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Restricted Sums of Three or Four Squares

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Abstract

The classical Gauss-Legendre theorem states that each $n = 0, 1, 2, \ldots$ not of the form $4^k(8m + 1)$ $(k, m = 0, 1, 2, \ldots)$ can be written as the sum of three squares. Lagrange's four-square theorem asserts that any natural number can be written as the sum of four squares. In this talk we introduce various conjectures refining these two theorems; for example, our 24-conjecture states that any nonnegative integers can be written as $x^2 + y^2 + z^2 + w^2$ with x, y, z, w nonnegative integers such that both x and $x + 24y$ are squares. We will also mention some recent results as well as related techniques in this direction.

Part I. The Birth of the 1-3-5-Conjecture

Lagrange's theorem

Lagrange's Theorem. Each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as the sum of four squares.

Examples. $3 = 1^2 + 1^2 + 1^2 + 0^2$ and $7 = 2^2 + 1^2 + 1^2 + 1^2$.

A. Diophantus (AD 299-215, or AD 285-201) was aware of this theorem as indicated by examples given in his book *Arithmetica*.

In 1621 Bachet translated Diophantus' book into Latin and stated the theorem in the notes of his translation.

In 1748 L. Euler found the four-square identity

$$\begin{aligned} &(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) \\ =&(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4)^2 + (x_1y_2 - x_2y_1 - x_3y_4 + x_4y_3)^2 \\ &+ (x_1y_3 - x_3y_1 + x_2y_4 - x_4y_2)^2 + (x_1y_4 - x_4y_1 - x_2y_3 + x_3y_2)^2. \end{aligned}$$

and hence reduced the theorem to the case with n prime.

The theorem was first proved by J. L. Lagrange in 1770.

The representation function $r_4(n)$

It is known that only the following numbers have a unique representation as the sum of four unordered squares:

$$1,\ 3,\ 5,\ 7,\ 11,\ 15,\ 23$$

and

$$2^{2k+1}m\ (k = 0, 1, 2, \ldots \text{ and } m = 1, 3, 7).$$

For example, $4^k \times 14 = (2^k 3)^2 + (2^{k+1})^2 + (2^k)^2 + 0^2$.

Jacobi considered the fourth power of the theta function

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and this led him to show that

$$r_4(n) = 8 \sum_{d \mid n\ \&\ 4 \nmid d} d \quad \text{for all } n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\},$$

where

$$r_4(n) := |\{(w, x, y, z) \in \mathbb{Z}^4 :\ w^2 + x^2 + y^2 + z^2 = n\}|.$$

Natural numbers as sums of polygonal numbers

For $m = 3, 4, 5, \ldots$, the *polygonal numbers of order m* (or *m-gonal numbers*) are given by

$$p_m(n) := (m-2)\binom{n}{2} + n \quad (n = 0, 1, 2, \ldots).$$

Clearly, $p_4(n) = n^2$, $p_5(n) = n(3n-1)/2$ and $p_6(n) = n(2n-1)$.

Fermat's Claim. Let $m \geqslant 3$ be an integer. Then any $n \in \mathbb{N}$ can be written as the sum of m polygonal numbers of order m.

This was proved by Lagrange in the case $m = 4$, by Gauss in the case $m = 3$, and by Cauchy in the case $m \geqslant 5$.

Conjecture (Z.-W. Sun, March 14, 2015). Each $n \in \mathbb{N}$ can be written as

$$p_5(x_1) + p_5(x_2) + p_5(x_3) + 2p_5(x_4) \quad (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

Theorem (conjectured by the speaker and proved by X.-Z. Meng and Z.-W. Sun [Acta Arith. 180(2017)]). $n \in \mathbb{N}$ can be written as

$$p_6(x_1) + p_6(x_2) + 2p_6(x_3) + 4p_6(x_4) \quad (x_1, x_2, x_3, x_4 \in \mathbb{N}).$$

Upgrade Waring's problem

In 1770 E. Waring proposed the following famous problem.

Waring's Problem. Whether for each integer $k > 1$ there is a positive integer $g(k) = r$ (as small as possible) such that every $n \in \mathbb{N}$ can be written as

$$x_1^k + x_2^k + \ldots + x_r^k \quad \text{with } x_1, \ldots, x_r \in \mathbb{N}.$$

In 1909 D. Hilbert proved that $g(k)$ always exists. It is conjectured that

$$g(k) = 2^k + \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 2.$$

New Problem (Z.-W. Sun, March 30-31, 2016). Determine $t(k)$ for any integer $k > 1$, where $t(k)$ is the smallest positive integer t such that

$$\{a_1x_1^k + a_2x_2^k + \ldots + a_tx_t^k : \ x_1, \ldots, x_t \in \mathbb{N}\} = \mathbb{N}$$

for some $a_1, \ldots, a_t \in \mathbb{Z}^+$ with $a_1 + a_2 + \ldots + a_t = g(k)$.

A conjecture

Conjecture (Z.-W. Sun, March 30-31, 2016). (i) $t(3) = 5$. In fact,

$$\{u^3 + v^3 + 2x^3 + 2y^3 + 3z^3 : \ u, v, x, y, z \in \mathbb{N}\} = \mathbb{N}.$$

(ii) $t(4) = 7$. In fact, we have

$$\{x_1^4 + x_2^4 + 2x_3^4 + 2x_4^4 + 3x_5^4 + 3x_6^4 + 7x_7^4 : \ x_1, \ldots, x_7 \in \mathbb{N}\} = \mathbb{N},$$

$$\{x_1^4 + x_2^4 + 2x_3^4 + 2x_4^4 + 3x_5^4 + 4x_6^4 + 6x_7^4 : \ x_1, \ldots, x_7 \in \mathbb{N}\} = \mathbb{N}.$$

(iii) $t(5) = 8$. In fact,

$$\{x_1^5+x_2^5+2x_3^5+3x_4^5+4x_5^5+5x_6^5+7x_7^5+14x_8^5 : \ x_1, \ldots, x_8 \in \mathbb{N}\} = \mathbb{N},$$

$$\{x_1^5+x_2^5+2x_3^5+3x_4^5+4x_5^5+6x_6^5+8x_7^5+12x_8^5 : \ x_1, \ldots, x_8 \in \mathbb{N}\} = \mathbb{N}.$$

(iv) $t(6) = 10$. In fact,

$$\{x_1^6+x_2^6+x_3^6+2x_4^6+3x_5^6+5x_6^6+6x_7^6+10x_8^6+18x_9^6+26x_{10}^6 : \ x_i \in \mathbb{N}\} = \mathbb{N}.$$

(v) In general, $t(k) \leqslant 2k - 1$ for any integer $k > 2$.

Discoveries on April 8, 2016

Motivated by my conjecture that any $n \in \mathbb{N}$ can be written as

$$x_1^3 + x_2^3 + 2x_3^3 + 2x_4^3 + 3x_5^3 \quad (x_1, x_2, x_3, x_4, x_5 \in \mathbb{N}),$$

on April 8, 2016 I considered to write $n \in \mathbb{N}$ as $\sum_{i=1}^5 a_i x_i^2$ $(x_i \in \mathbb{N})$ with certain restrictions on $x_1, \ldots, x_5$.

Conjecture (Z.-W. Sun) Let $n > 1$ be an integer.

(i) n can be written as

$$x_1^2+x_2^2+x_3^2+2x_4^2+2x_5^2 = x_1^2+x_2^2+x_3^2+(\underline{x_4+x_5})^2+(x_4-x_5)^2 \ (x_i \in \mathbb{N})$$

with $x_1 + x_2 + x_3 + \underline{x_4 + x_5}$ prime.

(ii) We can write n as

$$x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 5x_5^2 \quad (x_1, x_2, x_3, x_4, x_5 \in \mathbb{N})$$

with $x_1 + x_2 + x_3 + x_4$ a square.

Remark. Squares are sparser than prime numbers.

1-3-5-Conjecture (1350 US dollars for the first solution)

1-3-5-Conjecture (Z.-W. Sun, April 9, 2016): Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.

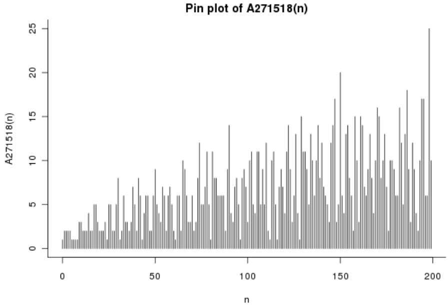
Examples.

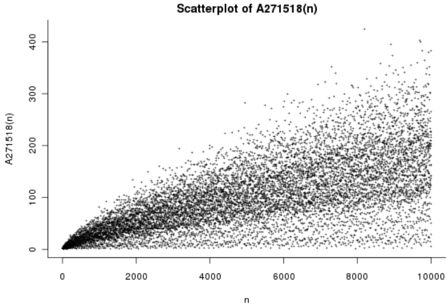
$$\begin{aligned} 7 =& 1^2 + 1^2 + 1^2 + 2^2 \text{ with } 1 + 3 \times 1 + 5 \times 1 = 3^2, \\ 8 =& 0^2 + 2^2 + 2^2 + 0^2 \text{ with } 0 + 3 \times 2 + 5 \times 2 = 4^2, \\ 31 =& 5^2 + 2^2 + 1^2 + 1^2 \text{ with } 5 + 3 \times 2 + 5 \times 1 = 4^2, \\ 43 =& 1^2 + 5^2 + 4^2 + 1^2 \text{ with } 1 + 3 \times 5 + 5 \times 4 = 6^2. \end{aligned}$$

The conjecture has been verified by Qing-Hu Hou for all $n \leqslant 10^{10}$.

We guess that if a, b, c are positive integers with $\gcd(a, b, c)$ squarefree such that each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $ax + by + cz$ a square then we must have $\{a, b, c\} = \{1, 3, 5\}$.

Graph for the number of such representations of n





无　解

数字几时有，
把酒问青天。
一二三四五，
自然藏玄机。

四个平方和，
遍历自然数。
奇妙一三五，
更上一层楼。

苍天捉弄人，
数论妙无穷。
吾辈虽努力，
难解一三五!

时势唤英雄，
攻关需豪杰。
人间若无解，
天神会证否？

Related conjectures

Conjecture (Z.-W. Sun, 2016): (i) Each $n \in \mathbb{N} \setminus \{7, 15, 23, 71, 97\}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 3y + 5z$ twice a square. Also, any $n \in \mathbb{N} \setminus \{7, 43, 79\}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $3x + 5y + 6z$ a square, and any $n \in \mathbb{N} \setminus \{5, 7, 15\}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $3x + 5y + 6z$ twice a square.

(ii) Let $a, b, c \in \mathbb{Z}^+$ with $\gcd(a, b, c)$ squarefree. If there are only finitely many positive integers which cannot be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $ax + by + cz$ a square, then $\{a, b, c\}$ must be among

$$\{1, 3, 5\},\ \{2, 6, 10\},\ \{3, 5, 6\},\ \{6, 10, 12\}.$$

Remark. Qing-Hu Hou at Tianjin Univ. has verified part (i) for n up to 10^9.

1-2-3-Conjecture (Companion of 1-3-5-Conjecture)

1-2-3-Conjecture (Z.-W. Sun, July 24, 2016). Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y + 3z$ is a square.

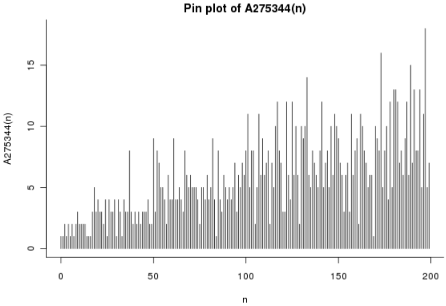
Examples:

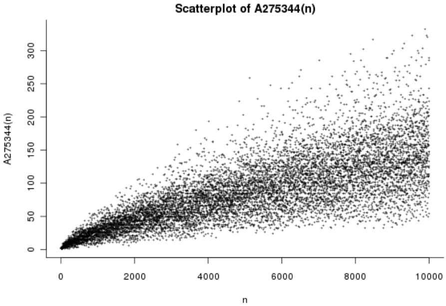
$$14 = 1^2 + 1^2 + 2^2 + 2 \times 2^2 \ \text{ with } 1 + 2 \times 1 + 3 \times 2 = 3^2,$$
$$30 = 3^2 + 2^2 + 3^2 + 2 \times 2^2 \ \text{ with } 3 + 2 \times 2 + 3 \times 3 = 4^2,$$
$$33 = 1^2 + 0^2 + 0^2 + 2 \times 4^2 \ \text{ with } 1 + 2 \times 0 + 3 \times 0 = 1^2,$$
$$84 = 4^2 + 6^2 + 0^2 + 2 \times 4^2 \ \text{ with } 4 + 2 \times 6 + 3 \times 0 = 4^2,$$
$$169 = 10^2 + 6^2 + 1^2 + 2 \times 4^2 \ \text{ with } 10 + 2 \times 6 + 3 \times 1 = 5^2,$$
$$225 = 10^2 + 6^2 + 9^2 + 2 \times 2^2 \ \text{ with } 10 + 2 \times 6 + 3 \times 9 = 7^2.$$

Another Conjecture (Sun, 2017). Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 3z = 1$.

Remark. This was proved by Hai-Liang Wu and Z.-W. Sun in 2017 for sufficient large integers n.

Graph for the number of such representations of n





Part II. Sums of Four Squares with Linear Restrictions

Diagonal ternary quadratic forms

For $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we define

$$E(a, b, c) := \{n \in \mathbb{N} : \ n \neq ax^2 + by^2 + cz^2 \text{ for any } x, y, z \in \mathbb{N}\}.$$

It is known that $E(a, b, c)$ is an infinite set.

Gauss-Legendre Theorem. $E(1, 1, 1) = \{4^k(8l + 7) : \ k, l \in \mathbb{N}\}$.

There are totally 102 diagonal ternary quadratic forms $ax^2 + by^2 + cz^2$ with $a, b, c \in \mathbb{Z}^+$ and $\gcd(a, b, c) = 1$ for which the structure of $E(a, b, c)$ is known explicitly. For example,

$$\begin{aligned}
E(1, 1, 2) =& \{4^k(16l + 14) : \ k, l \in \mathbb{N}\},\\
E(1, 1, 5) =& \{4^k(8l + 3) : \ k, l \in \mathbb{N}\},\\
E(1, 2, 3) =& \{4^k(16l + 10) : \ k, l \in \mathbb{N}\},\\
E(1, 2, 6) =& \{4^k(8l + 5) : \ k, l \in \mathbb{N}\}.
\end{aligned}$$

$n = x^2 + y^2 + z^2 + w^2$ with $P(x, y, z) = 0$

Theorem (Z.-W. Sun, 2016) (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z) = 0$, whenever $P(x, y, z)$ is among the polynomials

$$x(x-y),\ x(x-2y),\ (x-y)(x-2y),\ (x-y)(x-3y),$$
$$x(x+y-z),\ (x-y)(x+y-z),\ (x-2y)(x+y-z).$$

(ii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $(2x - 3y)(x + y - z) = 0$, provided that

$$\{x^2 + y^2 + 13z^2 :\ x, y, z \in \mathbb{N}\} \supseteq \{8q + 5 :\ q \in \mathbb{N}\}. \qquad (*)$$

Remark. We may require $x(x - y) = 0$ since $E(1, 1, 1) \cap E(1, 1, 2) = \emptyset$. Also, we may require that xy (or $2xy$, or $(x^2 + y^2)(x^2 + z^2)$) is a square. It seems that $(*)$ does hold.

Lemma. Let $n \in \mathbb{N}$. Then $n \notin E(1, 2, 6)$ if and only if $n = x^2 + y^2 + z^2 + w^2$ for some $x, y, z, w \in \mathbb{N}$ with $x + y = z$. Also, $n \notin E(1, 2, 3)$ if and only if $n = x^2 + y^2 + z^2 + w^2$ for some $x, y, z, w \in \mathbb{Z}$ with $x + y = 2z$.

$n = x^2 + y^2 + z^2 + w^2$ with $P(x, y, z, w) = 0$

Conjecture (Z.-W. Sun, 2016) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z, w) = 0$, whenever $P(x, y, z, w)$ is among the polynomials

$$(x-y)(x+y-3z),\ (x-y)(x+2y-z),\ (x-y)(x+2y-2z),$$
$$(x-y)(x+2y-7z),\ (x-y)(x+3y-3z),\ (x-y)(x+4y-6z),$$
$$(x-y)(x+5y-2z),\ (x-2y)(x+2y-z),\ (x-2y)(x+2y-2z),$$
$$(x-2y)(x+3y-3z),\ (x+y-z)(x+2y-2z),$$
$$(x-y)(x+y+3z-3w),\ (x-y)(x-y+3z-5w),$$
$$(x-y)(3x+3y+3z-5w),\ (x-y)(3x-3y+5z-7w),$$
$$(x-y)(3x-3y+7z-9w).$$

Sums of a fourth power and three squares

Theorem (Z.-W. Sun, March 27, 2016). Each $n \in \mathbb{N}$ can be written as $w^4 + x^2 + y^2 + z^2$ with $w, x, y, z \in \mathbb{N}$.

Proof. For $n = 0, 1, 2, \ldots, 15$, the result can be verified directly. Now let $n \geqslant 16$ be an integer and assume that the result holds for smaller values of n.

Case 1. $16 \mid n$.
By the induction hypothesis, we can write

$$\frac{n}{16} = x^4 + y^2 + z^2 + w^2 \ \text{ with } x, y, z, w \in \mathbb{N}.$$

It follows that $n = (2x)^4 + (4y)^2 + (4z)^2 + (4w)^2$.

Case 2. $n = 4^k q$ with $k \in \{0, 1\}$ and $q \equiv 7 \pmod 8$.
In this case, $n - 1 \notin E(1, 1, 1)$, and hence $n = 1^4 + y^2 + z^2 + w^2$ for some $y, z, w \in \mathbb{N}$.

Case 3. $16 \nmid n$ and $n \neq 4^k(8l + 7)$ for any $k \in \{0, 1\}$ and $l \in \mathbb{N}$.
In this case, $n \notin E(1, 1, 1)$ and hence there are $y, z, w \in \mathbb{N}$ such that $n = 0^4 + y^2 + z^2 + w^2$.

$aw^k + x^2 + y^2 + z^2$ with $a \in \{1, 4\}$ and $k \in \{4, 5, 6\}$

Via a similar method, we have proved the following result.

Theorem (Z.-W. Sun, March-June, 2016). Let $a \in \{1, 4\}$ and $k \in \{4, 5, 6\}$. Then, each $n \in \mathbb{N}$ can be written as $aw^k + x^2 + y^2 + z^2$ with $w, x, y, z \in \mathbb{N}$.

Conjecture (Z.-W. Sun) (i) (2015) Any $n \in \mathbb{N}$ can be written as $x^2 + y^3 + z^4 + 2w^4$ with $x, y, z, w \in \mathbb{N}$.

(ii) (2016) Each $n \in \mathbb{N}$ can be written as $x^5 + y^4 + z^2 + 3w^2$ with $x, y, z, w \in \mathbb{N}$. Also, any $n \in \mathbb{N}$ can be represented as $x^5 + y^4 + z^3 + T_w$ with $x, y, z, w \in \mathbb{N}$.

(iii) [JNT 171(2016)] Any positive integer can be written as $x^3 + y^2 + T_z$ with $x, y \in \mathbb{N}$ and $z \in \mathbb{Z}^+$ Also, each $n \in \mathbb{N}$ can be written as $x^4 + y(3y + 1)/2 + z(7z + 1)/2$ with $x, y, z \in \mathbb{Z}$.

Suitable polynomials

Definition (Z.-W. Sun, 2016). A polynomial $P(x, y, z, w)$ with integer coefficients is called *suitable* if any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z, w)$ is a square.

We have seen that both x and $2x$ are suitable polynomials. The 1-3-5-Conjecture says that $x + 3y + 5z$ is suitable.

We conjecture that there only finitely many $a, b, c, d \in \mathbb{Z}$ with $\gcd(a, b, c, d)$ squarefree such that $ax + by + cz + dw$ is suitable, and we have found all such quadruples (a, b, c, d).

$x - y$ and $2x - 2y$ are suitable

Let $a \in \{1, 2\}$. We claim that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $a(x - y)$ is a square, and want to prove this by induction.

For every $n = 0, 1, \ldots, 15$, we can verify the claim directly.

Now we fix an integer $n \geqslant 16$ and assume that the claim holds for smaller values of n.

Case 1. $16 \mid n$.
In this case, by the induction hypothesis, there are $x, y, z, w \in \mathbb{N}$ with $a(x - y)$ a square such that $n/16 = x^2 + y^2 + z^2 + w^2$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $a(4x - 4y)$ a square.

Case 2. $16 \nmid n$ and $n \notin E(1, 1, 2)$.
In this case, there are $x, y, z, w \in \mathbb{N}$ with $x = y$ and $n = x^2 + y^2 + z^2 + w^2$, thus $a(x - y) = 0^2$ is a square.

$x - y$ and $2x - 2y$ are suitable

Case 3. $16 \nmid n$ and $n \in E(1,1,2) = \{4^k(16l+14) : \ k, l \in \mathbb{N}\}$.

In this case, $n = 4^k(16l+14)$ for some $k \in \{0,1\}$ and $l \in \mathbb{N}$. Note that $n/2 - (2/a)^2 \notin E(1,1,1)$. So, $n/2 - (2/a)^2 = t^2 + u^2 + v^2$ for some $t, u, v \in \mathbb{N}$ with $t \geqslant u \geqslant v$. As $n/2 - (2/a)^2 \geqslant 8 - 4 > 3$, we have $t \geqslant 2 \geqslant 2/a$. Thus

$$\begin{aligned} n =& 2\left(\left(\frac{2}{a}\right)^2 + t^2\right) + 2(u^2+v^2) \\ =& \left(t+\frac{2}{a}\right)^2 + \left(t-\frac{2}{a}\right)^2 + (u+v)^2 + (u-v)^2 \end{aligned}$$

with

$$a\left(\left(t+\frac{2}{a}\right) - \left(t-\frac{2}{a}\right)\right) = 2^2.$$

This proves that $x - y$ and $2x - 2y$ are both suitable.

Suitable polynomials of the form $ax \pm by$

Conjecture (Z.-W. Sun, April 14,2016) Let $a, b \in \mathbb{Z}^+$ with $\gcd(a, b)$ squarefree.

(i) The polynomial $ax + by$ is suitable if and only if $\{a, b\} = \{1, 2\}, \{1, 3\}, \{1, 24\}$.

(ii) The polynomial $ax - by$ is suitable if and only if (a, b) is among the ordered pairs

$$(1, 1),\ (2, 1),\ (2, 2),\ (4, 3),\ (6, 2).$$

Remark. In 2016, I proved that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + 2y$ is a square (or a cube). In 2017, in a joint paper with my student Yu-Chen Sun, we managed to show that $x + 2y$ is indeed suitable.

Write $n = x^2 + y^2 + z^2 + w^2$ with $x + 3y$ a square

In 1916 Ramanujan conjectured that

(1) *the only positive even numbers not of the form* $x^2 + y^2 + 10z^2$ *are those* $4^k(16l + 6)$ $(k, l \in \mathbb{N})$

and

(2) *sufficiently large odd numbers are of the form* $x^2 + y^2 + 10z^2$.

In 1927 L. E. Dickson [Bull. AMS] proved (1). In 1990 W. Duke and R. Schulze-Pillot [Invent. Math.] confirmed (2). In 1997 K. Ono and K. Soundararajan [Invent. Math.] proved that under the GRH (Generalized Riemann Hypothesis) any odd number greater than 2719 has the form $x^2 + y^2 + 10z^2$.

With the help of the Ono-Soundararajan result, the speaker has proved the following result.

Theorem (Z.-W. Sun, 2016) Under the GRH, any $n \in \mathbb{N}$ can be written as $n = x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 3y$ a square.

Suitable $ax - by - cz$ or $ax + by - cz$

Conjecture (Z.-W. Sun, April 14, 2016): (i) Let $a, b, c \in \mathbb{Z}^+$ with $b \leqslant c$ and $\gcd(a, b, c)$ squarefree. Then $ax - by - cz$ is suitable if and only if (a, b, c) is among the five triples

$$(1,1,1),\ (2,1,1),\ (2,1,2),\ (3,1,2),\ (4,1,2).$$

(ii) Let $a, b, c \in \mathbb{Z}^+$ with $a \leqslant b$ and $\gcd(a, b, c)$ squarefree. Then $ax + by - cz$ is suitable if and only if (a, b, c) is among the following 52 triples

$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,3),\ (1,3,1),$
$(1,3,3),\ (1,4,4),\ (1,5,1),\ (1,6,6),\ (1,8,6),\ (1,12,4),\ (1,16,1),$
$(1,17,1),\ (1,18,1),\ (2,2,2),\ (2,2,4),\ (2,3,2),\ (2,3,3),\ (2,4,1),$
$(2,4,2),\ (2,6,1),\ (2,6,2),\ (2,6,6),\ (2,7,4),\ (2,7,7),\ (2,8,2),$
$(2,9,2),\ (2,32,2),\ (3,3,3),\ (3,4,2),\ (3,4,3),\ (3,8,3),\ (4,5,4),$
$(4,8,3),\ (4,9,4),\ (4,14,14),\ (5,8,5),\ (6,8,6),\ (6,10,8),\ (7,9,7),$
$(7,18,7), (7,18,12),\ (8,9,8),\ (8,14,14),\ (8,18,8),\ (14,32,14),$
$(16,18,16),\ (30,32,30),\ (31,32,31),\ (48,49,48),\ (48,121,48).$

Linear restrictions involving cubes

Conjecture (Z.-W. Sun, 2017). Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x+y-z$ an integer cube, where $x,y,z,w \in \mathbb{N}$, $x \geqslant y \leqslant z$ and $x \equiv y \pmod 2$.

Conjecture (Z.-W. Sun, 2016) For each $c=1,2,4$, any $n \in \mathbb{N}$ can be written as $w^2+x^2+y^2+z^2$ with $w,x,y,z \in \mathbb{N}$ and $y \leqslant z$ such that $2x+y-z=ct^3$ for some $t \in \mathbb{N}$.

Examples.

$$\begin{aligned} 8 =& 0^2+2^2+2^2+0^2 \ \text{ with } \ 2\times 0+2-2=0^3, \\ 13 =& 2^2+0^2+3^2+0^2 \ \text{ with } \ 2\times 2+0-3=1^3, \\ 2976 =& 20^2+16^2+48^2+4^2 \ \text{ with } \ 2\times 20+16-48=2^3. \end{aligned}$$

$n = x^2 + y^2 + z^2 + w^2$ with $x + y + z$ a square (or a cube)

Theorem (Z.-W. Sun, April-May, 2016) Let $c \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + cz = t^m$ for some $t \in \mathbb{Z}$.

Proof for the Case $c = 1$. For $n = 0, \ldots, 4^m - 1$ we can easily verify the desired result directly.

Now let $n \in \mathbb{N}$ with $n \geqslant 4^m$. Assume that any $r \in \{0, \ldots, n-1\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + z \in \{t^m : \ t \in \mathbb{Z}\}$. If $4^m \mid n$, then there are $x, y, z, w \in \mathbb{Z}$ with $x^2 + y^2 + z^2 + w^2 = n/4^m$ such that $x + y + z = t^m$ for some $t \in \mathbb{Z}$, and hence

$$n = (2^m x)^2 + (2^m y)^2 + (2^m z)^2 + (2^m w)^2$$

with $2^m x + 2^m y + (2^m z) = 2^m(x + y + z) = (2t)^m$. Below we suppose that $4^m \nmid n$.

Continued the proof

It suffices to show that there are $x, y, z \in \mathbb{Z}$ and $\delta \in \{0, 1, 2^m\}$ such that

$$n = x^2 + (y+z)^2 + (z-y)^2 + (\delta - 2z)^2 = x^2 + 2y^2 + 6z^2 - 4\delta z + \delta^2.$$

(Note that $(y+z) + (z-y) + (\delta - 2z) = \delta \in \{t^m : t \in \mathbb{Z}\}$.)
Suppose that this fails for $\delta = 0$. As

$$E(1,2,6) = \{4^k(8l+5) : k, l \in \mathbb{N}\},$$

$n = 4^k(8l+5)$ for some $k, l \in \mathbb{N}$ with $k < m$. Clearly,

$$3n - 1 = \begin{cases} 3(8l+5) - 1 = 2(12l+7) & \text{if } k = 0, \\ 3 \times 4(8l+5) - 1 = 8(12l+7) + 3 & \text{if } k = 1. \end{cases}$$

Thus, if $k \in \{0, 1\}$, then $3n - 1$ does not belong to

$$E(2,3,6) = \{3q+1 : q \in \mathbb{N}\} \cup \{4^k(8l+7) : k, l \in \mathbb{N}\},$$

Continue the proof

hence for some $x, y, z \in \mathbb{Z}$ we have

$$3n - 1 = 3x^2 + 6y^2 + 2(3z - 1)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 2z)) + 2$$

and thus

$$n = x^2 + 2y^2 + 6z^2 - 4z + 1 = x^2 + (y + z)^2 + (z - y)^2 + (1 - 2z)^2$$

as desired.

When $k = 2$ and $m = 3$, we have

$$3n - 64 = 3 \times 16(8l + 5) - 64 = 4^2(8(3l + 1) + 3) \notin E(2, 3, 6),$$

and hence there are $x, y, z \in \mathbb{Z}$ such that

$$3n - 4^3 = 3x^2 + 6y^2 + 2(3z - 8)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 16z)) + 2 \times 4^3$$

and thus

$$n = x^2 + 2y^2 + 6z^2 - 32z + 64 = x^2 + (y + z)^2 + (z - y)^2 + (2^3 - 2z)^2$$

as desired.

Suitable $ax + by + cz - dw$ or $ax + by - cz - dw$

Conjecture (Z.-W. Sun, April 14, 2016): Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c$ and $\gcd(a, b, c, d)$ squarefree. Then $ax + by + cz - dw$ is suitable if and only if (a, b, c, d) is among the 12 quadruples

$$(1,1,2,1),\ (1,2,3,1),\ (1,2,3,3),\ (1,2,4,2),$$
$$(1,2,4,4),\ (1,2,5,5),\ (1,2,6,2),\ (1,2,8,1),$$
$$(2,2,4,4),\ (2,4,6,4),\ (2,4,6,6),\ (2,4,8,2).$$

Conjecture (Z.-W. Sun, April 14, 2016): Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b$ and $c \leqslant d$, and $\gcd(a, b, c, d)$ squarefree. Then $ax + by - cz - dw$ is suitable if and only if (a, b, c, d) is among the 9 quadruples

$$(1,2,1,1),\ (1,2,1,2),\ (1,3,1,2),\ (1,4,1,3),$$
$$(2,4,1,2),\ (2,4,2,4),\ (8,16,7,8),\ (9,11,2,9),\ (9,16,2,7).$$

Conjecture (Z.-W. Sun, April 2016) For any $a, b, c, d \in \mathbb{Z}^+$ there are infinitely many positive integers not of the form $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $ax + by + cz + dw$ a square.

A general theorem joint with Yu-Chen Sun

Theorem (Yu-Chen Sun and Z.-W. Sun, 2016) Let $a, b, c, d \in \mathbb{Z}$ with a, b, c, d not all zero. Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$ Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}/(a^2 + b^2 + c^2 + d^2)$ such that $ax + by + cz + dw = \lambda r^m$ for some $r \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. By a result of Z.-W. Sun, we can write $(a^2 + b^2 + c^2 + d^2)n$ as $(\lambda r^m)^2 + t^2 + u^2 + v^2$ with $r, t, u, v \in \mathbb{N}$. Set $s = \lambda r^m$, and define x, y, z, w by

$$\begin{cases} x = & \frac{as-bt-cu-dv}{a^2+b^2+c^2+d^2}, \\ y = & \frac{bs+at+du-cv}{a^2+b^2+c^2+d^2}, \\ z = & \frac{cs-dt+au+bv}{a^2+b^2+c^2+d^2}, \\ w = & \frac{ds+ct-bu+av}{a^2+b^2+c^2+d^2}. \end{cases}$$

Proof of the general theorem

Then

$$\begin{cases} ax + by + cz + dw = & s, \\ ay - bx + cw - dz = & t, \\ az - bw - cx + dy = & u, \\ aw + bz - cy - dx = & v. \end{cases}$$

With the help of Euler's four-square identity,

$$x^2 + y^2 + z^2 + w^2 = \frac{s^2 + t^2 + u^2 + v^2}{a^2 + b^2 + c^2 + d^2} = n$$

and

$$ax + by + cz + dw = s = \lambda r^m.$$

This concludes the proof.

Joint work with Yu-Chen Sun

Theorem (Y.-C. Sun and Z.-W. Sun, 2016) (i) Let $m \in \mathbb{Z}^+$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + z + w$ an m-th power if and only if $m \leqslant 3$.

(ii) Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + z + 2w = \lambda r^m$ (or $x + y + 2z + 3w = \lambda r^m$) for some $r \in \mathbb{N}$.

(iii) Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 2y + 3z$ (or $x + y + 3z$, or $x + 2y + 2z$) in the set $\{\lambda r^m : \ r \in \mathbb{N}\}$.

(iv) (Progress on the 1-3-5-Conjecture) Let $\lambda \in \{1, 2\}$, $m \in \{2, 3\}$ and $n \in \mathbb{N}$. Then we can write n as $x^2 + y^2 + z^2 + w^2$ with $x, y, 5z, 5w \in \mathbb{Z}$ such that $x + 3y + 5z \in \{\lambda r^m : \ r \in \mathbb{N}\}$. Also, any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}/7$ such that $x + 3y + 5z \in \{\lambda r^m : \ r \in \mathbb{N}\}$.

A Lemma

The proof of the Theorem needs several lemmas and some previous results of Z.-W. Sun. Here is one of them.

Lemma. Define

$$\begin{cases} x = \frac{s-t-u-2v}{7}, \\ y = \frac{s+t+2u-v}{7}, \\ z = \frac{s-2t+u+v}{7}, \\ w = \frac{2s+t-u+v}{7}. \end{cases}$$

Then

$$x^2 + y^2 + z^2 + w^2 = \frac{s^2 + t^2 + u^2 + v^2}{7}.$$

Also,

$$\begin{aligned} x + y + z + 2w =& s, \\ w + 2x + 3z =& s - t, \\ x + 3y + 5w =& 2s + t. \end{aligned}$$

Joint work with Hai-Liang Wu

In 2017, Hai-Liang Wu and the speaker used the theory of ternary quadratic forms and modulo forms to obtain the following progress on the 1-3-5 conjecture.

Theorem (H.-L. Wu and Z.-W. Sun, 2017). There is a finite set A of positive integers such that any sufficiently large integer n not in the set $\{16^k a:\ a\in A,\ k\in\mathbb{N}\}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{Z}$ such that $x+3y+5z\in\{4^k:\ k\in\mathbb{N}\}$.

The speaker conjectured that any $n\in\mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w\in\mathbb{N}$ such that

$$|x+3y-5z|\in\{4^k:\ k\in\mathbb{N}\}.$$

Part III. Other Refinements of the Four-Square Theorem

Suitable polynomials of the form $ax^2 + by^2 + cz^2$

Conjecture (Z.-W. Sun, April 9, 2016): (i) Any natural number can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x \geqslant y$ such that $ax^2 + by^2 + cz^2$ is a square, provided that the triple (a, b, c) is among

$$(1,8,16),\ (4,21,24),\ (5,40,4),\ (9,63,7),\ (16,80,25),$$
$$(16,81,48),\ (20,85,16),\ (36,45,40),\ (40,72,9).$$

(ii) $ax^2 + by^2 + cz^2$ is suitable if (a, b, c) is among the triples

$$(1,3,12),\ (1,3,18),\ (1,3,21),\ (1,3,60),\ (1,5,15),$$
$$(1,8,24),\ (1,12,15),\ (1,24,56),\ (3,4,9),\ (3,9,13),$$
$$(4,5,12),\ (4,5,60),\ (4,9,60),\ (4,12,21),\ (4,12,45),\ (5,36,40).$$

(iii) If a, b, c are positive integers with $ax^2 + by^2 + cz^2$ suitable, then a, b, c cannot be pairwise coprime.

Suitable polynomials related to Pythagorean triples

Conjecture (Z.-W. Sun, April 12, 2016). Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $(10w + 5x)^2 + (12y + 36z)^2$ is a square.

Remark: In 2017, Yu-Chen Sun and Z.-W. Sun proved that any $n \in \mathbb{N}$ can be written as $w^2 + x^2 + y^2 + z^2$ with w, x, y, z *integers* such that $(10w + 5x)^2 + (12y + 36z)^2$ is a square.

Conjecture (Z.-W. Sun, May 15, 2016). (i) Any positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > z$ such that $(x + y)^2 + (4z)^2$ is a square.

(ii) Any integer $n > 5$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $8x + 12y$ and $15z$ are the two legs of a right triangle with positive integer sides.

Theorem (Z.-W. Sun, May 16, 2016). Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > 0$ such that $x + 4y + 4z$ and $9x + 3y + 3z$ are the two legs of a right triangle with positive integer sides.

A conjecture involving mixed terms

Conjecture (Z.-W. Sun, 2016) (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $xy + 2zw$ or $xy - 2zw$ is a square.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $w^2 + 4xy + 8yz + 32zx$ is a square.

(iii) Let $a, b, c \in \mathbb{N}$ with $1 \leqslant a \leqslant c$ and $\gcd(a, b, c)$ squarefree. Then $awx + bxy + cyz$ is suitable if and only if

$$(a, b, c) = (1, 2, 2),\ (2, 1, 4),\ (2, 8, 4).$$

(iv) Let $a, b, c \in \mathbb{Z}^+$ with $\gcd(a, b, c)$ squareferee. Then the polynomial $axy + byz + czx$ is suitable if and only if $\{a, b, c\}$ is among the sets

$$\{1, 2, 3\},\ \{1, 3, 8\},\ \{1, 8, 13\},\ \{2, 4, 45\},$$
$$\{4, 5, 7\},\ \{4, 7, 23\},\ \{5, 8, 9\},\ \{11, 16, 31\}.$$

(v) $36x^2y + 12y^2z + z^2x$, $w^2x^2 + 3x^2y^2 + 2y^2z^2$ and $w^2x^2 + 5x^2y^2 + 80y^2z^2 + 20z^2w^2$ are suitable.

Suitable polynomials of the form $ax^4 + by^3z$

The following conjecture sounds very mysterious!

Conjecture (Z.-W. Sun, 2016) Let a and b be nonzero integers with $\gcd(a, b)$ squarefree. Then the polynomial $ax^4 + by^3z$ is suitable if and only if (a, b) is among the ordered pairs

$$(1, 1),\ (1, 15),\ (1, 20),\ (1, 36),\ (1, 60),\ (1, 1680) \text{ and } (9, 260).$$

Examples:

$$9983 = 63^2 + 54^2 + 17^2 + 53^2$$

with $63^4 + 54^3 \times 17 = 4293^2$, and

$$20055 = 47^2 + 6^2 + 77^2 + 109^2$$

with $47^4 + 1680 \times 6^3 \times 77 = 5729^2$.

Other suitable polynomials

Theorem (i) (Conjectured by Z.-W. Sun and essentially proved by You-Ying Deng and Yu-Chen Sun) $x^2 - 4yz$, $x^2 + 4yz$ and $x^2 + 8yz$ are suitable.

(ii) (Z.-W. Sun, May 2016) $x^2y^2 + y^2z^2 + z^2x^2$ and $x^2y^2 + 4y^2z^2 + 4z^2x^2$ are suitable. Also, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $x^4 + 8yz(y^2 + z^2)$ (or $x^4 + 16yz(y^2 + 4z^2)$) is a fourth power.

Conjecture (Z.-W. Sun, 2016) (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ ($w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$) with $x^3 + 4yz(y - z)$ (or $x^3 + 8yz(2y - z)$) a square.

(ii) The polynomials $w(x + 2y + 3z)$, $w(x^2 + 8y^2 - z^2)$ and $x^2 + 3y^2 + 5z^2 - 8w^2$ are suitable.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $z < w$ such that $4x^2 + 5y^2 + 20zw$ is a square.

A new direction

Conjecture (Z.-W. Sun, August 7, 2016). (i) Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2(1 + y^2 + z^2)$ with $w, x, y, z \in \mathbb{N}$, $x > 0$ and $y \equiv z \pmod 2$. Moreover, any $n \in \mathbb{Z}^+ \setminus \{449\}$ can be written as $4^k(1 + x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$ and $x \equiv y \pmod 2$.

(ii) Each $n \in \mathbb{Z}^+$ can be written as $4^k(1 + x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$ and $x \leqslant y \leqslant z$.

Theorem (Z.-W. Sun, 2016) (i) Any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + 4x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$.

(ii) Under the GRH, any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + 5x^2 + y^2) + z^2$ with $k, x, y, z \in \mathbb{N}$, and also any $n \in \mathbb{Z}^+$ can be written as $4^k(1 + x^2 + y^2) + 5z^2$ with $k, x, y, z \in \mathbb{N}$.

Remark. Our proof of part (ii) uses the work of Ben Kane and Z.-W. Sun [Trans. AMS 362(2010), 6425–6455], where the authors determined for what $a, b, c \in \mathbb{Z}^+$ sufficiently large integers can be expressed as $ax^2 + by^2 + cz(z + 1)/2$ with $x, y, z \in \mathbb{Z}$.

Part IV. Further Conjectures on Sums of Four Squares

Restrictions involving powers of two

Theorem (Z.-W. Sun, arXiv:1701.05868) (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $|x - 2y| \in \{4^k : k \in \mathbb{N}\}$.
(ii) For each $c = 1, 2$, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y + cz \in \{c4^k : k \in \mathbb{N}\}$.
(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y + 2z \in \{4^k : k \in \mathbb{N}\}$.
(iv) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{4^k : k \in \mathbb{N}\}$.
(v) Any integer $n > 1$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{3 \times 2^k : k \in \mathbb{N}\}$.

Conjecture (Z.-W. Sun, 2016). Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y - 2z$ is a power of four (including $4^0 = 1$).

Remark. Qing-Hu Hou has verified this for n up to 10^9.

The 24-conjecture

24-Conjecture (Z.-W. Sun, Feb. 4, 2017). Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $x + 24y$ are squares.

Remark. Qing-Hu Hou has verified this for $n \leqslant 10^{10}$. I would like to offer 2400 US dollars as the prize for the first proof.

$$\begin{aligned}
12 =& 1^2 + 1^2 + 1^2 + 3^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 1 = 5^2,\\
23 =& 1^2 + 2^2 + 3^2 + 3^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 2 = 7^2,\\
24 =& 4^2 + 0^2 + 2^2 + 2^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 0 = 2^2,\\
47 =& 1^2 + 1^2 + 3^2 + 6^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 1 = 5^2,\\
71 =& 1^2 + 5^2 + 3^2 + 6^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 5 = 11^2,\\
168 =& 4^2 + 4^2 + 6^2 + 10^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 4 = 10^2,\\
344 =& 4^2 + 0^2 + 2^2 + 18^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 0 = 2^2,\\
632 =& 0^2 + 6^2 + 14^2 + 20^2 \text{ with } 0 = 0^2 \text{ and } 0 + 24 \times 6 = 12^2,\\
1724 =& 25^2 + 1^2 + 3^2 + 33^2 \text{ with } 25 = 5^2 \text{ and } 25 + 24 \times 1 = 7^2.
\end{aligned}$$

A similar conjecture

Conjecture (Z.-W. Sun, Feb. 4, 2017). (i) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $49x + 48(y - z)$ are squares.

(ii) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $121x + 48(y - z)$ are squares.

(iii) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $-7x - 8y + 8z + 16w$ are squares.

Remark. Qing-Hu Hou has verified parts (i)-(ii) and part (iii) for n up to 10^9 and 10^8.

Other conjectures involving joint restrictions

Conjecture (Z.-W. Sun, Feb. 2017). (i) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x \equiv y \pmod 2$ such that both x and $x^2 + 62xy + y^2$ are squares.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^4 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that $8y^2 - 8yz + 9z^2$ is a square. Also, each $n \in \mathbb{N}$ can be written as $4x^4 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $79y^2 - 220yz + 205z^2$ is a square.

(iii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both $2x - y$ and $4z^2 + 724zw + w^2$ (or $9z^2 + 666zw + w^2$) are squares.

(iv) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both $2x - y$ and $64z^2 - 84zw + 21w^2$ (or $81z^2 - 112zw + 56w^2$) are squares.

Remark. Qing-Hu Hou has verified part (iii) for n up to 10^9.

Other conjectures involving joint restrictions

Conjecture (Z.-W. Sun, March 2, 2017). (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, w \in \mathbb{N}$ and $y, z \in \mathbb{Z}$ such that both $x + 2y$ and $z + 2w$ are squares.

(ii) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{Z}$ and $w \in \mathbb{N}$ such that both $x + 3y$ and $z + 3w$ are squares.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that both $2x + y$ and $2x + z$ are squares.

Conjectures involving cubic diophantine equations

Conjecture (Z.-W. Sun, March 2017). (i) Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ and $y \equiv z \pmod 2$ such that $72x^3+(y-z)^3$ is a square.

(ii) Let $a,b \in \mathbb{Z}^+$ with $\gcd(a,b)$ squarefree. Then, each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that $ax^3+b(y-z)^3$ is a square, if and only if (a,b) is among the ordered pairs

$$(1,1),\ (1,9),\ (2,18),\ (8,1),\ (9,5),\ (9,8),$$
$$(9,40),\ (16,2),\ (18,16),\ (25,16),\ (72,1).$$

(iii) Let a and $b \geqslant a$ be positive integers with $\gcd(a,b)$ squarefree. Then, every $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with ax^3+by^3 a square, if and only if (a,b) is among the ordered pairs

$$(1,2),\ (1,8),\ (2,16),\ (4,23),\ (4,31),\ (5,9),$$
$$(8,9),\ (8,225),\ (9,47),\ (25,88),\ (50,54).$$

Part V. Restricted Sums of Three Squares

On the representation $n = x^2 + y^2 + z(3z - 1)/2$

Those numbers $z(3z-1)/2$ ($z \in \mathbb{Z}$) are called *generalized pentagonal numbers*.

In a paper published in 2015, I noted that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z(3z-1)/2$ with $x, y, z \in \mathbb{Z}$. Surprising, this can be further refined.

Conjecture (Z.-W. Sun, March 3, 2017). Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z(3z-1)/2$ with $x, y, z \in \mathbb{Z}$ such that $x + 2y$ is a square.

Example.

$$803 = (-17)^2 + 13^2 + \frac{(-15)(3(-15)-1)}{2} \text{ with } (-17) + 2 \times 13 = 3^2.$$

Refining the Gauss-Legendre theorem

Gauss-Legendre Theorem. A nonnegative integer n can be expressed as the sum of three squares if and only if it is not of the form $4^k(8l+7)$ with $k, l \in \mathbb{N}$.

Conjecture (Z.-W. Sun, March 4, 2017). (i) Any $n \in \mathbb{Z}^+$ with $\mathrm{ord}_2(n)$ odd can be written as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{Z}$ such that $x+3y+5z$ is a square.

(ii) Let $n \in \mathbb{N} \setminus \{63\}$. Then $4n+1$ can be written as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{Z}$ such that $x+3y+5z$ is a square.

(iii) Any $n \in \mathbb{N}$ not of the form $4^k(8l+7)$ with $k, l \in \mathbb{N}$ can be written as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{Z}$ such that $x+2y+3z$ is a square or twice a square.

Example.

$$1430 = (-13)^2+(-6)^2+35^2 \text{ with } (-13)+3\times(-6)+5\times 35 = 12^2.$$

Restricted representations for positive odd numbers

It is known that any positive odd numbers can be written as $x^2+y^2+2z^2$ with $x,y,z\in\mathbb{Z}$. Also, we can replace $x^2+y^2+2z^2$ by $x^2+2y^2+3z^2$.

Conjecture (Z.-W. Sun, March 5, 2017). (i) Any positive odd integer can be written as $x^2+y^2+2z^2$ with $x,y,z\in\mathbb{Z}$ such that $2x+y+z$ is a square or a power of two.

(ii) Any positive odd integer can be written as $x^2+2y^2+3z^2$ with $x,y,z\in\mathbb{Z}$ such that $x+y+z$ is a square or twice a square.

Examples:

$2\times1143+1=(-22)^2+2\times30^2+3\times1^2$ with $(-22)+30+1=3^2$,

and

$$2\times6408+1=(-22)^2+2\times75^2+3\times19^2$$

with

$$(-22)+75+19=2\times6^2.$$

References

For the main sources of my above conjectures and related results, you may look at two recent preprints:

1. Zhi-Wei Sun, *Refining Lagrange's four-square theorem*, J. Number Theory 175(2017), 167–190. `arXiv:1604.06723`

2. Yu-Chen Sun and Zhi-Wei Sun, *Some refinements of Lagrange's four-square theorem*, `arXiv:1605.03074`, `http://arxiv.org/abs/1605.03074`.

3. Zhi-Wei Sun, *Restricted sums of four squares*, `arXiv:1701.05868` , `http://arxiv.org/abs/1701.05868`.

4. Hai-Liang Wu and Zhi-Wei Sun, *On the 1-3-5 conjecture and related topics*, `arXiv:1710.08763`, available from the website `http://arxiv.org/abs/1710.08763`.

Thank you!