) \) and hence there are \( y, z, w \in \mathbb{N} \) such that \( n = 0^4 + y^2 + z^2 + w^2 \).
via a similar method, we have proved the following result.

**Theorem** (Z.-W. Sun, March-June, 2016). Let \( a \in \{1, 4\} \) and \( k \in \{4, 5, 6\} \). Then, each \( n \in \mathbb{N} \) can be written as
\[
aw^k + x^2 + y^2 + z^2
\]
with \( w, x, y, z \in \mathbb{N} \).

**Conjecture** (Z.-W. Sun) (i) (2015) Any \( n \in \mathbb{N} \) can be written as
\[
x^2 + y^3 + z^4 + 2w^4
\]
with \( x, y, z, w \in \mathbb{N} \).

(ii) (2016) Each \( n \in \mathbb{N} \) can be written as
\[
x^5 + y^4 + z^2 + 3w^2
\]
with \( x, y, z, w \in \mathbb{N} \). Also, any \( n \in \mathbb{N} \) can be represented as
\[
x^5 + y^4 + z^3 + T_w
\]
with \( x, y, z, w \in \mathbb{N} \).

(iii) [JNT 171(2016)] Any positive integer can be written as
\[
x^3 + y^2 + T_z
\]
with \( x, y \in \mathbb{N} \) and \( z \in \mathbb{Z}^+ \). Also, each \( n \in \mathbb{N} \) can be written as
\[
x^4 + y(3y + 1)/2 + z(7z + 1)/2
\]
with \( x, y, z \in \mathbb{Z} \).
Suitable polynomials

**Definition** (Z.-W. Sun, 2016). A polynomial $P(x, y, z, w)$ with integer coefficients is called *suitable* if any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z, w)$ is a square.

We have seen that both $x$ and $2x$ are suitable polynomials. The 1-3-5-Conjecture says that $x + 3y + 5z$ is suitable.

We conjecture that there only finitely many $a, b, c, d \in \mathbb{Z}$ with $\gcd(a, b, c, d)$ squarefree such that $ax + by + cz + dw$ is suitable, and we have found all such quadruples $(a, b, c, d)$. 
Let $a \in \{1, 2\}$. We claim that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $a(x - y)$ is a square, and want to prove this by induction.

For every $n = 0, 1, \ldots, 15$, we can verify the claim directly.

Now we fix an integer $n \geq 16$ and assume that the claim holds for smaller values of $n$.

Case 1. $16 | n$.
In this case, by the induction hypothesis, there are $x, y, z, w \in \mathbb{N}$ with $a(x - y)$ a square such that $n/16 = x^2 + y^2 + z^2 + w^2$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $a(4x - 4y)$ a square.

Case 2. $16 \nmid n$ and $n \not\in E(1, 1, 2)$.
In this case, there are $x, y, z, w \in \mathbb{N}$ with $x = y$ and $n = x^2 + y^2 + z^2 + w^2$, thus $a(x - y) = 0^2$ is a square.
Case 3. $16 \mid n$ and $n \in E(1, 1, 2) = \{4^k(16l + 14) : k, l \in \mathbb{N}\}$.

In this case, $n = 4^k(16l + 14)$ for some $k \in \{0, 1\}$ and $l \in \mathbb{N}$. Note that $n/2 - (2/a)^2 \notin E(1, 1, 1)$. So, $n/2 - (2/a)^2 = t^2 + u^2 + v^2$ for some $t, u, v \in \mathbb{N}$ with $t \geq u \geq v$. As $n/2 - (2/a)^2 \geq 8 - 4 > 3$, we have $t \geq 2 \geq 2/a$. Thus

$$n = 2 \left( \left( \frac{2}{a} \right)^2 + t^2 \right) + 2(u^2 + v^2)$$

$$= \left( t + \frac{2}{a} \right)^2 + \left( t - \frac{2}{a} \right)^2 + (u + v)^2 + (u - v)^2$$

with

$$a \left( \left( t + \frac{2}{a} \right) - \left( t - \frac{2}{a} \right) \right) = 2^2.$$ 

This proves that $x - y$ and $2x - 2y$ are both suitable.
Suitable polynomials of the form $ax \pm by$

**Conjecture** (Z.-W. Sun, April 14, 2016) Let $a, b \in \mathbb{Z}^+$ with $\gcd(a, b)$ squarefree.

(i) The polynomial $ax + by$ is suitable if and only if
\[
\{a, b\} = \{1, 2\}, \{1, 3\}, \{1, 24\}.
\]

(ii) The polynomial $ax - by$ is suitable if and only if $(a, b)$ is among the ordered pairs
\[
(1, 1), (2, 1), (2, 2), (4, 3), (6, 2).
\]

**Remark.** In 2016, I proved that any $n \in \mathbb{N}$ can be written as
\[
x^2 + y^2 + z^2 + w^2 \text{ with } x, y, z, w \in \mathbb{Z} \text{ such that } x + 2y \text{ is a square (or a cube). In 2017, in a joint paper with my student Yu-Chen Sun, we managed to show that } x + 2y \text{ is indeed suitable.}
Write \( n = x^2 + y^2 + z^2 + w^2 \) with \( x + 3y \) a square

In 1916 Ramanujan conjectured that

(1) the only positive even numbers not of the form \( x^2 + y^2 + 10z^2 \) are those \( 4^k(16l + 6) \) \((k, l \in \mathbb{N})\)

and

(2) sufficiently large odd numbers are of the form \( x^2 + y^2 + 10z^2 \).

In 1927 L. E. Dickson [Bull. AMS] proved (1). In 1990 W. Duke and R. Schulze-Pillot [Invent. Math.] confirmed (2). In 1997 K. Ono and K. Soundararajan [Invent. Math.] proved that under the GRH (Generalized Riemann Hypothesis) any odd number greater than 2719 has the form \( x^2 + y^2 + 10z^2 \).

With the help of the Ono-Soundararajan result, the speaker has proved the following result.

**Theorem** (Z.-W. Sun, 2016) Under the GRH, any \( n \in \mathbb{N} \) can be written as \( n = x^2 + y^2 + z^2 + w^2 \) \((x, y, z, w \in \mathbb{Z})\) with \( x + 3y \) a square.
Suitable $ax − by − cz$ or $ax + by − cz$

**Conjecture** (Z.-W. Sun, April 14, 2016): (i) Let $a, b, c ∈ \mathbb{Z}^+$ with $b ≤ c$ and $\gcd(a, b, c)$ squarefree. Then $ax − by − cz$ is suitable if and only if $(a, b, c)$ is among the five triples

$$(1, 1, 1), (2, 1, 1), (2, 1, 2), (3, 1, 2), (4, 1, 2).$$

(ii) Let $a, b, c ∈ \mathbb{Z}^+$ with $a ≤ b$ and $\gcd(a, b, c)$ squarefree. Then $ax + by − cz$ is suitable if and only if $(a, b, c)$ is among the following 52 triples

$$(1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (1, 2, 3), (1, 3, 1),
(1, 3, 3), (1, 4, 4), (1, 5, 1), (1, 6, 6), (1, 8, 6), (1, 12, 4), (1, 16, 1),
(1, 17, 1), (1, 18, 1), (2, 2, 2), (2, 2, 4), (2, 3, 2), (2, 3, 3), (2, 4, 1),
(2, 4, 2), (2, 6, 1), (2, 6, 2), (2, 6, 6), (2, 7, 4), (2, 7, 7), (2, 8, 2),
(2, 9, 2), (2, 32, 2), (3, 3, 3), (3, 4, 2), (3, 4, 3), (3, 8, 3), (4, 5, 4),
(4, 8, 3), (4, 9, 4), (4, 14, 14), (5, 8, 5), (6, 8, 6), (6, 10, 8), (7, 9, 7),
(7, 18, 7), (7, 18, 12), (8, 9, 8), (8, 14, 14), (8, 18, 8), (14, 32, 14),
(16, 18, 16), (30, 32, 30), (31, 32, 31), (48, 49, 48), (48, 121, 48).
### Conjecture (Z.-W. Sun, 2017)

Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x + y - z$ an integer cube, where $x, y, z, w \in \mathbb{N}$, $x \geq y \leq z$ and $x \equiv y \pmod{2}$.

### Conjecture (Z.-W. Sun, 2016)

For each $c = 1, 2, 4$, any $n \in \mathbb{N}$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w, x, y, z \in \mathbb{N}$ and $y \leq z$ such that $2x + y - z = ct^3$ for some $t \in \mathbb{N}$.

### Examples

- $8 = 0^2 + 2^2 + 2^2 + 0^2$ with $2 \times 0 + 2 - 2 = 0^3$,
- $13 = 2^2 + 0^2 + 3^2 + 0^2$ with $2 \times 2 + 0 - 3 = 1^3$,
- $2976 = 20^2 + 16^2 + 48^2 + 4^2$ with $2 \times 20 + 16 - 48 = 2^3$. 

\[ n = x^2 + y^2 + z^2 + w^2 \text{ with } x + y + z \text{ a square (or a cube)} \]

**Theorem** (Z.-W. Sun, April-May, 2016) Let \( c \in \{1, 2\} \) and \( m \in \{2, 3\} \). Then any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{Z} \) such that \( x + y + cz = t^m \) for some \( t \in \mathbb{Z} \).

**Proof for the Case** \( c = 1 \). For \( n = 0, \ldots, 4^m - 1 \) we can easily verify the desired result directly.

Now let \( n \in \mathbb{N} \) with \( n \geq 4^m \). Assume that any \( r \in \{0, \ldots, n - 1\} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{Z} \) such that \( x + y + z \in \{t^m : t \in \mathbb{Z}\} \). If \( 4^m \mid n \), then there are \( x, y, z, w \in \mathbb{Z} \) with \( x^2 + y^2 + z^2 + w^2 = n/4^m \) such that \( x + y + z = t^m \) for some \( t \in \mathbb{Z} \), and hence

\[
 n = (2^m x)^2 + (2^m y)^2 + (2^m z)^2 + (2^m w)^2
\]

with \( 2^m x + 2^m y + (2^m z) = 2^m (x + y + z) = (2t)^m \). Below we suppose that \( 4^m \nmid n \).
Continued the proof

It suffices to show that there are $x, y, z \in \mathbb{Z}$ and $\delta \in \{0, 1, 2^m\}$ such that

$$n = x^2 + (y + z)^2 + (z - y)^2 + (\delta - 2z)^2 = x^2 + 2y^2 + 6z^2 - 4\delta z + \delta^2.$$ 

(Note that $(y + z) + (z - y) + (\delta - 2z) = \delta \in \{t^m : t \in \mathbb{Z}\}$.)

Suppose that this fails for $\delta = 0$. As

$$E(1, 2, 6) = \{4^k(8l + 5) : k, l \in \mathbb{N}\},$$

$n = 4^k(8l + 5)$ for some $k, l \in \mathbb{N}$ with $k < m$. Clearly,

$$3n - 1 = \begin{cases} 
3(8l + 5) - 1 = 2(12l + 7) & \text{if } k = 0, \\
3 \times 4(8l + 5) - 1 = 8(12l + 7) + 3 & \text{if } k = 1.
\end{cases}$$

Thus, if $k \in \{0, 1\}$, then $3n - 1$ does not belong to

$$E(2, 3, 6) = \{3q + 1 : q \in \mathbb{N}\} \cup \{4^k(8l + 7) : k, l \in \mathbb{N}\},$$
Continue the proof

hence for some \( x, y, z \in \mathbb{Z} \) we have

\[
3n - 1 = 3x^2 + 6y^2 + 2(3z - 1)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 2z)) + 2
\]

and thus

\[
n = x^2 + 2y^2 + 6z^2 - 4z + 1 = x^2 + (y + z)^2 + (z - y)^2 + (1 - 2z)^2
\]

as desired.

When \( k = 2 \) and \( m = 3 \), we have

\[
3n - 64 = 3 \times 16(8l + 5) - 64 = 4^2(8(3l + 1) + 3) \notin E(2, 3, 6),
\]

and hence there are \( x, y, z \in \mathbb{Z} \) such that

\[
3n - 4^3 = 3x^2 + 6y^2 + 2(3z - 8)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 16z)) + 2 \times 4^3
\]

and thus

\[
n = x^2 + 2y^2 + 6z^2 - 32z + 64 = x^2 + (y + z)^2 + (z - y)^2 + (2^3 - 2z)^2
\]

as desired.
Suitable $ax + by + cz - dw$ or $ax + by - cz - dw$

**Conjecture** (Z.-W. Sun, April 14, 2016): Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leq b \leq c$ and $\gcd(a, b, c, d)$ squarefree. Then $ax + by + cz - dw$ is suitable if and only if $(a, b, c, d)$ is among the 12 quadruples

$$(1, 1, 2, 1), (1, 2, 3, 1), (1, 2, 3, 3), (1, 2, 4, 2), (1, 2, 4, 4), (1, 2, 5, 5), (1, 2, 6, 2), (1, 2, 8, 1), (2, 2, 4, 4), (2, 4, 6, 4), (2, 4, 6, 6), (2, 4, 8, 2).$$

**Conjecture** (Z.-W. Sun, April 14, 2016): Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leq b$ and $c \leq d$, and $\gcd(a, b, c, d)$ squarefree. Then $ax + by - cz - dw$ is suitable if and only if $(a, b, c, d)$ is among the 9 quadruples

$$(1, 2, 1, 1), (1, 2, 1, 2), (1, 3, 1, 2), (1, 4, 1, 3), (2, 4, 1, 2), (2, 4, 2, 4), (8, 16, 7, 8), (9, 11, 2, 9), (9, 16, 2, 7).$$

**Conjecture** (Z.-W. Sun, April 2016) For any $a, b, c, d \in \mathbb{Z}^+$ there are infinitely many positive integers not of the form $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{N}$) with $ax + by + cz + dw$ a square.
A general theorem joint with Yu-Chen Sun

**Theorem** (Yu-Chen Sun and Z.-W. Sun, 2016) Let $a, b, c, d \in \mathbb{Z}$ with $a, b, c, d$ not all zero. Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$ Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}/(a^2 + b^2 + c^2 + d^2)$ such that $ax + by + cz + dw = \lambda r^m$ for some $r \in \mathbb{N}$.

**Proof.** Let $n \in \mathbb{N}$. By a result of Z.-W. Sun, we can write $(a^2 + b^2 + c^2 + d^2)n$ as $(\lambda r^m)^2 + t^2 + u^2 + v^2$ with $r, t, u, v \in \mathbb{N}$. Set $s = \lambda r^m$, and define $x, y, z, w$ by

$$
\begin{align*}
  x &= \frac{as - bt - cu - dv}{a^2 + b^2 + c^2 + d^2}, \\
  y &= \frac{bs + at + du - cv}{a^2 + b^2 + c^2 + d^2}, \\
  z &= \frac{cs - dt + au + bv}{a^2 + b^2 + c^2 + d^2}, \\
  w &= \frac{ds + ct - bu + av}{a^2 + b^2 + c^2 + d^2}.
\end{align*}
$$
Proof of the general theorem

Then

\[
\begin{align*}
ax + by + cz + dw &= s, \\
ay - bx + cw - dz &= t, \\
az - bw - cx + dy &= u, \\
aw + bz - cy - dx &= v.
\end{align*}
\]

With the help of Euler’s four-square identity,

\[
x^2 + y^2 + z^2 + w^2 = \frac{s^2 + t^2 + u^2 + v^2}{a^2 + b^2 + c^2 + d^2} = n
\]

and

\[
ax + by + cz + dw = s = \lambda r^m.
\]

This concludes the proof.
Theorem (Y.-C. Sun and Z.-W. Sun, 2016) (i) Let $m \in \mathbb{Z}^+$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + z + w$ an $m$-th power if and only if $m \leq 3$.

(ii) Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + y + z + 2w = \lambda r^m$ (or $x + y + 2z + 3w = \lambda r^m$) for some $r \in \mathbb{N}$.

(iii) Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ ($x, y, z, w \in \mathbb{Z}$) with $x + 2y + 3z$ (or $x + y + 3z$, or $x + 2y + 2z$) in the set $\{\lambda r^m : r \in \mathbb{N}\}$.

(iv) (Progress on the 1-3-5-Conjecture) Let $\lambda \in \{1, 2\}$, $m \in \{2, 3\}$ and $n \in \mathbb{N}$. Then we can write $n$ as $x^2 + y^2 + z^2 + w^2$ with $x, y, 5z, 5w \in \mathbb{Z}$ such that $x + 3y + 5z \in \{\lambda r^m : r \in \mathbb{N}\}$. Also, any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}/7$ such that $x + 3y + 5z \in \{\lambda r^m : r \in \mathbb{N}\}$. 

A Lemma

The proof of the Theorem needs several lemmas and some previous results of Z.-W. Sun. Here is one of them.

**Lemma.** Define

\[
\begin{align*}
    x &= \frac{s-t-u-2v}{7}, \\
    y &= \frac{s+t+2u-v}{7}, \\
    z &= \frac{s-2t+u+v}{7}, \\
    w &= \frac{2s+t-u+v}{7}.
\end{align*}
\]

Then

\[
x^2 + y^2 + z^2 + w^2 = \frac{s^2 + t^2 + u^2 + v^2}{7}.
\]

Also,

\[
\begin{align*}
    x + y + z + 2w &= s, \\
    w + 2x + 3z &= s - t, \\
    x + 3y + 5w &= 2s + t.
\end{align*}
\]
Joint work with Hai-Liang Wu

In 2017, Hai-Liang Wu and the speaker used the theory of ternary quadratic forms and modulo forms to obtain the following progress on the 1-3-5 conjecture.

**Theorem** (H.-L. Wu and Z.-W. Sun, 2017). There is a finite set $A$ of positive integers such that any sufficiently large integer $n$ not in the set $\{16^k a : a \in A, \ k \in \mathbb{N}\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + 3y + 5z \in \{4^k : \ k \in \mathbb{N}\}$.

The speaker conjectured that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that

$$|x + 3y - 5z| \in \{4^k : \ k \in \mathbb{N}\}.$$
Part III. Other Refinements of the Four-Square Theorem
Suitable polynomials of the form $ax^2 + by^2 + cz^2$

**Conjecture** (Z.-W. Sun, April 9, 2016): (i) Any natural number can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x \geq y$ such that $ax^2 + by^2 + cz^2$ is a square, provided that the triple $(a, b, c)$ is among

$$(1, 8, 16), (4, 21, 24), (5, 40, 4), (9, 63, 7), (16, 80, 25),$$
$$(16, 81, 48), (20, 85, 16), (36, 45, 40), (40, 72, 9).$$

(ii) $ax^2 + by^2 + cz^2$ is suitable if $(a, b, c)$ is among the triples

$$(1, 3, 12), (1, 3, 18), (1, 3, 21), (1, 3, 60), (1, 5, 15),$$
$$(1, 8, 24), (1, 12, 15), (1, 24, 56), (3, 4, 9), (3, 9, 13),$$
$$(4, 5, 12), (4, 5, 60), (4, 9, 60), (4, 12, 21), (4, 12, 45), (5, 36, 40).$$

(iii) If $a, b, c$ are positive integers with $ax^2 + by^2 + cz^2$ suitable, then $a, b, c$ cannot be pairwise coprime.
Suitable polynomials related to Pythagorean triples

**Conjecture (Z.-W. Sun, April 12, 2016).** Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $(10w + 5x)^2 + (12y + 36z)^2$ is a square.

*Remark:* In 2017, Yu-Chen Sun and Z.-W. Sun proved that any $n \in \mathbb{N}$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w, x, y, z$ integers such that $(10w + 5x)^2 + (12y + 36z)^2$ is a square.

**Conjecture (Z.-W. Sun, May 15, 2016).** (i) Any positive integer $n$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > z$ such that $(x + y)^2 + (4z)^2$ is a square.

(ii) Any integer $n > 5$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $8x + 12y$ and $15z$ are the two legs of a right triangle with positive integer sides.

**Theorem (Z.-W. Sun, May 16, 2016).** Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > 0$ such that $x + 4y + 4z$ and $9x + 3y + 3z$ are the two legs of a right triangle with positive integer sides.
A conjecture involving mixed terms

**Conjecture** (Z.-W. Sun, 2016) (i) Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that \( xy + 2zw \) or \( xy - 2zw \) is a square.

(ii) Any \( n \in \mathbb{Z}^+ \) can be written as \( w^2 + x^2 + y^2 + z^2 \) with \( w \in \mathbb{Z}^+ \) and \( x, y, z \in \mathbb{N} \) such that \( w^2 + 4xy + 8yz + 32zx \) is a square.

(iii) Let \( a, b, c \in \mathbb{N} \) with \( 1 \leq a \leq c \) and \( \gcd(a, b, c) \) squarefree. Then \( awx + bxy + cyz \) is suitable if and only if

\[
(a, b, c) = (1, 2, 2), \ (2, 1, 4), \ (2, 8, 4).
\]

(iv) Let \( a, b, c \in \mathbb{Z}^+ \) with \( \gcd(a, b, c) \) squarefree. Then the polynomial \( axy + byz + czx \) is suitable if and only if \( \{a, b, c\} \) is among the sets

\[
\{1, 2, 3\}, \ \{1, 3, 8\}, \ \{1, 8, 13\}, \ \{2, 4, 45\}, \\
\{4, 5, 7\}, \ \{4, 7, 23\}, \ \{5, 8, 9\}, \ \{11, 16, 31\}.
\]

(v) \( 36x^2y + 12y^2z + z^2x, \ w^2x^2 + 3x^2y^2 + 2y^2z^2 \) and \( w^2x^2 + 5x^2y^2 + 80y^2z^2 + 20z^2w^2 \) are suitable.
Suitable polynomials of the form $ax^4 + by^3z$

The following conjecture sounds very mysterious!

**Conjecture** (Z.-W. Sun, 2016) Let $a$ and $b$ be nonzero integers with $\gcd(a, b)$ squarefree. Then the polynomial $ax^4 + by^3z$ is suitable if and only if $(a, b)$ is among the ordered pairs

$(1, 1)$, $(1, 15)$, $(1, 20)$, $(1, 36)$, $(1, 60)$, $(1, 1680)$ and $(9, 260)$.

**Examples:**

$$9983 = 63^2 + 54^2 + 17^2 + 53^2$$

with $63^4 + 54^3 \times 17 = 4293^2$, and

$$20055 = 47^2 + 6^2 + 77^2 + 109^2$$

with $47^4 + 1680 \times 6^3 \times 77 = 5729^2$. 
Other suitable polynomials

**Theorem** (i) (Conjectured by Z.-W. Sun and essentially proved by You-Ying Deng and Yu-Chen Sun) $x^2 - 4yz$, $x^2 + 4yz$ and $x^2 + 8yz$ are suitable.

(ii) (Z.-W. Sun, May 2016) $x^2y^2 + y^2z^2 + z^2x^2$ and $x^2y^2 + 4y^2z^2 + 4z^2x^2$ are suitable. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $x^4 + 8yz(y^2 + z^2)$ (or $x^4 + 16yz(y^2 + 4z^2)$) is a fourth power.

**Conjecture** (Z.-W. Sun, 2016) (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ ($w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$) with $x^3 + 4yz(y - z)$ (or $x^3 + 8yz(2y - z)$) a square.

(ii) The polynomials $w(x + 2y + 3z)$, $w(x^2 + 8y^2 - z^2)$ and $x^2 + 3y^2 + 5z^2 - 8w^2$ are suitable.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $z < w$ such that $4x^2 + 5y^2 + 20zw$ is a square.
A new direction

**Conjecture** (Z.-W. Sun, August 7, 2016). (i) Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2(1 + y^2 + z^2)$ with $w, x, y, z \in \mathbb{N}$, $x > 0$ and $y \equiv z \pmod{2}$. Moreover, any $n \in \mathbb{Z}^+ \setminus \{449\}$ can be written as $4^k(1 + x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$ and $x \equiv y \pmod{2}$.

(ii) Each $n \in \mathbb{Z}^+$ can be written as $4^k(1 + x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$ and $x \leq y \leq z$.

**Theorem** (Z.-W. Sun, 2016) (i) Any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + 4x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$.

(ii) Under the GRH, any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + 5x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$, and also any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + x^2 + y^2) + 5z^2$ with $k, x, y, z \in \mathbb{N}$.

**Remark.** Our proof of part (ii) uses the work of Ben Kane and Z.-W. Sun [Trans. AMS 362(2010), 6425–6455], where the authors determined for what $a, b, c \in \mathbb{Z}^+$ sufficiently large integers can be expressed as $ax^2 + by^2 + cz(z + 1)/2$ with $x, y, z \in \mathbb{Z}$. 
Part IV. Further Conjectures on Sums of Four Squares
Restrictions involving powers of two

**Theorem (Z.-W. Sun, arXiv:1701.05868)**

(i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $|x - 2y| \in \{4^k : k \in \mathbb{N}\}$.

(ii) For each $c = 1, 2$, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y + cz \in \{c4^k : k \in \mathbb{N}\}$.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y + 2z \in \{4^k : k \in \mathbb{N}\}$.

(iv) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{4^k : k \in \mathbb{N}\}$.

(v) Any integer $n > 1$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{3 \times 2^k : k \in \mathbb{N}\}$.

**Conjecture (Z.-W. Sun, 2016).** Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y - 2z$ is a power of four (including $4^0 = 1$).

**Remark.** Qing-Hu Hou has verified this for $n$ up to $10^9$. 
The 24-conjecture

24-Conjecture (Z.-W. Sun, Feb. 4, 2017). Each \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that both \( x \) and \( x + 24y \) are squares.

Remark. Qing-Hu Hou has verified this for \( n \leq 10^{10} \). I would like to offer 2400 US dollars as the prize for the first proof.

\[
\begin{align*}
12 &= 1^2 + 1^2 + 1^2 + 3^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 1 = 5^2, \\
23 &= 1^2 + 2^2 + 3^2 + 3^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 2 = 7^2, \\
24 &= 4^2 + 0^2 + 2^2 + 2^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 0 = 2^2, \\
47 &= 1^2 + 1^2 + 3^2 + 6^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 1 = 5^2, \\
71 &= 1^2 + 5^2 + 3^2 + 6^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 5 = 11^2, \\
168 &= 4^2 + 4^2 + 6^2 + 10^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 4 = 10^2, \\
344 &= 4^2 + 0^2 + 2^2 + 18^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 0 = 2^2, \\
632 &= 0^2 + 6^2 + 14^2 + 20^2 \text{ with } 0 = 0^2 \text{ and } 0 + 24 \times 6 = 12^2, \\
1724 &= 25^2 + 1^2 + 3^2 + 33^2 \text{ with } 25 = 5^2 \text{ and } 25 + 24 \times 1 = 7^2.
\end{align*}
\]
A similar conjecture

**Conjecture** (Z.-W. Sun, Feb. 4, 2017). (i) Each \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that both \( x \) and \( 49x + 48(y - z) \) are squares.

(ii) Each \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that both \( x \) and \( 121x + 48(y - z) \) are squares.

(iii) Each \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that both \( x \) and \( -7x - 8y + 8z + 16w \) are squares.

**Remark.** Qing-Hu Hou has verified parts (i)-(ii) and part (iii) for \( n \) up to \( 10^9 \) and \( 10^8 \).
Other conjectures involving joint restrictions

**Conjecture** (Z.-W. Sun, Feb. 2017). (i) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x \equiv y \pmod{2}$ such that both $x$ and $x^2 + 62xy + y^2$ are squares.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^4 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that $8y^2 - 8yz + 9z^2$ is a square. Also, each $n \in \mathbb{N}$ can be written as $4x^4 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $79y^2 - 220yz + 205z^2$ is a square.

(iii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both $2x - y$ and $4z^2 + 724zw + w^2$ (or $9z^2 + 666zw + w^2$) are squares.

(iv) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both $2x - y$ and $64z^2 - 84zw + 21w^2$ (or $81z^2 - 112zw + 56w^2$) are squares.

**Remark.** Qing-Hu Hou has verified part (iii) for $n$ up to $10^9$. 
Other conjectures involving joint restrictions

**Conjecture** (Z.-W. Sun, March 2, 2017). (i) Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, w \in \mathbb{N} \) and \( y, z \in \mathbb{Z} \) such that both \( x + 2y \) and \( z + 2w \) are squares.

(ii) Each \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z \in \mathbb{Z} \) and \( w \in \mathbb{N} \) such that both \( x + 3y \) and \( z + 3w \) are squares.

(iii) Any \( n \in \mathbb{Z}^+ \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z \in \mathbb{Z} \) and \( w \in \mathbb{Z}^+ \) such that both \( 2x + y \) and \( 2x + z \) are squares.
Conjectures involving cubic diophantine equations

**Conjecture** (Z.-W. Sun, March 2017). (i) Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) and \( y \equiv z \pmod{2} \) such that \( 72x^3 + (y - z)^3 \) is a square.

(ii) Let \( a, b \in \mathbb{Z}^+ \) with \( \gcd(a, b) \) squarefree. Then, each \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) with \( x, y, z, w \in \mathbb{N} \) such that \( ax^3 + b(y - z)^3 \) is a square, if and only if \( (a, b) \) is among the ordered pairs

\[
(1, 1), (1, 9), (2, 18), (8, 1), (9, 5), (9, 8), (9, 40), (16, 2), (18, 16), (25, 16), (72, 1).
\]

(iii) Let \( a \) and \( b \geq a \) be positive integers with \( \gcd(a, b) \) squarefree. Then, every \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 + w^2 \) \( (x, y, z, w \in \mathbb{Z}) \) with \( ax^3 + by^3 \) a square, if and only if \( (a, b) \) is among the ordered pairs

\[
(1, 2), (1, 8), (2, 16), (4, 23), (4, 31), (5, 9), (8, 9), (8, 225), (9, 47), (25, 88), (50, 54).
\]
Part V. Restricted Sums of Three Squares
On the representation \( n = x^2 + y^2 + z(3z - 1)/2 \)

Those numbers \( z(3z - 1)/2 \) \((z \in \mathbb{Z})\) are called \textit{generalized pentagonal numbers}.

In a paper published in 2015, I noted that any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z(3z - 1)/2 \) with \( x, y, z \in \mathbb{Z} \). Surprising, this can be further refined.

\begin{flushleft} \textbf{Conjecture} (Z.-W. Sun, March 3, 2017). Any \( n \in \mathbb{N} \) can be written as \( x^2 + y^2 + z(3z - 1)/2 \) with \( x, y, z \in \mathbb{Z} \) such that \( x + 2y \) is a square. \end{flushleft}

\textbf{Example}.

\[ 803 = (-17)^2 + 13^2 + \frac{(-15)(3(-15) - 1)}{2} \text{ with } (-17) + 2 \times 13 = 3^2. \]
Refining the Gauss-Legendre theorem

**Gauss-Legendre Theorem.** A nonnegative integer \( n \) can be expressed as the sum of three squares if and only if it is not of the form \( 4^k(8l + 7) \) with \( k, l \in \mathbb{N} \).

**Conjecture** (Z.-W. Sun, March 4, 2017). (i) Any \( n \in \mathbb{Z}^+ \) with \( \text{ord}_2(n) \) odd can be written as \( x^2 + y^2 + z^2 \) with \( x, y, z \in \mathbb{Z} \) such that \( x + 3y + 5z \) is a square.

(ii) Let \( n \in \mathbb{N} \setminus \{63\} \). Then \( 4n + 1 \) can be written as \( x^2 + y^2 + z^2 \) with \( x, y, z \in \mathbb{Z} \) such that \( x + 3y + 5z \) is a square.

(iii) Any \( n \in \mathbb{N} \) not of the form \( 4^k(8l + 7) \) with \( k, l \in \mathbb{N} \) can be written as \( x^2 + y^2 + z^2 \) with \( x, y, z \in \mathbb{Z} \) such that \( x + 2y + 3z \) is a square or twice a square.

**Example.**

\[
1430 = (-13)^2 + (-6)^2 + 35^2 \quad \text{with} \quad (-13) + 3 \times (-6) + 5 \times 35 = 12^2.
\]
Restricted representations for positive odd numbers

It is known that any positive odd numbers can be written as $x^2 + y^2 + 2z^2$ with $x, y, z \in \mathbb{Z}$. Also, we can replace $x^2 + y^2 + 2z^2$ by $x^2 + 2y^2 + 3z^2$.

**Conjecture** (Z.-W. Sun, March 5, 2017). (i) Any positive odd integer can be written as $x^2 + y^2 + 2z^2$ with $x, y, z \in \mathbb{Z}$ such that $2x + y + z$ is a square or a power of two.

(ii) Any positive odd integer can be written as $x^2 + 2y^2 + 3z^2$ with $x, y, z \in \mathbb{Z}$ such that $x + y + z$ is a square or twice a square.

**Examples:**

$$2 \times 1143 + 1 = (-22)^2 + 2 \times 30^2 + 3 \times 1^2 \text{ with } (-22) + 30 + 1 = 3^2,$$
and

$$2 \times 6408 + 1 = (-22)^2 + 2 \times 75^2 + 3 \times 19^2$$

with

$$(-22) + 75 + 19 = 2 \times 6^2.$$
References

For the main sources of my above conjectures and related results, you may look at two recent preprints:


Thank you!