Conjectures and Results on Super Congruences

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Part A. Previous Work by Others

What are super congruences ?

A *super congruence* is a p-adic congruence which happens to hold not just modulo a prime p but a higher power of p.

Example. (Wolstenholme) For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}$$

and

$$\binom{2p-1}{p-1} = \frac{1}{2}\binom{2p}{p} \equiv 1 \pmod{p^3}.$$

Remark. It is easy to see that

$$\sum_{k=1}^{p-1} \frac{1}{k} = \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k} + \frac{1}{p-k}\right) = \sum_{k=1}^{(p-1)/2} \frac{p}{k(p-k)} \equiv 0 \pmod{p}$$

and

$$\binom{2p-1}{p-1} = \prod_{k=1}^{p-1} \frac{p+k}{k} = \prod_{k=1}^{p-1} \left(1 + \frac{p}{k}\right) \equiv 1 + \sum_{k=1}^{p-1} \frac{p}{k} \equiv 1 \pmod{p^2}.$$

Supper Congruences for Apéry Numbers

In 1978 Apéry proved that $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is irrational! During his proof he used the sequence $\{B(n)/A(n)\}_{n=1}^{\infty}$ of rational numbers to approximate $\zeta(3)$, where

$$A(0) = 1,\ A(1) = 5,\ B(0) = 0,\ B(1) = 6,$$

and both $\{A(n)\}_{n\geqslant 0}$ and $\{B(n)\}_{n\geqslant 0}$ satisfy the recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}\ (n = 1, 2, \ldots).$$

In fact,

$$A(n) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2$$

and these numbers are called *Apéry numbers*.

Dedekind eta function in the theory of modular forms:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n) \quad \text{with } q = e^{2\pi i \tau}$$

Note that $q| < 1$ if $\tau \in \mathbb{H} = \{z \in \mathbb{C} :\ \mathrm{Im}(z) > 0\}$.

Beukers' Conjecture (1985). For any prime $p > 3$ we have the super congruence

$$A\left(\frac{p-1}{2}\right) \equiv a(p) \pmod{p^2},$$

where $a(n)$ $(n = 1, 2, 3, \dots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4 = \sum_{n=1}^{\infty} a(n)q^n.$$

S. Ahlgren and Ken Ono [J. Reine Angew. Math. 518(2000)]: The Beukers conjecture is true!

Outline of their proof. First show that $a(p)$ can be expressed as a special value of the Gauss hypergeometric function ${}_4F_3(\lambda)$ defined in terms of Jacobi sums. Then rewrite Jacobi sums in terms of Gauss' sums and apply the Gross-Koblitz formula to express Gauss sums in terms of the p-adic Gamma function $\Gamma_p(x)$. Finally use combinatorial properties of $\Gamma_p(x)$ and some sophisticated combinatorial identities involving harmonic numbers $H_n = \sum_{0<k\leqslant n} 1/k$.

Gaussian hypergeometric series

The rising factorial (or Pochhammer symbol):

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

Note that $(1)_n = n!$.

Classical Gaussian hypergeometric series:

$${}_{r+1}F_r\left(\alpha_0, \ldots, \alpha_r; \beta_1, \ldots, \beta_r \mid x\right) = \sum_{n=0}^{\infty} \frac{(\alpha_0)_n(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_r)_n} \cdot \frac{x^n}{n!},$$

where $0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r < 1$ and $0 \leqslant \beta_1 \leqslant \cdots \leqslant \beta_r < 1$.

Greene's character sum analogue of hypergeometric series

Let p be an odd prime. For two Dirichlet characters A and B mod p, define the *normalized Jacobi sum*

$$\binom{A}{B} := \frac{B(-1)}{p} J(A, \bar{B}) = \frac{B(-1)}{p} \sum_{x=0}^{p-1} A(x)\bar{B}(1-x).$$

Given Dirichlet characters $A_0, A_1, \ldots, A_r$ and $B_1, \ldots, B_r$ mod p, Greene [Trans. AMS, 1987] defined

$$\begin{aligned} &{}_{r+1}F_r(A_0, A_1, \ldots, A_r; B_1, \ldots, B_r \mid x)_p \\ =& \frac{p}{p-1} \sum_{\chi} \binom{A_0\chi}{\chi} \prod_{i=1}^{r} \binom{A_i\chi}{B_i\chi} \chi(x). \end{aligned}$$

$$\begin{aligned} &{}_{r+1}F_r(A_0, A_1, \ldots, A_r; B_1, \ldots, B_r \mid x)_p \\ =& \frac{A_rB_r(-1)}{p} \sum_{y=0}^{p-1} {}_rF_{r-1}(A_0, A_1, \ldots, A_{r-1}; B_1, \ldots, B_{r-1} \mid xy)_p \\ & \times A_r(y)\bar{A}_rB_r(1-y) \quad \text{(Greene, 1987).} \end{aligned}$$

Connections to elliptic curves over $\mathbb{F}_p$

Let ε_p be the trivial character with $\varepsilon_p(x) = 1$ for all $x \not\equiv 0 \pmod{p}$. Let ϕ_p be the Legendre character given by $\phi_p(x) = (\frac{x}{p})$. Set

$$_{r+1}F_r(x)_p := {}_{r+1}F_r(\phi_p, \phi_p, \ldots, \phi_p; \varepsilon_p, \ldots, \varepsilon_p \mid x)_p.$$

Consider the curve over $\mathbb{Q}$ defined by

$$_2E_1(\lambda): \; y^2 = x(x-1)(x-\lambda).$$

If $\lambda \in \mathbb{Q} \setminus \{0, 1\}$, then $_2E_1(\lambda)$ is an elliptic curve with

$$\Delta({}_2E_1(\lambda)) = 16\lambda^2(\lambda - 1)^2 \;\; \text{(discriminant)}$$

$$j({}_2E_1(\lambda)) = \frac{256(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2} \;\; (j - invariant).$$

Connections to elliptic curves over $\mathbb{F}_p$

Suppose $\lambda(\lambda - 1) \not\equiv 0 \pmod{p}$. Then p is a prime of good reduction for ${}_2E_1(\lambda)$, and we define

$$ {}_2a_1(p;\lambda) := p + 1 - |{}_2E_1(\lambda)_p| $$

where $|{}_2E_1(\lambda)_p|$ denotes the number of $\mathbb{F}_p$-points of ${}_2E_1(\lambda)_p$ including the point at infinity. It is known that

$$ {}_2a_1(p;\lambda) = -\sum_{x=0}^{p-1} \phi_p(x(x-1)(x-\lambda)) $$

and

$$ {}_2F_1(\lambda)_p = -\frac{\phi_p(-1)}{p} {}_2a_1(p;\lambda) = \left(\frac{-1}{p}\right) \sum_{x=0}^{p-1} \left(\frac{x(x-1)(x-\lambda)}{p}\right). $$

Conjectures of Rodriguez-Villegas

In 2001 Rodriguez-Villegas conjectured 22 congruences which relate truncated hypergeometric series to the number of $\mathbb{F}_p$-points of some family of Calabi-Yau manifolds. Here we list some of them.

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) = (-1)^{(p-1)/2} \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} \equiv [q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6 \pmod{p^2}.$$

A theorem of Stienstra and Beukers

J. Stienstra and F. Beukers [Math. Ann. 27(1985)]:

$$[q^p]q\prod_{n=1}^{\infty}(1-q^{4n})^6$$
$$=\begin{cases}4x^2-2p & \text{if } p=1 \pmod 4 \ \& \ p=x^2+y^2 \text{ with } 2\nmid x \ \& \ 2\mid y,\\ 0 & \text{if } p\equiv 3 \pmod 4;\end{cases}$$

$$[q^p]q\prod_{n=1}^{\infty}(1-q^n)^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2$$
$$=\begin{cases}4x^2-2p & \text{if } p=1,3 \pmod 8 \ \& \ p=x^2+2y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 5,7 \pmod 8;\end{cases}$$

$$[q^p]q\prod_{n=1}^{\infty}(1-q^{2n})^3(1-q^{6n})^3$$
$$=\begin{cases}4x^2-2p & \text{if } p=1 \pmod 3 \ \& \ p=x^2+3y^2 \text{ with } x,y\in\mathbb{Z},\\ 0 & \text{if } p\equiv 2 \pmod 3.\end{cases}$$

Progress on Rodriguez-Villegas conjectures

The congruences we list have been confirmed, see,

E. Motenson, J. Number Theory 99(2003); Trans. AMS 355(2003); Proc. AMS 133(2005).

Many of the 22 conjectures remain open.

Ramanujan's series for $1/\pi$

Here are 5 of the 17 Ramanujan series recorded by him in 1914:

$$\sum_{k=0}^{\infty}(-1)^k(4k+1)\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{6k+1}{4^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}=\frac{4}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{6k+1}{(-8)^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{43k+5}{64^k}\cdot\frac{(1/2)_k^3}{(1)_k^3}=\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{20k+3}{(-4)^k}\cdot\frac{(1/2)_k(1/4)_k(3/4)_k}{(1)_k^3}=\sum_{k=0}^{\infty}(20k+3)\frac{\binom{4k}{k,k,k,k}}{(-1024)^k}=\frac{8}{\pi}.$$

Remark. The first one was actually proved by G. Bauer in 1859.

Hamme's Conjectures

L. Van Hamme [1997] conjectured the p-adic analogues of the above first 4 identities and W. Zudilin [JNT, 2009] obtained the p-adic analogue of the last identity.

$$\sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{-1}{p}\right)p \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{256^k} \equiv \left(\frac{-1}{p}\right)p \pmod{p^4},$$

$$\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k} \equiv \left(\frac{-2}{p}\right)p \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k} \equiv \left(\frac{-1}{p}\right)5p \pmod{p^4},$$

$$\sum_{k=0}^{p-1}(20k+3)\frac{\binom{4k}{k,k,k,k}}{(-1024)^k} \equiv \left(\frac{-1}{p}\right)3p \pmod{p^3}.$$

Progress on Hamme's conjectures

The first of the above congruence was proved by E. Mortenson [Proc. AMS 136(2008)] and the second one was recently shown by Ling Long, while the last was confirmed by Zudilin via the WZ method. The third and the fourth remain open.

The p-adic Gamma function plays an important role in Hamme's formulation of those conjectures. It is defined in the following way:

$$\Gamma_p(n) := (-1)^n \prod_{\substack{1<k<n \\ p\nmid n}} k \quad (n = 1, 2, 3, \ldots)$$

and

$$\Gamma_p(x) = \lim_{n\to x} \Gamma_p(n) \text{ for any } p-\text{adic integer } x.$$

Part B. My Results and Conjectures

Some Joint Work

H. Pan and Z. W. Sun [Discrete Math. 2006].

$$\sum_{k=0}^{p-1}\binom{2k}{k+d}\equiv\left(\frac{p-d}{3}\right)\pmod p\quad(d=0,\ldots,p),$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod p\quad\text{for } p>3.$$

Sun & R. Tauraso [arXiv:0709.1665, Adv. in Appl. Math.].

$$\sum_{k=0}^{p^a-1}\binom{2k}{k}\equiv\left(\frac{p^a}{3}\right)\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv\frac{8}{9}p^2B_{p-3}\pmod{p^3}\quad\text{for } p>3,$$

L. L. Zhao, H. Pan and Z. W. Sun [Proc. AMS, 2010]

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv 0\pmod p.$$

My own results

Observe that if $p/2 < k < p$ then

$$\binom{2k}{k} = \frac{(2k)!}{(k!)^2} \equiv 0 \pmod{p}$$

and so

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \pmod{p},$$

where m is an integer with $p \nmid m$.

In 2009 I [arXiv:0909.5648, arXiv:0911.3060, 0909.3808] determined

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \bmod p^2, \quad \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{m^k} \bmod p^2, \quad \sum_{k=0}^{p-1} \frac{\binom{3k}{k}}{m^k} \bmod p$$

in terms of linear recurrences.

Some particular congruences due to me

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},\quad \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{2}{p}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{3^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},\quad \sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{16^k}\equiv\left(\frac{3}{p}\right)\pmod{p^2}.$$

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}\equiv\left(\frac{p}{5}\right)(1-2F_{p-(\frac{p}{5})})\pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(-16)^k}\equiv\left(\frac{p}{5}\right)\left(1+\frac{1}{2}F_{p-(\frac{p}{5})}\right)\pmod{p^2},$$

where $\{F_n\}_{n\geqslant 0}$ is the Fibonacci sequence defined by

$$F_0=0,\ F_1=1,\ F_{n+1}=F_n+F_{n-1}\ (n=1,2,3,\ldots).$$

Some particular congruences due to me

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{8^k}\equiv\frac{3}{4}\left(\left(\frac{p}{5}\right)-1\right)\pmod p,$$

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{7^k}\equiv\begin{cases}-2\pmod p & \text{if } p\equiv\pm2\pmod 7,\\ 1\pmod p & \text{otherwise.}\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k}}{5^k}\equiv\begin{cases}1\pmod p & \text{if } p\equiv 1\pmod 5\ \&\ p\neq 11,\\ -1/11\pmod p & \text{if } p\equiv 2,3\pmod 5,\\ -9/11\pmod p & \text{if } p\equiv 4\pmod 5.\end{cases}$$

If $p\equiv 1\pmod 3$ then

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}}{6^k}\equiv 2^{(p-1)/3}\pmod p.$$

My recent results on super congruences

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0 = 1, \ \sum_{2|k} \binom{n}{k} E_{n-k} = 0 \ (n = 1, 2, 3, \ldots).$$

It is known that $E_1 = E_3 = E_5 = \cdots = 0$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \ \left(|x| < \frac{\pi}{2}\right).$$

Z. W. Sun [arXiv:1001.4453].

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}}{8^k} \equiv \left(\frac{2}{p}\right) + \left(\frac{-2}{p}\right) \frac{p^2}{4} E_{p-3} \pmod{p^3}.$$

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My recent results on super congruences

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\equiv -2p\sum_{k=1}^{(p-1)/2}\frac{1}{k^2\binom{2k}{k}}\pmod{p^2},$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)-\left(\frac{-1}{p}\right)\frac{3}{8}p\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\pmod{p^4},$$

$$\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{16^k}\equiv\frac{(-1)^{(p+1)/2}}{4}+\frac{p^2}{2}(2^{p-1}-1)$$
$$+(-1)^{(p-1)/2}\frac{3}{32}p\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\pmod{p^4}.$$

Remark. Via some advanced tools, R. Osburn and C. Schneider [Math. Comp. 78(2009)] proved that

$$(-1)^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv 1-\frac{3}{8}p\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\pmod{p^3}.$$

Conjecture 1

I have formulated over 50 conjectures on super congruences, see Z. W. Sun, *Open Conjectures on Congruences*, `arXiv:0911.5665`.

Conjecture 1. For any prime $p > 3$ we have

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} \frac{4}{3} E_{p-3} \pmod{p},$$

$$\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2} + p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}^2}{16^k} \equiv -2p^2 E_{p-3} \pmod{p^3},$$

$$\sum_{p/2<k<p} \frac{k\binom{2k}{k}^2}{16^k} \equiv \frac{p^2}{2} E_{p-3} \pmod{p^3}.$$

A Remark to Conjecture 1

Remark. I have proved that the first, the second and the third congruences are equivalent. Note that

$$\lim_{k\to+\infty}\frac{k\binom{2k}{k}^2}{16^k}=\frac{1}{\pi}$$

and

$$\sum_{k=0}^{\infty}\frac{1}{k^2\binom{2k}{k}}=\frac{\pi^2}{18}.$$

Conjecture 2

Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1 \ \&\ p = x^2 + 7y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1. \end{cases}$$

Moreover,

$$\sum_{k=0}^{p-1} (21k+8) \binom{2k}{k}^3 \equiv 8p \pmod{p^4}$$

and

$$\sum_{k=0}^{(p-1)/2} (21k+8) \binom{2k}{k}^3 \equiv 8p + \left(\frac{-1}{p}\right) 32p^3 E_{p-3} \pmod{p^4}.$$

Remark. Quite recently M. Jameson and K. Ono claimed to have a proof of the first congruence via the theory of modular forms and $K3$ surfaces related to Calabi-Yau manifolds.

Conjecture 3

Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}$$
$$\equiv\begin{cases}x^2-2p \pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{11})=-1,\ \text{i.e., } p\equiv 2,6,7,8,10 \pmod{11}.\end{cases}$$

Furthermore,

$$\sum_{k=0}^{p-1}(11k+3)\frac{\binom{2k}{k}^2\binom{3k}{k}}{64^k}\equiv 3p \pmod{p^4}.$$

Remark. It is well-known that the quadratic field $\mathbb{Q}(\sqrt{-11})$ has class number one and hence for any odd prime p with $(\frac{p}{11})=1$ we can write $4p=x^2+11y^2$ with $x,y\in\mathbb{Z}$.

Conjecture 4

Let $p>3$ be a prime. If $p\equiv 1 \pmod 3$ and $p=x^2+3y^2$ with $x,y\in\mathbb{Z}$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv 4x^2-2p \pmod{p^2} \text{ and } \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{16^k}\equiv p-\frac{4x^2}{3} \pmod{p^2}.$$

If $p\equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv 0 \pmod{p^2} \text{ and } \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^3}{16^k}\equiv \frac{p}{3} \pmod{p^2}.$$

Furthermore,

$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^3}{16^k}\equiv p \pmod{p^4}$$

and

$$\sum_{k=0}^{(p-1)/2}(3k+1)\frac{\binom{2k}{k}^3}{16^k}\equiv p+2\left(\frac{-1}{p}\right)p^3E_{p-3} \pmod{p^4}.$$

Conjecture 5

Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k,k,k}}{24^k} \equiv \sum_{k=0}^{p-1}\frac{\binom{3k}{k,k,k}}{(-216)^k} \equiv \begin{cases}\binom{2(p-1)/3}{(p-1)/3} \pmod{p^2} & \text{if } p \equiv 1 \pmod 6,\\ 0 \pmod p & \text{if } p \equiv 5 \pmod 6.\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k,k,k}}{(-27)^k}$$

$$\equiv \begin{cases}4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1,4 \pmod{15}\ \&\ p = x^2 + 15y^2,\\ 20x^2 - 2p \pmod{p^2} & \text{if } p \equiv 2,8 \pmod{15}\ \&\ p = 5x^2 + 3y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15}) = -1.\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k} \equiv 4p\left(\frac{p}{3}\right) \pmod{p^3}.$$

Conjecture 6

Let $p > 5$ be a prime. Then

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{k,k,k,k}}{(-2^{10})^k}$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 9 \pmod{20} \ \&\ p = x^2 + 5y^2, \\ 2(p - x^2) \pmod{p^2} & \text{if } p \equiv 3, 7 \pmod{20} \ \&\ 2p = x^2 + 5y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 11, 13, 17, 19 \pmod{20}. \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{3k}{k}}{216^k}$$

$$\equiv \begin{cases} x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \ \&\ 4p = x^2 + 6y^2, \\ 2x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \ \&\ 2p = x^2 + 6y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 13, 17, 19, 23 \pmod{24}. \end{cases}$$

$$\sum_{k=0}^{p-1} (6k+1) \frac{\binom{2k}{k}^2 \binom{3k}{k}}{216^k} \equiv \left(\frac{p}{3}\right) p \pmod{p^3}.$$

Conjecture 7

Let $p > 3$ be a prime. If $(\frac{p}{7}) = 1$ and $p = x^2 + 7y^2$ with $(\frac{x}{7}) = 1$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{63^k} \equiv \left(\frac{p}{3}\right)\left(2x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{k\binom{2k}{k}\binom{4k}{2k}}{63^k} \equiv 8\left(\frac{p}{3}\right)\left(\frac{p}{2x} - x\right) \pmod{p^2}.$$

If $(\frac{p}{7}) = -1$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{63^k} \equiv 0 \pmod{p} \quad \text{and} \quad \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}^2}{63^k} \equiv 0 \pmod{p}.$$

Conjecture 8

Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^3}{(-8)^k} \equiv p\left(\frac{-1}{p}\right) + p^3 E_{p-3} \pmod{p^4}.$$

If $p \equiv 1 \pmod 4$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\left(1-\frac{1}{(-8)^k}\right) \equiv 0 \pmod{p^3}.$$

If $p \equiv 3 \pmod 4$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k} \equiv 0 \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\left(1+\frac{1}{(-2)^k}\right) \equiv 0 \pmod{p^3}.$$

Conjecture 9

Let $p > 5$ be a prime. If $p \equiv 1 \pmod 4$, then

$$\sum_{k=0}^{p-1} \frac{k^3\binom{2k}{k}^3}{64^k} \equiv 0 \pmod{p^2}.$$

If $p \equiv 3 \pmod 4$. Then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{\binom{2k}{k}^2}{(-8)^k} \equiv 0 \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv 0 \pmod{p^2}.$$

Remark. The last congruence is equivalent to

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{64^k} H_k \equiv 0 \pmod p \quad \text{with} \quad H_k = \sum_{0<j\leqslant k} \frac{1}{j}.$$

Conjecture 10

Let $p > 3$ be a prime. If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod 4$, then

$$\sum_{k=0}^{p-1} \left(\frac{k}{3}\right) \frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-3)/4} \left(4y - \frac{p}{3y}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \left(\frac{k}{3}\right) \frac{k\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p+1)/4} y \pmod{p^2}.$$

If $p \equiv 1 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \left(\frac{k}{3}\right) \frac{\binom{2k}{k}^2}{16^k} \equiv 0 \pmod{p^2}.$$

Recall that the Pell sequence $\{P_n\}_{n\geqslant 0}$ is given by

$$P_0 = 0,\ \ P_1 = 1, \text{and } P_{n+1} = 2P_n + P_{n-1}\ \ (n = 1, 2, 3, \dots).$$

Conjecture 11 Let p be an odd prime. If $p \equiv 1, 3 \pmod 8$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1, 3 \pmod 8$, then

$$\sum_{k=0}^{p-1} \frac{P_k}{(-8)^k} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } 8 \mid p-1, \\ (-1)^{(p-3)/8}(\frac{p}{2x} - 2x) \pmod{p^2} & \text{if } 8 \mid p-3. \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{kP_k}{(-8)^k} \binom{2k}{k}^2 \equiv \frac{(-1)^{(x+1)/2}}{2} \left(x + \frac{p}{2x}\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{p-1} \frac{P_k}{(-8)^k} \binom{2k}{k}^2 \equiv 0 \pmod p \ \ \text{if } p \equiv 5 \pmod 8,$$

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \frac{P_k}{8^k} \binom{2k}{k}^2 \equiv 0 \pmod{p^2} \ \ \text{if } p \equiv 7 \pmod 8.$$

Define the sequence $\{S_n\}_{n\geqslant 0}$ is given by

$$S_0 = 0,\ S_1 = 1, \text{and } S_{n+1} = 4S_n - S_{n-1}\ (n = 1, 2, 3, \ldots).$$

Conjecture 12. Let $p > 3$ be a prime. If $p \equiv 7 \pmod{12}$ and $p = x^2 + 3y^2$ with $y \equiv 1 \pmod 4$, then

$$\sum_{k=0}^{p-1} \frac{S_k}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p+1)/4} \left(4y - \frac{p}{3y}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1} \frac{kS_k}{4^k} \binom{2k}{k}^2 \equiv (-1)^{(p-3)/4} \left(6y - \frac{7p}{3y}\right) \pmod{p^2}.$$

We also have

$$\sum_{k=0}^{p-1} \frac{S_k}{4^k} \binom{2k}{k}^2 \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}, \\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

If $p \equiv \pm 1 \pmod{12}$, then

$$\sum_{k=0}^{p-1} \binom{p-1}{k} \binom{2k}{k} (-1)^k S_k \equiv (-1)^{(p-1)/2} S_{p-1} \pmod{p^3}.$$

Conjecture 13. For any $n \in \mathbb{Z}^+$ we have

$$\frac{(-1)^{\lfloor n/5\rfloor-1}}{(2n+1)n^2\binom{2n}{n}}\sum_{k=0}^{n-1}F_{2k+1}\binom{2k}{k}\equiv\begin{cases}6 \pmod{25} & \text{if } n\equiv 0 \pmod 5,\\ 4 \pmod{25} & \text{if } n\equiv 1 \pmod 5,\\ 1 \pmod{25} & \text{if } n\equiv 2,4 \pmod 5,\\ 9 \pmod{25} & \text{if } n\equiv 3 \pmod 5.\end{cases}$$

Also, if $a, b \in \mathbb{Z}^+$ and $a \geqslant b$ then the sum

$$\frac{1}{5^{2a}}\sum_{k=0}^{5^a-1}F_{2k+1}\binom{2k}{k}$$

modulo 5^b only depends on b.

Remark. I have proved that if $p \neq 2, 5$ is a prime then

$$\sum_{k=0}^{p-1}F_{2k}\binom{2k}{k}\equiv(-1)^{\lfloor p/5\rfloor}\left(1-\left(\frac{p}{5}\right)\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}F_{2k+1}\binom{2k}{k}\equiv(-1)^{\lfloor p/5\rfloor}\left(\frac{p}{5}\right) \pmod{p^2}.$$

Conjecture 14. For any prime p and positive integer n we have

$$\nu_p\bigg(\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\bigg)\geqslant \nu_p(n)$$

and

$$\nu_p\bigg(\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^k\binom{(p-1)k}{k,\ldots,k}\bigg)\geqslant \nu_p(n),$$

where $\nu_p(n)=\max\{a\in\mathbb{N}:\ p^a\mid n\}$ is the p-adic order of n.

Remark. I have proved that an integer $p>1$ is a prime if and only if

$$\sum_{k=0}^{p-1}\binom{(p-1)k}{k,\ldots,k}\equiv 0 \pmod{p}.$$

He also showed that if $n\in\mathbb{Z}^+$ is a multiple of a prime p then

$$\sum_{k=0}^{n-1}\binom{(p-1)k}{k,\ldots,k}\equiv 0 \pmod{p}.$$

Conjectures 15 and 16

Conjecture 15(Sun and Tauraso [Adv. in Appl. Math., in press]) Let $p \neq 2, 5$ be a prime and let $a \in \mathbb{Z}^+$. Then

$$\sum_{k=0}^{p^a-1} (-1)^k \binom{2k}{k} \equiv \left(\frac{p^a}{5}\right) \left(1 - 2F_{p^a - (\frac{p^a}{5})}\right) \pmod{p^3},$$

where $\{F_n\}_{n \geqslant 0}$ is the Fibonacci sequence.

Remark. I have proved the congruence mod p^2.

Conjecture 16 Let p be an odd prime. If $p \equiv 1, 2 \pmod 5$, then

$$\sum_{k=0}^{\lfloor \frac{4}{5}p \rfloor} (-1)^k \binom{2k}{k} \equiv \left(\frac{5}{p}\right) \pmod{p^2}.$$

If $p \equiv 1, 3 \pmod 5$, then

$$\sum_{k=0}^{\lfloor \frac{3}{5}p \rfloor} (-1)^k \binom{2k}{k} \equiv \left(\frac{5}{p}\right) \pmod{p^2}.$$

Conjectures 17 and 18

Conjecture 17 Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod{3}$ or $a > 1$, then

$$\sum_{k=0}^{\lfloor \frac{5}{6} p^a \rfloor} \frac{\binom{2k}{k}}{16^k} \equiv \left(\frac{3}{p^a}\right) \pmod{p^2}.$$

Remark. The author [S09f] proved that $\sum_{k=0}^{\lfloor p^a/2 \rfloor} \binom{2k}{k}/16^k \equiv (\frac{3}{p^a}) \pmod{p^2}$ for odd prime p and $a \in \mathbb{Z}^+$.

Conjecture 18. For any nonnegative integer n we have

$$\frac{1}{(2n+1)^2 \binom{2n}{n}} \sum_{k=0}^{n} \frac{\binom{2k}{k}}{16^k} \equiv \begin{cases} 1 \pmod{9} & \text{if } 3 \mid n, \\ 4 \pmod{9} & \text{if } 3 \nmid n. \end{cases}$$

. Also,

$$\frac{1}{3^{2a}} \sum_{k=0}^{(3^a-1)/2} \frac{\binom{2k}{k}}{16^k} \equiv (-1)^a 10 \pmod{27}$$

for every $a = 1, 2, 3, \ldots$.

Conjectures 19 and 20

Conjecture 19. Let p be an odd prime. Then we have

$$\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv -3p\,q_p^2(2)\pmod{p^2},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Remark. Zhao, Pan and Sun [Proc. AMS, 2010] proved that $\sum_{k=1}^{p-1}\frac{2^k}{k}\binom{3k}{k}\equiv 0\pmod{p}$ for any odd prime p.

Conjecture 20. Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{2k}{k}((-1)^k-(-3)^{-k})\equiv\left(\frac{p}{3}\right)(3^{p-1}-1)\pmod{p^3}.$$

Remark. I have proved the congruence mod p^2.

More Conjectures on Congruences

For more conjectures of mine on congruences, see

Z. W. Sun, *Open Conjectures on Congruences*,

`arXiv:0911.5665`.

You are welcome to solve my conjectures!

Thank you!