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Problems and Results on Sums of Squares

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Abstract

Due to the efforts of Fermat, Euler, Legendre and Gauss, it is known what natural numbers can be written as the sum of two squares or three squares. Lagrange's four-square theorem proved in 1770 states that each natural number can be expressed as the sum of four squares. In the talk we will first review classical results on sums of two or three or four squares. Then we turn to the speaker's recent discoveries which refine Lagrange's four-square theorem or the Gauss-Legendre theorem on sums of three squares. In particular we will introduce our results refining Lagrange's four-square theorem as well as some recent conjectures of the speaker one of which states that any integer greater than one can be written as the sum of two squares, a power of 3 and a power of 5.

Part I. Sums of Two, Three, Four Squares

An assertion of Fermat

Squares: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, clearly

$(2n)^2 \equiv 0 \pmod 4$ and $(2n+1)^2 = 4n(n+1)+1 \equiv 1 \pmod 8$.

If $x, y \in \mathbb{Z}$, then $x^2 + y^2$ is congruent to $0+0$ or $0+1$ or $1+1$ modulo 4, and hence $x^2 + y^2 \not\equiv 3 \mod 4$. Clearly, $2 = 1^2 + 1^2$.

Theorem (claimed by Fermat in 1640 and proved by Euler in 1752). Each prime $p \equiv 1 \pmod 4$ can be written as the sum of two squares.

If $p = x^2 + y^2$ with $x, y \in \mathbb{Z}$, then $(xy^*)^2 \equiv -(yy^*)^2 \equiv -1 \pmod p$ where y^* is the unique integer in $\{1, \ldots, p-1\}$ with $yy^* \equiv 1 \pmod p$. Let $q = \frac{p-1}{2}!$. Then

$$q^2 \equiv \prod_{r=1}^{(p-1)/2} r(p-r) = (p-1)! \equiv -1 \pmod p$$

with the help of Wilson's theorem.

Proof of Fermat's assertion

Two of the $(\lfloor\sqrt{p}\rfloor + 1)^2 > p$ numbers

$$x + qy \ (x, y = 0, \ldots, \lfloor\sqrt{p}\rfloor),$$

say $x_1 + qy_1$ and $x_2 + qy_2$ (with $x_1, y_1, x_2, y_2 \in \{0, \ldots, \lfloor\sqrt{p}\rfloor\}$, and $x_1 \neq x_2$ or $y_1 \neq y_2$), are congruent modulo p. Thus

$$(x_1 - x_2)^2 \equiv (q(y_2 - y_1))^2 \equiv -(y_1 - y_2)^2 \pmod{p}.$$

Let $x = x_1 - x_2$ and $y = y_1 - y_2$. Then $x^2 + y^2 \equiv 0 \pmod{p}$ and

$$0 < x^2 + y^2 \leqslant \lfloor\sqrt{p}\rfloor^2 + \lfloor\sqrt{p}\rfloor^2 < 2p.$$

So we must have $x^2 + y^2 = p$.

Jacobsthal's Theorem

It can be shown that any prime $p \equiv 1 \pmod 4$ can be written **uniquely** in the form $x^2 + y^2$ with $x, y \in \mathbb{N}$ and $x \leqslant y$.

Recall that for an odd prime p and an integer a, the Legendre symbol $(\frac{a}{p})$ is defined as follows:

$$\left(\frac{a}{p}\right) = \begin{cases} 0 & \text{if } p \mid a, \\ 1 & \text{if } p \nmid a \text{ and } x^2 \equiv a \pmod p \text{ for some } x \in \mathbb{Z}, \\ -1 & \text{if } p \nmid a \text{ and } x^2 \not\equiv a \pmod p \text{ for all } x \in \mathbb{Z}. \end{cases}$$

Jacobsthal's Theorem (1907). Let p be any prime congruent to 1 mod 4. Let $s, t \in \mathbb{Z}$ with $(\frac{s}{p}) = 1$ and $(\frac{t}{p}) = -1$. Then

$$p = \left(\sum_{x=0}^{(p-1)/2} \left(\frac{x(x^2+s)}{p} \right) \right)^2 + \left(\sum_{x=0}^{(p-1)/2} \left(\frac{x(x^2+t)}{p} \right) \right)^2.$$

Sums of Two Squares

The set $\{x^2 + y^2 : \ x, y \in \mathbb{Z}\}$ is closed under multiplication. In fact,

$$(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2.$$

Thus those $n \in \mathbb{N}$ with $\mathrm{ord}_p(n)$ even for any prime $p \equiv 3 \pmod 4$ can be written as the sum of two squares. In 1749 Euler showed that other numbers cannot be written as the sum of two squares.

Theorem (Jacobi, 1834). Let $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$ and define

$$r_2(n) := |\{(x, y) \in \mathbb{Z}^2 : \ x^2 + y^2 = n\}|.$$

Then we have

$$r_2(n) = 4 \sum_{2 \nmid d \mid n} (-1)^{(d-1)/2}.$$

Note that $\sum_{n=0}^{\infty} r_2(n) q^n = \varphi(q)^2$, where

$$\varphi(q) := \sum_{x=-\infty}^{+\infty} q^{x^2}.$$

Four-square theorem

Four-Square Theorem. Each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as the sum of four squares.

Examples. $3 = 1^2 + 1^2 + 1^2 + 0^2$ and $7 = 2^2 + 1^2 + 1^2 + 1^2$.

A. Diophantus (AD 299-215, or AD 285-201) was aware of this theorem as indicated by examples given in his book *Arithmetica*.

In 1621 Bachet translated Diophantus' book into Latin and stated the theorem in the notes of his translation.

In 1748 L. Euler found the four-square identity

$$\begin{aligned} &(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) \\ =&(x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4)^2 + (x_1 y_2 - x_2 y_1 - x_3 y_4 + x_4 y_3)^2 \\ &+ (x_1 y_3 - x_3 y_1 + x_2 y_4 - x_4 y_2)^2 + (x_1 y_4 - x_4 y_1 - x_2 y_3 + x_3 y_2)^2. \end{aligned}$$

and hence reduced the theorem to the case with n prime.

Euler also proved for each odd prime p there is a positive integer $m < p$ such that $pm = x^2 + y^2 + 1^2 + 0^2$ for some $x, y \in \mathbb{Z}$.

Lagrange first proved the Four-Square Theorem

On the basis of Euler's work, in 1770 J. L. Lagrange first completed the proof of the four-square theorem. The celebrated theorem is now well known as *Lagrange's four-square theorem*.

A crucial step in Lagrange's proof is to show that for any odd prime p if $pm = x^2 + y^2 + z^2 + w^2$ for some $m \in \{2, \ldots, p-1\}$ and $x, y, z, w \in \mathbb{Z}$ then there is a positive integer $m_0 < m$ such that pm_0 can be written as the sum of four squares.

It is known that only the following numbers have a unique representation as the sum of four unordered squares:

$$1,\ 3,\ 5,\ 7,\ 11,\ 15,\ 23$$

and

$$2^{2k+1}m\ (k = 0, 1, 2, \ldots \text{ and } m = 1, 3, 7).$$

For example, $4^k \times 14 = (2^k 3)^2 + (2^{k+1})^2 + (2^k)^2 + 0^2$.

The representation function $r_4(n)$

Jacobi used his triple-product formula

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1}z)(1+q^{2n-1}z^{-1}) = \sum_{n=-\infty}^{+\infty} z^n q^{n^2} \ \ (|q|<1,\ z\neq 0)$$

to study the fourth power of $\varphi(q)=\sum_{n=-\infty}^{\infty} q^{n^2}$, and this led him to deduce that

$$r_4(n) = 8\sum_{d\mid n\ \&\ 4\nmid d} d \quad \text{for all } n\in\mathbb{Z}^+,$$

where

$$r_4(n) := |\{(w,x,y,z)\in\mathbb{Z}^4:\ w^2+x^2+y^2+z^2=n\}|.$$

This is related to modular forms of weight two. Let τ be a complex number with positive real part and set $\theta(\tau)=\varphi(e^{2\pi i\tau})$. Then

$$\theta\left(\frac{\tau}{4\tau+1}\right)=\sqrt{4\tau+1}\,\theta(\tau) \text{ and hence } \theta^4\left(\frac{\tau}{4\tau+1}\right)=(4\tau+1)^2\theta^4(\tau).$$

Sums of three squares

Suppose that $n = x^2 + y^2 + z^2$ with $x, y, z \in \mathbb{Z}$. If $4 \mid n$, then x, y, z are all even and

$$\frac{n}{4} = \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2.$$

If $n \equiv 3 \mod 4$, then x, y, z are odd and hence $n \equiv 3 \pmod 8$.

Gauss-Legendre Theorem. $n \in \mathbb{N}$ can be written as the sum of three squares if and only if n is not of the form $4^k(8l + 7)$ with $k, l \in \mathbb{N}$.

Let $n \in \mathbb{Z}^+$ and let $h(-n)$ denote the class number of the field $\mathbb{Q}(\sqrt{-n})$. For

$R_3(n) := |\{(x, y, z) \in \mathbb{Z}^3 : \; x^2 + y^2 + z^2 = n \text{ and } \gcd(x, y, z) = 1\}|,$

Gauss proved that

$$R_3(n) = \begin{cases} 12h(-n) & \text{if } n > 3 \text{ and } n \equiv 1, 2 \pmod 4, \\ 24h(-n) & \text{if } n > 3 \text{ and } n \equiv 3 \pmod 8, \\ 0 & \text{if } 4 \mid n \text{ or } n \equiv 7 \pmod 8. \end{cases}$$

Part II. New Problems for Sums of Four Squares

Universal sums of four mixed powers

If any $n \in \mathbb{N}$ can be written as $f(x_1, \ldots, x_n)$ with $x_1, \ldots, x_n$ in $\mathbb{N}$ (or $\mathbb{Z}$), then we say that f is *universal over* $\mathbb{N}$ (or $\mathbb{Z}$).

Theorem (Sun [J. Number Theory 175(2017)]) For any $a \in \{1, 4\}$ and $k \in \{4, 5, 6\}$, $aw^k + x^2 + y^2 + z^2$ is universal over $\mathbb{N}$.

Theorem (Z.-W. Sun [Nanjing Univ. J. Math. Biquarterly 34(2017)]) Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c \leqslant d$, and let $h, i, j, k \in \{2, 3, \ldots\}$ with at most one of h, i, j, k equal to two. Suppose that $h \leqslant i$ if $a = b$, $i \leqslant j$ if $b = c$, and $j \leqslant k$ if $c = d$. If $f(w, x, y, z) = aw^h + bx^i + cy^j + dz^k$ is universal over $\mathbb{N}$, then $f(w, x, y, z)$ must be among the following 9 polynomials

$$w^2 + x^3 + y^4 + 2z^3,\ w^2 + x^3 + y^4 + 2z^4,\ w^2 + x^3 + 2y^3 + 3z^3,$$

$$w^2 + x^3 + 2y^3 + 3z^4,\ w^2 + x^3 + 2y^3 + 4z^3,\ w^2 + x^3 + 2y^3 + 5z^3,$$

$$w^2 + x^3 + 2y^3 + 6z^3,\ w^2 + x^3 + 2y^3 + 6z^4,\ w^3 + x^4 + 2y^2 + 4z^3.$$

Conjecture (Sun [Nanjing Univ. J. Math. Biquarterly 34(2017)]) All the 9 polynomials are universal over $\mathbb{N}$.

Discoveries on April 8, 2016

Motivated by my conjecture that any $n \in \mathbb{N}$ can be written as

$$x_1^3 + x_2^3 + 2x_3^3 + 2x_4^3 + 3x_5^3 \quad (x_1, x_2, x_3, x_4, x_5 \in \mathbb{N})$$

(which is stronger than the result $g(3) = 9$ for Waring's problem), on April 8, 2016 I considered to write $n \in \mathbb{N}$ as $\sum_{i=1}^5 a_i x_i^2$ $(x_i \in \mathbb{N})$ with certain restrictions on $x_1, \ldots, x_5$.

Conjecture (Z.-W. Sun) Let $n > 1$ be an integer.

(i) n can be written as

$$x_1^2+x_2^2+x_3^2+2x_4^2+2x_5^2 = x_1^2+x_2^2+x_3^2+(\underline{x_4+x_5})^2+(x_4-x_5)^2 \ (x_i \in \mathbb{N})$$

with $x_1 + x_2 + x_3 + \underline{x_4 + x_5}$ prime.

(ii) We can write n as

$$x_1^2 + 2x_2^2 + 3x_3^2 + 4x_4^2 + 5x_5^2 \quad (x_1, x_2, x_3, x_4, x_5 \in \mathbb{N})$$

with $x_1 + x_2 + x_3 + x_4$ a square.

Remark. Squares are sparser than prime numbers.

1-3-5-Conjecture (1350 US dollars for the first solution)

1-3-5-Conjecture (Z.-W. Sun, April 9, 2016): Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.

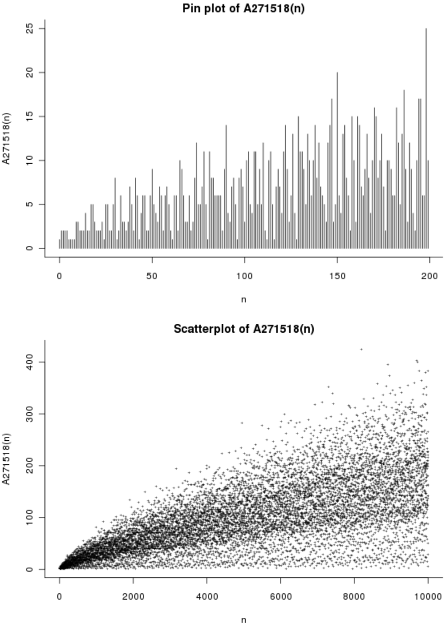
Examples.

$$\begin{aligned} 7 =& 1^2 + 1^2 + 1^2 + 2^2 \text{ with } 1 + 3 \times 1 + 5 \times 1 = 3^2, \\ 8 =& 0^2 + 2^2 + 2^2 + 0^2 \text{ with } 0 + 3 \times 2 + 5 \times 2 = 4^2, \\ 31 =& 5^2 + 2^2 + 1^2 + 1^2 \text{ with } 5 + 3 \times 2 + 5 \times 1 = 4^2, \\ 43 =& 1^2 + 5^2 + 4^2 + 1^2 \text{ with } 1 + 3 \times 5 + 5 \times 4 = 6^2. \end{aligned}$$

The conjecture has been verified by Qing-Hu Hou for all $n \leqslant 10^{10}$.

We guess that, if a, b, c are positive integers with $\gcd(a, b, c)$ squarefree such that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $ax + by + cz$ a square, then we must have $\{a, b, c\} = \{1, 3, 5\}$.

Graph for the number of such representations of n



无　解

数字几时有，
把酒问青天。
一二三四五，
自然藏玄机。

四个平方和，
遍历自然数。
奇妙一三五，
更上一层楼。

苍天捉弄人，
数论妙无穷。
吾辈虽努力，
难解一三五!

时势唤英雄，
攻关需豪杰。
人间若无解，
天神会证否?

Refinements of the 1-3-5 conjecture

Conjecture (Z.-W. Sun, Dec. 12, 2016): For any positive integer n, there is a prime p such that there are exactly n ways to write p as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $x + 3y + 5z$ a square.

Remark. I have verified this for $n = 1, \ldots, 500$.

Conjecture (Z.-W. Sun, Feb. 17, 2017): Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ such that $x + 3y + 5z$ and (at least) one of y, z, w are squares.

Conjecture (Z.-W. Sun, March 12, 2018): Each $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ such that $x + 3y + 5z$ is a positive square and $2x$ (or $3x$) or y or z is a square.

Examples:

$$8 = 0^2 + 2^2 + 2^2 + 0^2 \text{ with } 0 + 3 \times 2 + 5 \times 2 = 4^2 \text{ and } 0 = 0^2;$$
$$188 = 7^2 + 9^2 + 3^2 + 7^2 \text{ with } 7 + 3 \cdot 9 + 5 \cdot 3 = 7^2 \text{ and } 9 = 3^2;$$
$$248 = 10^2 + 2^2 + 0^2 + 12^2 \text{ with } 10 + 3 \cdot 2 + 5 \cdot 0 = 4^2 \text{ and } 0 = 0^2;$$
$$808 = 12^2 + 14^2 + 18^2 + 12^2 \text{ with } 12 + 3 \cdot 14 + 5 \cdot 18 = 12^2 \,\&\, 3 \cdot 12 = 6^2.$$

1-2-3-Conjecture (Companion of 1-3-5-Conjecture)

1-2-3-Conjecture (Z.-W. Sun, July 24, 2016). Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y + 3z$ is a square.

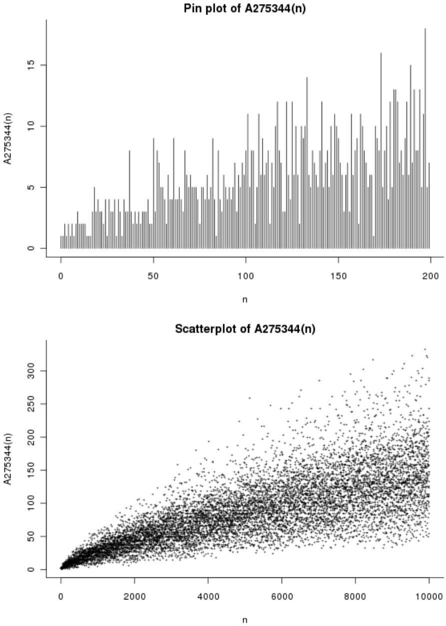
Examples:

$$\begin{aligned} 14 =& 1^2 + 1^2 + 2^2 + 2 \times 2^2 \ \text{ with } 1 + 2 \times 1 + 3 \times 2 = 3^2, \\ 30 =& 3^2 + 2^2 + 3^2 + 2 \times 2^2 \ \text{ with } 3 + 2 \times 2 + 3 \times 3 = 4^2, \\ 33 =& 1^2 + 0^2 + 0^2 + 2 \times 4^2 \ \text{ with } 1 + 2 \times 0 + 3 \times 0 = 1^2, \\ 84 =& 4^2 + 6^2 + 0^2 + 2 \times 4^2 \ \text{ with } 4 + 2 \times 6 + 3 \times 0 = 4^2, \\ 169 =& 10^2 + 6^2 + 1^2 + 2 \times 4^2 \ \text{ with } 10 + 2 \times 6 + 3 \times 1 = 5^2, \\ 225 =& 10^2 + 6^2 + 9^2 + 2 \times 2^2 \ \text{ with } 10 + 2 \times 6 + 3 \times 9 = 7^2. \end{aligned}$$

Another Conjecture (Sun, 2017). Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + 2w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 3z = 1$.

Remark. This was proved by Hai-Liang Wu and Z.-W. Sun in 2017 for sufficient large integers n.

Graph for the number of such representations of n



Diagonal ternary quadratic forms

For $a, b, c \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we define

$$E(a, b, c) := \{n \in \mathbb{N} : \ n \neq ax^2 + by^2 + cz^2 \text{ for any } x, y, z \in \mathbb{N}\}.$$

It is known that $E(a, b, c)$ is an infinite set.

Gauss-Legendre Theorem. $E(1, 1, 1) = \{4^k(8l + 7) : \ k, l \in \mathbb{N}\}$.

There are totally 102 diagonal ternary quadratic forms $ax^2 + by^2 + cz^2$ with $a, b, c \in \mathbb{Z}^+$ and $\gcd(a, b, c) = 1$ for which the structure of $E(a, b, c)$ is known explicitly. For example,

$$\begin{aligned}
E(1,1,2) =& \{4^k(16l + 14) : \ k, l \in \mathbb{N}\},\\
E(1,1,5) =& \{4^k(8l + 3) : \ k, l \in \mathbb{N}\},\\
E(1,2,3) =& \{4^k(16l + 10) : \ k, l \in \mathbb{N}\},\\
E(1,2,6) =& \{4^k(8l + 5) : \ k, l \in \mathbb{N}\}.
\end{aligned}$$

Sums of a fourth power and three squares

Theorem (Z.-W. Sun, March 27, 2016). Each $n \in \mathbb{N}$ can be written as $w^4 + x^2 + y^2 + z^2$ with $w, x, y, z \in \mathbb{N}$.

Proof. For $n = 0, 1, 2, \ldots, 15$, the result can be verified directly. Now let $n \geqslant 16$ be an integer and assume that the result holds for smaller values of n.

Case 1. $16 \mid n$.
By the induction hypothesis, we can write

$$\frac{n}{16} = x^4 + y^2 + z^2 + w^2 \ \text{ with } x, y, z, w \in \mathbb{N}.$$

It follows that $n = (2x)^4 + (4y)^2 + (4z)^2 + (4w)^2$.

Case 2. $n = 4^k q$ with $k \in \{0, 1\}$ and $q \equiv 7 \pmod 8$.
In this case, $n - 1 \notin E(1, 1, 1)$, and hence $n = 1^4 + y^2 + z^2 + w^2$ for some $y, z, w \in \mathbb{N}$.

Case 3. $16 \nmid n$ and $n \neq 4^k(8l + 7)$ for any $k \in \{0, 1\}$ and $l \in \mathbb{N}$.
In this case, $n \notin E(1, 1, 1)$ and hence there are $y, z, w \in \mathbb{N}$ such that $n = 0^4 + y^2 + z^2 + w^2$.

Suitable polynomials

Definition (Z.-W. Sun, 2016). A polynomial $P(x, y, z, w)$ with integer coefficients is called *suitable* if any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $P(x, y, z, w)$ is a square.

We have seen that both x and $2x$ are suitable polynomials. The 1-3-5-Conjecture says that $x + 3y + 5z$ is suitable.

We conjecture that there only finitely many $a, b, c, d \in \mathbb{Z}$ with $\gcd(a, b, c, d)$ squarefree such that $ax + by + cz + dw$ is suitable, and we have found all such quadruples (a, b, c, d).

$x - y$ and $2x - 2y$ are suitable

Let $a \in \{1, 2\}$. We claim that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $a(x - y)$ is a square, and want to prove this by induction.

For every $n = 0, 1, \ldots, 15$, we can verify the claim directly.

Now we fix an integer $n \geqslant 16$ and assume that the claim holds for smaller values of n.

Case 1. $16 \mid n$.
In this case, by the induction hypothesis, there are $x, y, z, w \in \mathbb{N}$ with $a(x - y)$ a square such that $n/16 = x^2 + y^2 + z^2 + w^2$, and hence $n = (4x)^2 + (4y)^2 + (4z)^2 + (4w)^2$ with $a(4x - 4y)$ a square.

Case 2. $16 \nmid n$ and $n \notin E(1, 1, 2)$.
In this case, there are $x, y, z, w \in \mathbb{N}$ with $x = y$ and $n = x^2 + y^2 + z^2 + w^2$, thus $a(x - y) = 0^2$ is a square.

$x - y$ and $2x - 2y$ are suitable

Case 3. $16 \nmid n$ and $n \in E(1,1,2) = \{4^k(16l + 14) : \ k, l \in \mathbb{N}\}$.

In this case, $n = 4^k(16l + 14)$ for some $k \in \{0, 1\}$ and $l \in \mathbb{N}$. Note that $n/2 - (2/a)^2 \notin E(1,1,1)$. So, $n/2 - (2/a)^2 = t^2 + u^2 + v^2$ for some $t, u, v \in \mathbb{N}$ with $t \geqslant u \geqslant v$. As $n/2 - (2/a)^2 \geqslant 8 - 4 > 3$, we have $t \geqslant 2 \geqslant 2/a$. Thus

$$\begin{aligned} n =& 2\left(\left(\frac{2}{a}\right)^2 + t^2\right) + 2(u^2 + v^2) \\ =& \left(t + \frac{2}{a}\right)^2 + \left(t - \frac{2}{a}\right)^2 + (u + v)^2 + (u - v)^2 \end{aligned}$$

with

$$a\left(\left(t + \frac{2}{a}\right) - \left(t - \frac{2}{a}\right)\right) = 2^2.$$

This proves that $x - y$ and $2x - 2y$ are both suitable.

Suitable polynomials of the form $ax \pm by$

Conjecture (Z.-W. Sun, April 14,2016) Let $a, b \in \mathbb{Z}^+$ with $\gcd(a, b)$ squarefree.

(i) The polynomial $ax + by$ is suitable if and only if $\{a, b\} = \{1, 2\}, \{1, 3\}, \{1, 24\}$.

(ii) The polynomial $ax - by$ is suitable if and only if (a, b) is among the ordered pairs

$$(1, 1),\ (2, 1),\ (2, 2),\ (4, 3),\ (6, 2).$$

Remark. In 2016, I proved that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + 2y$ is a square (or a cube). In a joint paper with my student Yu-Chen Sun [Acta Arith. 183(2018)], we managed to show that $x + 2y$ is indeed suitable.

Write $n = x^2 + y^2 + z^2 + w^2$ with $x + 3y$ a square

In 1916 Ramanujan conjectured that

(1) *the only positive even numbers not of the form* $x^2 + y^2 + 10z^2$ *are those* $4^k(16l + 6)$ $(k, l \in \mathbb{N})$

and

(2) *sufficiently large odd numbers are of the form* $x^2 + y^2 + 10z^2$.

In 1927 L. E. Dickson [Bull. AMS] proved (1). In 1990 W. Duke and R. Schulze-Pillot [Invent. Math.] confirmed (2). In 1997 K. Ono and K. Soundararajan [Invent. Math.] proved that under the GRH (Generalized Riemann Hypothesis) any odd number greater than 2719 has the form $x^2 + y^2 + 10z^2$.

With the help of the Ono-Soundararajan result, the speaker has proved the following result.

Theorem (Z.-W. Sun, 2016) Under the GRH, any $n \in \mathbb{N}$ can be written as $n = x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 3y$ a square.

Suitable $ax - by - cz$ or $ax + by - cz$

Conjecture (Z.-W. Sun, April 14, 2016): (i) Let $a, b, c \in \mathbb{Z}^+$ with $b \leqslant c$ and $\gcd(a, b, c)$ squarefree. Then $ax - by - cz$ is suitable if and only if (a, b, c) is among the five triples

$$(1,1,1),\ (2,1,1),\ (2,1,2),\ (3,1,2),\ (4,1,2).$$

(ii) Let $a, b, c \in \mathbb{Z}^+$ with $a \leqslant b$ and $\gcd(a, b, c)$ squarefree. Then $ax + by - cz$ is suitable if and only if (a, b, c) is among the following 52 triples

$$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,3),\ (1,3,1),$$
$$(1,3,3),\ (1,4,4),\ (1,5,1),\ (1,6,6),\ (1,8,6),\ (1,12,4),\ (1,16,1),$$
$$(1,17,1),\ (1,18,1),\ (2,2,2),\ (2,2,4),\ (2,3,2),\ (2,3,3),\ (2,4,1),$$
$$(2,4,2),\ (2,6,1),\ (2,6,2),\ (2,6,6),\ (2,7,4),\ (2,7,7),\ (2,8,2),$$
$$(2,9,2),\ (2,32,2),\ (3,3,3),\ (3,4,2),\ (3,4,3),\ (3,8,3),\ (4,5,4),$$
$$(4,8,3),\ (4,9,4),\ (4,14,14),\ (5,8,5),\ (6,8,6),\ (6,10,8),\ (7,9,7),$$
$$(7,18,7),(7,18,12),\ (8,9,8),\ (8,14,14),\ (8,18,8),\ (14,32,14),$$
$$(16,18,16),\ (30,32,30),\ (31,32,31),\ (48,49,48),\ (48,121,48).$$

Linear restrictions involving cubes

Conjecture (Z.-W. Sun, 2016). For each $c = 1, 2, 4$, any $n \in \mathbb{N}$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w, x, y, z \in \mathbb{N}$ and $y \leqslant z$ such that $2x + y - z = ct^3$ for some $t \in \mathbb{N}$.

Examples.

$$13 = 2^2 + 0^2 + 3^2 + 0^2 \quad \text{with} \quad 2 \times 2 + 0 - 3 = 1^3,$$
$$2976 = 20^2 + 16^2 + 48^2 + 4^2 \quad \text{with} \quad 2 \times 20 + 16 - 48 = 2^3.$$

Conjecture (Z.-W. Sun, March 17, 2018). Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $x + 3y + 9z = 2^k m^3$ for some $k, m \in \mathbb{N}$.

Examples.

$$79 = 2^2 + 7^2 + 1^2 + 5^2 \quad \text{with} \quad 2 + 3 \times 7 + 9 \times 1 = 2^2 3^3,$$
$$496 = 4^2 + 8^2 + 4^2 + 20^2 \quad \text{with} \quad 4 + 3 \times 8 + 9 \times 4 = 4^3.$$

$n = x^2 + y^2 + z^2 + w^2$ with $x + y + z$ a square (or a cube)

Theorem (Z.-W. Sun, April-May, 2016) Let $c \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + cz = t^m$ for some $t \in \mathbb{Z}$.

Proof for the Case $c = 1$. For $n = 0, \ldots, 4^m - 1$ we can easily verify the desired result directly.

Now let $n \in \mathbb{N}$ with $n \geqslant 4^m$. Assume that any $r \in \{0, \ldots, n-1\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + y + z \in \{t^m : \ t \in \mathbb{Z}\}$. If $4^m \mid n$, then there are $x, y, z, w \in \mathbb{Z}$ with $x^2 + y^2 + z^2 + w^2 = n/4^m$ such that $x + y + z = t^m$ for some $t \in \mathbb{Z}$, and hence

$$n = (2^m x)^2 + (2^m y)^2 + (2^m z)^2 + (2^m w)^2$$

with $2^m x + 2^m y + (2^m z) = 2^m(x + y + z) = (2t)^m$. Below we suppose that $4^m \nmid n$.

Continued the proof

It suffices to show that there are $x, y, z \in \mathbb{Z}$ and $\delta \in \{0, 1, 2^m\}$ such that

$$n = x^2 + (y+z)^2 + (z-y)^2 + (\delta - 2z)^2 = x^2 + 2y^2 + 6z^2 - 4\delta z + \delta^2.$$

(Note that $(y+z) + (z-y) + (\delta - 2z) = \delta \in \{t^m : \ t \in \mathbb{Z}\}$.)
Suppose that this fails for $\delta = 0$. As

$$E(1, 2, 6) = \{4^k(8l+5) : \ k, l \in \mathbb{N}\},$$

$n = 4^k(8l+5)$ for some $k, l \in \mathbb{N}$ with $k < m$. Clearly,

$$3n - 1 = \begin{cases} 3(8l+5) - 1 = 2(12l+7) & \text{if } k = 0, \\ 3 \times 4(8l+5) - 1 = 8(12l+7) + 3 & \text{if } k = 1. \end{cases}$$

Thus, if $k \in \{0, 1\}$, then $3n - 1$ does not belong to

$$E(2, 3, 6) = \{3q + 1 : \ q \in \mathbb{N}\} \cup \{4^k(8l+7) : \ k, l \in \mathbb{N}\},$$

Continue the proof

hence for some $x, y, z \in \mathbb{Z}$ we have

$$3n - 1 = 3x^2 + 6y^2 + 2(3z-1)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 2z)) + 2$$

and thus

$$n = x^2 + 2y^2 + 6z^2 - 4z + 1 = x^2 + (y+z)^2 + (z-y)^2 + (1-2z)^2$$

as desired.

When $k = 2$ and $m = 3$, we have

$$3n - 64 = 3 \times 16(8l+5) - 64 = 4^2(8(3l+1)+3) \notin E(2,3,6),$$

and hence there are $x, y, z \in \mathbb{Z}$ such that

$$3n - 4^3 = 3x^2 + 6y^2 + 2(3z-8)^2 = 3(x^2 + 2y^2 + 2(3z^2 - 16z)) + 2 \times 4^3$$

and thus
$$n = x^2 + 2y^2 + 6z^2 - 32z + 64 = x^2 + (y+z)^2 + (z-y)^2 + (2^3 - 2z)^2$$
as desired.

Suitable $ax + by + cz - dw$ or $ax + by - cz - dw$

Conjecture (Z.-W. Sun, April 14, 2016): Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c$ and $\gcd(a, b, c, d)$ squarefree. Then $ax + by + cz - dw$ is suitable if and only if (a, b, c, d) is among the 12 quadruples

$$(1,1,2,1),\ (1,2,3,1),\ (1,2,3,3),\ (1,2,4,2),$$
$$(1,2,4,4),\ (1,2,5,5),\ (1,2,6,2),\ (1,2,8,1),$$
$$(2,2,4,4),\ (2,4,6,4),\ (2,4,6,6),\ (2,4,8,2).$$

Conjecture (Z.-W. Sun, April 14, 2016): Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b$ and $c \leqslant d$, and $\gcd(a, b, c, d)$ squarefree. Then $ax + by - cz - dw$ is suitable if and only if (a, b, c, d) is among the 9 quadruples

$$(1,2,1,1),\ (1,2,1,2),\ (1,3,1,2),\ (1,4,1,3),$$
$$(2,4,1,2),\ (2,4,2,4),\ (8,16,7,8),\ (9,11,2,9),\ (9,16,2,7).$$

Conjecture (Z.-W. Sun, April 2016) For any $a, b, c, d \in \mathbb{Z}^+$ there are infinitely many positive integers not of the form $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ with $ax + by + cz + dw$ a square.

A general theorem joint with Yu-Chen Sun

Theorem (Yu-Chen Sun and Z.-W. Sun, 2016) Let $a, b, c, d \in \mathbb{Z}$ with a, b, c, d not all zero. Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$ Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}/(a^2 + b^2 + c^2 + d^2)$ such that $ax + by + cz + dw = \lambda r^m$ for some $r \in \mathbb{N}$.

Proof. Let $n \in \mathbb{N}$. By a result of Z.-W. Sun, we can write $(a^2 + b^2 + c^2 + d^2)n$ as $(\lambda r^m)^2 + t^2 + u^2 + v^2$ with $r, t, u, v \in \mathbb{N}$. Set $s = \lambda r^m$, and define x, y, z, w by

$$\begin{cases} x = & \frac{as-bt-cu-dv}{a^2+b^2+c^2+d^2}, \\ y = & \frac{bs+at+du-cv}{a^2+b^2+c^2+d^2}, \\ z = & \frac{cs-dt+au+bv}{a^2+b^2+c^2+d^2}, \\ w = & \frac{ds+ct-bu+av}{a^2+b^2+c^2+d^2}. \end{cases}$$

Proof of the general theorem

Then

$$\begin{cases} ax + by + cz + dw = & s, \\ ay - bx + cw - dz = & t, \\ az - bw - cx + dy = & u, \\ aw + bz - cy - dx = & v. \end{cases}$$

With the help of Euler's four-square identity,

$$x^2 + y^2 + z^2 + w^2 = \frac{s^2 + t^2 + u^2 + v^2}{a^2 + b^2 + c^2 + d^2} = n$$

and

$$ax + by + cz + dw = s = \lambda r^m.$$

This concludes the proof.

Joint work with Yu-Chen Sun

Theorem (Y.-C. Sun and Z.-W. Sun [Acta Arith. 183(2018)]) (i) Let $m \in \mathbb{Z}^+$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + z + w$ an m-th power if and only if $m \leqslant 3$.

(ii) Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + y + z + 2w = \lambda r^m$ (or $x + y + 2z + 3w = \lambda r^m$) for some $r \in \mathbb{N}$.

(iii) Let $\lambda \in \{1, 2\}$ and $m \in \{2, 3\}$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{Z})$ with $x + 2y + 3z$ (or $x + y + 3z$, or $x + 2y + 2z$) in the set $\{\lambda r^m : \ r \in \mathbb{N}\}$.

(iv) (Progress on the 1-3-5-Conjecture) Let $\lambda \in \{1, 2\}$, $m \in \{2, 3\}$ and $n \in \mathbb{N}$. Then we can write n as $x^2 + y^2 + z^2 + w^2$ with $x, y, 5z, 5w \in \mathbb{Z}$ such that $x + 3y + 5z \in \{\lambda r^m : \ r \in \mathbb{N}\}$. Also, any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}/7$ such that $x + 3y + 5z \in \{\lambda r^m : \ r \in \mathbb{N}\}$.

A Lemma

The proof of the Theorem needs several lemmas and some previous results of Z.-W. Sun. Here is one of them.

Lemma. Define

$$\begin{cases} x = \frac{s-t-u-2v}{7}, \\ y = \frac{s+t+2u-v}{7}, \\ z = \frac{s-2t+u+v}{7}, \\ w = \frac{2s+t-u+v}{7}. \end{cases}$$

Then

$$x^2 + y^2 + z^2 + w^2 = \frac{s^2 + t^2 + u^2 + v^2}{7}.$$

Also,

$$\begin{aligned} x + y + z + 2w =& s, \\ w + 2x + 3z =& s - t, \\ x + 3y + 5w =& 2s + t. \end{aligned}$$

Joint work with Hai-Liang Wu

Besides the 1-3-5 conjecture, I also conjectured that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that

$$|x + 3y - 5z| \in \{4^k : \ k \in \mathbb{N}\}.$$

In 2017, Hai-Liang Wu and the speaker used the theory of ternary quadratic forms and modular forms to obtain the following progress on the 1-3-5 conjecture.

Theorem (H.-L. Wu and Z.-W. Sun, 2017). Any sufficiently large integer n not divisible by 16 can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $x + 3y + 5z \in \{1, 4\}$.

Let

$$A = \bigcup_{k \in \mathbb{N}} \{4k+1, 4k+2, 8k+4\} \ \text{ and } \ B = \bigcup_{k \in \mathbb{N}} \{4k+3, 16k+8\}.$$

Then $A \cup B = \{n \in \mathbb{N} : \ 16 \nmid n\}$.

Wu and Sun's work

Suppose that $n \in A$ is large enough. Then we are able to show that $70n - 2 = 5r_0^2 + 7r_1^2 + 70w^2$ for some $r_0, r_1, w \in \mathbb{Z}$. As $r_0^2 \equiv 1 \pmod 7$ and $r_1^2 \equiv 2^2 \pmod 5$, without loss of generality we may assume that $r_0 \equiv 1 \pmod 7$ and $r_1 \equiv 2 \pmod 5$. Apparently, $r_0 \equiv r_1 \pmod 2$.

If $r_0 \equiv 1 \pmod{14}$, then $r_0 = 14u + 1$ and $r_1 = 10v - 3$ for some $u, v \in \mathbb{Z}$, hence $70n - 2 = 5(14u + 1)^2 + 7(10v - 3)^2 + 70w^2$ and thus $n = x^2 + y^2 + z^2 + w^2$, where $x = 1 + u - 3v$, $y = 3u + v$ and $z = -2u$. Note that $x + 3y + 5z = 1$.

If $r_0 \equiv 8 \pmod{14}$, then $r_0 = 14u + 8$ and $r_1 = 10v + 2$ for some $u, v \in \mathbb{Z}$, hence $70n - 2 = 5(14u + 8)^2 + 7(10v + 2)^2 + 70w^2$ and thus $n = x^2 + y^2 + z^2 + w^2$, where $x = u - 3v$, $y = 2 + 3u + v$ and $z = -1 - 2u$. Obviously, $x + 3y + 5z = 1$.

Similarly, we can show that if $n \in B$ is large enough then there are $x, y, z, w \in \mathbb{Z}$ such that $n = x^2 + y^2 + z^2 + w^2$ and $x + 3y + 5z = 4$.

Suitable polynomials of the form $ax^2 + by^2 + cz^2$

Conjecture (Z.-W. Sun, April 9, 2016): (i) Any natural number can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x \geqslant y$ such that $ax^2 + by^2 + cz^2$ is a square, provided that the triple (a, b, c) is among

$$(1, 8, 16),\ (4, 21, 24),\ (5, 40, 4),\ (9, 63, 7),\ (16, 80, 25),$$
$$(16, 81, 48),\ (20, 85, 16),\ (36, 45, 40),\ (40, 72, 9).$$

(ii) $ax^2 + by^2 + cz^2$ is suitable if (a, b, c) is among the triples

$$(1, 3, 12),\ (1, 3, 18),\ (1, 3, 21),\ (1, 3, 60),\ (1, 5, 15),$$
$$(1, 8, 24),\ (1, 12, 15),\ (1, 24, 56),\ (3, 4, 9),\ (3, 9, 13),$$
$$(4, 5, 12),\ (4, 5, 60),\ (4, 9, 60),\ (4, 12, 21),\ (4, 12, 45),\ (5, 36, 40).$$

(iii) If a, b, c are positive integers with $ax^2 + by^2 + cz^2$ suitable, then a, b, c cannot be pairwise coprime.

Suitable polynomials related to Pythagorean triples

Conjecture (Z.-W. Sun, April 12, 2016). Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $(10w + 5x)^2 + (12y + 36z)^2$ is a square.

Remark: In 2017, Yu-Chen Sun and Z.-W. Sun proved that any $n \in \mathbb{N}$ can be written as $w^2 + x^2 + y^2 + z^2$ with w, x, y, z *integers* such that $(10w + 5x)^2 + (12y + 36z)^2$ is a square.

Conjecture (Z.-W. Sun, May 15, 2016). (i) Any positive integer n can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > z$ such that $(x + y)^2 + (4z)^2$ is a square.

(ii) Any integer $n > 5$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $8x + 12y$ and $15z$ are the two legs of a right triangle with positive integer sides.

Theorem (Z.-W. Sun, May 16, 2016). Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $y > 0$ such that $x + 4y + 4z$ and $9x + 3y + 3z$ are the two legs of a right triangle with positive integer sides.

A conjecture involving mixed terms

Conjecture (Z.-W. Sun, 2016) (i) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $xy + 2zw$ or $xy - 2zw$ is a square.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $w^2 + x^2 + y^2 + z^2$ with $w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$ such that $w^2 + 4xy + 8yz + 32zx$ is a square.

(iii) Let $a, b, c \in \mathbb{N}$ with $1 \leqslant a \leqslant c$ and $\gcd(a, b, c)$ squarefree. Then $awx + bxy + cyz$ is suitable if and only if

$$(a, b, c) = (1, 2, 2),\ (2, 1, 4),\ (2, 8, 4).$$

(iv) Let $a, b, c \in \mathbb{Z}^+$ with $\gcd(a, b, c)$ squareferee. Then the polynomial $axy + byz + czx$ is suitable if and only if $\{a, b, c\}$ is among the sets

$$\{1, 2, 3\},\ \{1, 3, 8\},\ \{1, 8, 13\},\ \{2, 4, 45\},$$
$$\{4, 5, 7\},\ \{4, 7, 23\},\ \{5, 8, 9\},\ \{11, 16, 31\}.$$

(v) $36x^2y + 12y^2z + z^2x$, $w^2x^2 + 3x^2y^2 + 2y^2z^2$ and $w^2x^2 + 5x^2y^2 + 80y^2z^2 + 20z^2w^2$ are suitable.

Suitable polynomials of the form $ax^4 + by^3z$

The following conjecture sounds very mysterious!

Conjecture (Z.-W. Sun, 2016) Let a and b be nonzero integers with $\gcd(a, b)$ squarefree. Then the polynomial $ax^4 + by^3z$ is suitable if and only if (a, b) is among the ordered pairs

$$(1, 1),\ (1, 15),\ (1, 20),\ (1, 36),\ (1, 60),\ (1, 1680) \text{ and } (9, 260).$$

Examples:

$$9983 = 63^2 + 54^2 + 17^2 + 53^2$$

with $63^4 + 54^3 \times 17 = 4293^2$, and

$$20055 = 47^2 + 6^2 + 77^2 + 109^2$$

with $47^4 + 1680 \times 6^3 \times 77 = 5729^2$.

Other suitable polynomials

Conjecture (Z.-W. Sun, 2016) (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ ($w \in \mathbb{Z}^+$ and $x, y, z \in \mathbb{N}$) with $x^3 + 4yz(y - z)$ (or $x^3 + 8yz(2y - z)$) a square.

(ii) The polynomials $w(x + 2y + 3z)$, $w(x^2 + 8y^2 - z^2)$ and $x^2 + 3y^2 + 5z^2 - 8w^2$ are suitable.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $z < w$ such that $4x^2 + 5y^2 + 20zw$ is a square.

Restrictions involving powers of two

Theorem (Z.-W. Sun, arXiv:1701.05868) (i) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $|x + y - z| \in \{4^k : \ k \in \mathbb{N}\}$ (or $|x - 2y| \in \{4^k : \ k \in \mathbb{N}\}$).

(ii) For each $c = 1, 2$, any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + y + 2z \in \{c4^k : \ k \in \mathbb{N}\}$.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{4^k : \ k \in \mathbb{N}\}$.

(iv) Any integer $n > 1$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ and $x + 2y + 2z \in \{3 \times 2^k : \ k \in \mathbb{N}\}$.

Conjecture (Z.-W. Sun, 2016). Any $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 2y - 2z$ is a power of four (including $4^0 = 1$).

Remark. Qing-Hu Hou has verified this for n up to 10^9.

The 24-conjecture

24-Conjecture (Z.-W. Sun, Feb. 4, 2017). Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $x + 24y$ are squares.

Remark. Qing-Hu Hou has verified this for $n \leqslant 10^{10}$. I would like to offer 2400 US dollars as the prize for the first proof.

$$\begin{aligned}
12 =& 1^2 + 1^2 + 1^2 + 3^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 1 = 5^2,\\
23 =& 1^2 + 2^2 + 3^2 + 3^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 2 = 7^2,\\
24 =& 4^2 + 0^2 + 2^2 + 2^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 0 = 2^2,\\
47 =& 1^2 + 1^2 + 3^2 + 6^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 1 = 5^2,\\
71 =& 1^2 + 5^2 + 3^2 + 6^2 \text{ with } 1 = 1^2 \text{ and } 1 + 24 \times 5 = 11^2,\\
168 =& 4^2 + 4^2 + 6^2 + 10^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 4 = 10^2,\\
344 =& 4^2 + 0^2 + 2^2 + 18^2 \text{ with } 4 = 2^2 \text{ and } 4 + 24 \times 0 = 2^2,\\
632 =& 0^2 + 6^2 + 14^2 + 20^2 \text{ with } 0 = 0^2 \text{ and } 0 + 24 \times 6 = 12^2,\\
1724 =& 25^2 + 1^2 + 3^2 + 33^2 \text{ with } 25 = 5^2 \text{ and } 25 + 24 \times 1 = 7^2.
\end{aligned}$$

A similar conjecture

Conjecture (Z.-W. Sun, Feb. 4, 2017). (i) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $49x + 48(y - z)$ are squares.

(ii) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $121x + 48(y - z)$ are squares.

(iii) Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that both x and $-7x - 8y + 8z + 16w$ are squares.

Remark. Qing-Hu Hou has verified parts (i)-(ii) and part (iii) for n up to 10^9 and 10^8.

Other conjectures involving joint restrictions

Conjecture (Z.-W. Sun, Feb. 2017). (i) Each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ and $x \equiv y \pmod 2$ such that both x and $x^2+62xy+y^2$ are squares.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^4+y^2+z^2+w^2$ with $x,y,z \in \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that $8y^2-8yz+9z^2$ is a square.

Conjecture (Z.-W. Sun, March 2, 2017). (i) Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,w \in \mathbb{N}$ and $y,z \in \mathbb{Z}$ such that both $x+2y$ and $z+2w$ are squares.

(ii) Each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z \in \mathbb{Z}$ and $w \in \mathbb{N}$ such that both $x+3y$ and $z+3w$ are squares.

(iii) Any $n \in \mathbb{Z}^+$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z \in \mathbb{Z}$ and $w \in \mathbb{Z}^+$ such that both $2x+y$ and $2x+z$ are squares.

Other conjectures involving joint restrictions

Conjecture (Z.-W. Sun, March 13-4 2018). (i) Each $n \in \mathbb{Z}^+$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that both $(3x)^2 + (4y)^2 + (12z)^2$ is a square and one of $z, 2z, 3z$ is a square.

(ii) Any $n \in \mathbb{Z}^+$ can be written as $x^4 + y^2 + z^2 + w^2$ with $x, y, z \in \mathbb{N}$ and $w \in \mathbb{Z}^+$ such that $(12x)^2 + (15y)^2 + (20z)^2$ is a square and one of x, y, z is a square.

(iii) Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $(12x)^2 + (21y)^2 + (28z)^2$ is a square and one of $x, 2y, z$ is a square.

Examples:

$671 = 18^2 + 17^2 + 3^2 + 7^2$ with $(3 \cdot 18)^2 + (4 \cdot 17)^2 + (12 \cdot 3)^2 = 94^2$;

$39 = 5^2 + 2^2 + 1^2 + 3^2$ with $(12 \cdot 5)^2 + (15 \cdot 2)^2 + (20 \cdot 1)^2 = 70^2$;

and $1244 = 14^2 + 2^2 + 12^2 + 30^2$ with

$$(12 \cdot 14)^2 + (21 \cdot 2)^2 + (28 \cdot 12)^2 = 378^2.$$

Conjectures involving cubic diophantine equations

Conjecture (Z.-W. Sun, March 2017). (i) Any $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ and $y \equiv z \pmod 2$ such that $72x^3+(y-z)^3$ is a square.

(ii) Let $a,b \in \mathbb{Z}^+$ with $\gcd(a,b)$ squarefree. Then, each $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ with $x,y,z,w \in \mathbb{N}$ such that $ax^3+b(y-z)^3$ is a square, if and only if (a,b) is among the ordered pairs

$$(1,1),\ (1,9),\ (2,18),\ (8,1),\ (9,5),\ (9,8),$$
$$(9,40),\ (16,2),\ (18,16),\ (25,16),\ (72,1).$$

(iii) Let a and $b \geqslant a$ be positive integers with $\gcd(a,b)$ squarefree. Then, every $n \in \mathbb{N}$ can be written as $x^2+y^2+z^2+w^2$ $(x,y,z,w \in \mathbb{Z})$ with ax^3+by^3 a square, if and only if (a,b) is among the ordered pairs

$$(1,2),\ (1,8),\ (2,16),\ (4,23),\ (4,31),\ (5,9),$$
$$(8,9),\ (8,225),\ (9,47),\ (25,88),\ (50,54).$$

Restricted sums of four squares involving primes

The following conjecture implies the twin prime conjecture.

Conjecture (Z.-W. Sun, August 23, 2017). Any positive odd integer can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $p = x^2 + 3y^2 + 5z^2 + 7w^2$ and $p - 2$ are twin prime.

Example.

$$39 = 1^2 + 3^2 + 5^2 + 2^2$$

with $1^2 + 3 \cdot 3^2 + 5 \cdot 5^2 + 7 \cdot 2^2 = 181$ and $181 - 2 = 179$ twin prime. Also,

$$123 = 7^2 + 3^2 + 7^2 + 4^2$$

with $7^2 + 3 \cdot 3^2 + 5 \cdot 7^2 + 7 \cdot 4^2 = 433$ and $433 - 2 = 431$ twin prime.

Conjecture (Z.-W. Sun, August 28, 2017). Any integer $n > 1$ not divisible by 4 can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ such that $p = x + 2y + 5z$, $p - 2$, $p + 4$ and $p + 10$ are all prime.

Restricted sums of four squares involving primes

The following conjecture implies the twin prime conjecture, and I have verified it for all odd numbers below 4×10^7.

Conjecture (Z.-W. Sun, August 23, 2017). Any positive odd integer can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{Z}$ such that $p = x^2 + 3y^2 + 5z^2 + 7w^2$ and $p - 2$ are twin prime.

Example.

$$39 = 1^2 + 3^2 + 5^2 + 2^2$$

with $1^2 + 3 \cdot 3^2 + 5 \cdot 5^2 + 7 \cdot 2^2 = 181$ and $181 - 2 = 179$ twin prime. Also,

$$123 = 7^2 + 3^2 + 7^2 + 4^2$$

with $7^2 + 3 \cdot 3^2 + 5 \cdot 7^2 + 7 \cdot 4^2 = 433$ and $433 - 2 = 431$ twin prime.

Conjecture (Z.-W. Sun, August 28, 2017). Any integer $n > 1$ not divisible by 4 can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ such that $p = x + 2y + 5z$, $p - 2$, $p - 4$ and $p + 10$ are all prime.

Restricted sums of four squares involving primes

Conjecture (Z.-W. Sun, August 19, 2017). Any positive odd integer can be written as $x^2 + y^2 + z^2 + 4w^2$ with $x, y, z, w \in \mathbb{N}$ such that $2^x + 2^y + 2^z + 1$ is prime.

Example. $143 = 1^2 + 5^2 + 9^2 + 4 \cdot 3^2$ with $2^1 + 2^5 + 2^9 + 1 = 547$ prime.

Conjecture (Z.-W. Sun, August 20, 2017). Any odd integer $n > 1$ can be written as $x^2 + y^2 + z^2 + w^2$ $(x, y, z, w \in \mathbb{N})$ such that $2^{x+y} + 2^{z+w} + 1$ is prime.

Example. $197 = 6^2 + 6^2 + 2^2 + 11^2$ with $2^{6+6} + 2^{2+11} + 1 = 12289$ prime. And

$$2 \times 6998538 + 1 = 122^2 + 220^2 + 208^2 + 3727^2$$

with $2^{122+220} + 2^{208+3727} + 1 = 2^{342} + 2^{3935} + 1$ a prime of 1185 decimal digits.

I have verified both conjectures for positive odd integers not more than 2×10^7.

Part III. Restricted Sums of Three Squares

On the representation $n = x^2 + y^2 + z(3z-1)/2$

Those numbers $z(3z-1)/2$ ($z \in \mathbb{Z}$) are called *generalized pentagonal numbers*.

In a paper published in 2015, I noted that any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z(3z-1)/2$ with $x, y, z \in \mathbb{Z}$. Surprising, this can be further refined.

Conjecture (Z.-W. Sun, March 3, 2017). Any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z(3z-1)/2$ with $x, y, z \in \mathbb{Z}$ such that $x + 2y$ is a square.

Example.

$$803 = (-17)^2 + 13^2 + \frac{(-15)(3(-15)-1)}{2} \text{ with } (-17) + 2 \times 13 = 3^2.$$

Refining the Gauss-Legendre theorem

Gauss-Legendre Theorem. A nonnegative integer n can be expressed as the sum of three squares if and only if it is not of the form $4^k(8l+7)$ with $k, l \in \mathbb{N}$.

Conjecture (Z.-W. Sun, March 4, 2017). (i) Any $n \in \mathbb{Z}^+$ with $\mathrm{ord}_2(n)$ odd can be written as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{Z}$ such that $x+3y+5z$ is a square.

(ii) Let $n \in \mathbb{N} \setminus \{63\}$. Then $4n+1$ can be written as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{Z}$ such that $x+3y+5z$ is a square.

(iii) Any $n \in \mathbb{N}$ not of the form $4^k(8l+7)$ with $k, l \in \mathbb{N}$ can be written as $x^2+y^2+z^2$ with $x, y, z \in \mathbb{Z}$ such that $x+2y+3z$ is a square or twice a square.

Example.

$$1430 = (-13)^2+(-6)^2+35^2 \text{ with } (-13)+3\times(-6)+5\times 35 = 12^2.$$

Restricted representations for positive odd numbers

It is known that any positive odd numbers can be written as $x^2+y^2+2z^2$ with $x,y,z\in\mathbb{Z}$. Also, we can replace $x^2+y^2+2z^2$ by $x^2+2y^2+3z^2$.

Conjecture (Z.-W. Sun, March 5, 2017). (i) Any positive odd integer can be written as $x^2+y^2+2z^2$ with $x,y,z\in\mathbb{Z}$ such that $2x+y+z$ is a square or a power of two.

(ii) Any positive odd integer can be written as $x^2+2y^2+3z^2$ with $x,y,z\in\mathbb{Z}$ such that $x+y+z$ is a square or twice a square.

Examples:

$2\times1143+1=(-22)^2+2\times30^2+3\times1^2$ with $(-22)+30+1=3^2$,

and

$$2\times6408+1=(-22)^2+2\times75^2+3\times19^2$$

with

$$(-22)+75+19=2\times6^2.$$

Part IV. Sums of Two Squares and Two Other Terms

Write $n > 5$ as $x^2 + y^2 + 2^z + 5 \times 2^w$

Conjecture (Z.-W. Sun, Feb. 22, 2018). For any $n \in \mathbb{Z}^+$ we can write n^2 as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ and $x + 2y + 3z \in \{4^k : k \in \mathbb{N}\}$.

Conjecture (Z.-W. Sun, April 22, 2018). Any integer $n > 3$ can be written as $x^2 + 2y^2 + 3 \times 2^z + 4^w$ with $x, y, z, w \in \mathbb{N}$.

Remark. I have verified this for n up to 10^{10}. The first value of $n > 1$ with n^2 not of the form $x^2 + 2y^2 + 3 \times 2^z$ is 5884015571.

R. Crocker [Colloq. Math. 2008]: For each integer $a \geqslant 2$, there are infinitely many positive integers none of which is the sum of two squares and at most two powers of a.

Conjecture (Z.-W. Sun, April 28, 2018). Any integer $n > 5$ can be written as $x^2 + y^2 + 2^z + 5 \times 2^w$ with $x, y, z, w \in \mathbb{N}$.

Remark. I have verified this for n up to 5×10^9.

Write $n = a^2 + b^2 + 3^c + 5^d$

Conjecture (Z.-W. Sun, April 28, 2018). Any integer $n > 1$ can be written as $a^2 + b^2 + 3^c + 5^d$ with $a, b, c, d \in \mathbb{N} = \{0, 1, 2, \ldots\}$.

Remark. I have verified this for n up to 2×10^{10}, and I'd like to offer 3500 US dollars as the prize for the first proof of this conjecture. I also conjecture that 5^d in the conjecture can be replaced by 2^d.

Example.

$$2 = 0^2 + 0^2 + 3^0 + 5^0, \ 5 = 0^2 + 1^2 + 3^1 + 5^0, \ 25 = 1^2 + 4^2 + 3^1 + 5^1.$$

Conjecture (Z.-W. Sun, April 26, 2018). Any integer $n > 1$ can be written as the sum of two squares and two central binomial coefficients.

Remark. I have verified this for n up to 10^{10}.

Example.

$$2435 = 32^2 + 33^2 + \binom{2 \times 4}{4} + \binom{2 \times 5}{5}.$$

Sums of two triangular numbers and two powers of 5

Recall that those $T_n = n(n+1)/2$ with $n \in \mathbb{N}$ are called triangular numbers. As claimed by Fermat and proved by Gauss, each $n \in \mathbb{N}$ is the sum of three triangular numbers.

Conjecture (Z.-W. Sun, April 23, 2018). Any integer $n > 1$ can be written as $T_a + T_b + 5^c + 5^d$ with $a, b, c, d \in \mathbb{N}$.

Remark. I have verified this for n up to 10^{10}.

Conjecture (Z.-W. Sun, April 23, 2018). Any integer $n > 1$ can be written as the sum of $p_5(a) + p_5(b) + 3^c + 3^d$ with $a, b, c, d \in \mathbb{N}$, where $p_5(k)$ denotes the pentagonal number $k(3k-1)/2$.

Remark. I have verified this for n up to 7×10^6.

Example.

$$285 = p_5(1) + p_5(11) + 3^3 + 3^4, \ 13372 = p_5(17) + p_5(65) + 3^4 + 3^8.$$

Two new conjectures involving primes

Conjecture (Z.-W. Sun, May 2018) Any even number $n > 2$ can be written as the sum of a prime, a power of 2 and a power of 5. Furthermore, we can write every integer $n > 7$ as $p + 2^k + (1 + (n \text{ mod } 2)) \times 5^m$, where p is an odd prime, and $k, m \in \mathbb{N}$ with $2^k + (1 + (n \text{ mod } 2)) \times 5^m$ squarefree.

Remark. I have verified this for n up to 2×10^{10}.

Recall that the Fibonacci numbers $F_0, F_1, \ldots$ and the Lucas numbers $L_0, L_1, \ldots$ are respectively given by

$$F_0 = 0,\ F_1 = 1,\ F_{n+1} = F_n + F_{n-1}\ (n = 1, 2, 3, \ldots)$$

and

$$L_0 = 2,\ L_1 = 1,\ L_{n+1} = L_n + L_{n-1}\ (n = 1, 2, 3, \ldots).$$

Conjecture (Z.-W. Sun, June 2018) Any integer $n > 3$ can be written as $p + F_k L_m$, where p is an odd prime, and k and m are positive integers.

Remark. I have verified this for n up to 5×10^9.

References

For the main sources of my above conjectures and related results, you may look at the following papers:

1. Zhi-Wei Sun, *Refining Lagrange's four-square theorem*, J. Number Theory 175(2017), 167–190. arXiv:1604.06723

2. Yu-Chen Sun and Zhi-Wei Sun, *Some variants of Lagrange's four squares theorem*, Acta Arith. 183(2018), no.4, 339-356. See also http://arxiv.org/abs/1605.03074.

3. Zhi-Wei Sun, *Restricted sums of four squares*, arXiv:1701.05868 , http://arxiv.org/abs/1701.05868.

4. Hai-Liang Wu and Zhi-Wei Sun, *On the 1-3-5 conjecture and related topics*, arXiv:1710.08763, available from the website http://arxiv.org/abs/1710.08763.

Thank you!