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A Survey of Arithmetic Properties of Combinatorial Quantities

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Abstract

There are many basic combinatorial quantities arising naturally from enumerative combinatorics, such as Bell numbers, derangement numbers, central trinomial coefficients, Motzkin numbers, Delannoy numbers and Schröder numbers. Surprisingly such combinatorial quantities have nice arithmetic properties involving arithmetic means and congruences. This talk is a survey of problems and results in this field.

Part I. On a curious property of Bell numbers

Bell numbers

Bell numbers are named after E. T. Bell who studied them in the 1930s. The history of Bell numbers goes back to the 19th century. For $n = 1, 2, 3, \ldots$, the n-th Bell number B_n denotes the number of partitions of a set of cardinality n, i.e., B_n is the number of equivalent relations on the set $\{1, 2, \ldots, n\}$. In addition, $B_0 := 1$. For example, $B_3 = 5$ since there are totally 5 partitions of $\{1, 2, 3\}$:

$\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}, \{\{1, 2, 3\}\}.$

Here are values of $B_1, \ldots, B_{10}$:

$$1,\ 2,\ 5,\ 15,\ 52,\ 203,\ 877,\ 4140,\ 21147,\ 115975.$$

Recursion:

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k \quad (n = 0, 1, 2, \ldots).$$

Exponential Generating Function:

$$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1}.$$

Dobinski's formula and Touchard's congruence

Dobinski's Formula (1877). For any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, we have

$$B_n = \frac{1}{e} \sum_{m=0}^{\infty} \frac{m^n}{m!}.$$

Touchard's Congruence: For any prime p and $m, n \in \mathbb{N}$ we have

$$B_{p^m+n} \equiv mB_n + B_{n+1} \pmod{p}.$$

In particular,

$$B_p \equiv B_0 + B_1 = 2 \pmod{p}.$$

A direct proof of $B_p \equiv 2 \pmod{p}$

Note that

$$B_n = \sum_{k=0}^{n} S(n,k),$$

where $S(n,k)$ (a Stirling number of the second kind) is the number of ways to partition $\{1,\ldots,n\}$ into k nonempty sets. Clearly, $S(n,0) = 0$ and $S(n,1) = S(n,n) = 1$ for any $n \in \mathbb{Z}^+ = \{1,2,3,\ldots\}$. It is well known that

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n \text{ for all } n \in \mathbb{Z}^+ \text{ and } k \in \mathbb{N}.$$

Let p be a prime. With the help of Fermat's little theorem, if $1 < k < p$ then $S(p,k) \equiv S(1,k) = 0 \pmod{p}$. Therefore

$$B_p \equiv S(p,1) + S(p,p) = 2 \pmod{p}.$$

A curious discovery concerning Bell numbers

Conjecture (Z.-W. Sun, July 17, 2010). For any positive integer n there is a unique integer $a(n)$ such that

$$\sum_{k=0}^{p-1}\frac{B_k}{(-n)^k}\equiv a(n)\pmod p\ \text{ for any prime } p\nmid n.$$

In particular,

$$a(2)=1,\ a(3)=2,\ a(4)=-1,\ a(5)=10,\ a(6)=-43,$$
$$a(7)=266,\ a(8)=-1853,\ a(9)=14834,\ a(10)=-133495.$$

But I was unable to figure out the pattern of $a(n)$. The sequence $a(n)$ does not appear in OEIS.

Remark. It is easy to see that $a(1)=2$. In fact, if p is prime then

$$\begin{aligned}\sum_{k=0}^{p-1}(-1)^kB_k\equiv&\sum_{k=0}^{p-1}\binom{p-1}{k}B_k=B_p\\ \equiv&B_0+B_1=2\pmod p\ \text{(by Touchard's congruence)}.\end{aligned}$$

Meet Don Zagier in Taiwan (August 2010)

On the invitation of Prof. W. Wenching Li, I visited the National Center for Theoretical Sciences at Hsinchu (Taiwan) during August 1-August 8, 2010. In that period, Prof. Don Zagier, a famous number theorist was also visiting the center. On August 4, 2010 I presented a talk at the center and D. Zagier attended my lecture.

When he heard my above conjecture on Bell numbers, he did not believe it. After my lecture, he made further computation to check my conjecture. (By the way, this mathematician is very good at computation!)

Instead of $a(n)$, Zagier investigated the sequence $u_n = a(n) - 1$ for which

$$\sum_{k=1}^{p-1} \frac{B_k}{(-n)^k} \equiv u_n \pmod{p} \ \text{ for any prime } p \nmid n.$$

Derangement numbers occur

Note that

$$u_1 = 1,\ u_2 = 0,\ u_3 = 1,\ u_4 = -2,\ u_5 = 9,\ u_6 = -44,\ u_7 = 265,\ \ldots.$$

Zagier recognized the pattern of u_n:

$$u_{n+1} = 1 - nu_n \quad \text{for } n = 1, 2, \ldots.$$

After he told me this, I immediately realized that $u_n = (-1)^{n-1}D_{n-1}$, where $D_0, D_1, \ldots$ are derangement numbers in enumerative combinatorics!

For $n = 1, 2, 3, \ldots$ the n-th derangement number D_n is the number of permutations σ of $\{1, \ldots, n\}$ with $\sigma(i) \neq i$ for all $i = 1, \ldots, n$; in addition, $D_0 := 1$. It is well known that

$$\frac{D_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \ \text{ for all } n \in \mathbb{N}.$$

Sun and Zagier's results on Bell numbers

Theorem (Sun and D. Zagier [Bull. Austral. Math. Soc., 84(2011)]) (i) For every positive integer n we have

$$\sum_{k=1}^{p-1}\frac{B_k}{(-n)^k}\equiv(-1)^{n-1}D_{n-1}\pmod p$$

for any prime p not dividing n.

(ii) Let p be any prime. Then for all $n=1,\ldots,p-1$ we have

$$B_n\equiv\sum_{k=1}^{p-1}(-1)^kD_{k-1}(-k)^n\pmod{p}.$$

Consequently, $B_{p-1}\equiv D_{p-1}+1 \pmod p$ for any prime p because

$$\begin{aligned}B_{p-1}\equiv&\sum_{k=1}^{p-1}(-1)^kD_{k-1}=(-1)^{p-1}D_{p-1}-\sum_{j=0}^{p-1}(-1)^jD_j\\ \equiv&D_{p-1}-\sum_{j=0}^{p-1}\binom{p-1}{j}D_j=D_{p-1}-(p-1)!\equiv D_{p-1}+1 \pmod p.\end{aligned}$$

Part (i) implies part (ii)

Part (i) implies part (ii). For $k, n \in \{1, \ldots, p-1\}$ with $k \neq n$, as $p - 1 \nmid n - k$ we have

$$\sum_{m=1}^{p-1} (-m)^{n-k} \equiv 0 \pmod{p}.$$

Thus, with the help of part (i), if $n \in \{1, \ldots, p-1\}$ then

$$\begin{aligned} -B_n &\equiv \sum_{k=1}^{p-1} B_k \sum_{m=1}^{p-1} (-m)^{n-k} = \sum_{m=1}^{p-1} (-m)^n \sum_{k=1}^{p-1} \frac{B_k}{(-m)^k} \\ &\equiv \sum_{m=1}^{p-1} (-m)^n (-1)^{m-1} D_{m-1} \pmod{p}. \end{aligned}$$

Prove part (i) by induction

For any prime p,

$$\sum_{k=1}^{p-1}(-1)^k B_k \equiv \sum_{k=1}^{p-1}\binom{p-1}{k}B_k = B_p - B_0 \equiv 1 \pmod{p}.$$

So the desired result holds when $n = 1$.

Now fix $n \in \mathbb{Z}^+$ and suppose that

$$\sum_{k=1}^{p-1}\frac{B_k}{(-n)^k} \equiv (-1)^{n-1}D_{n-1} \pmod{p}$$

for every prime $p \nmid n$.

Let p be any prime not dividing $n+1$. Recall the easy identity

$$D_n = nD_{n-1} + (-1)^n.$$

If $p \mid n$, then $D_n \equiv (-1)^n \pmod{p}$ and hence

$$\sum_{k=1}^{p-1}\frac{B_k}{(-n-1)^k} \equiv \sum_{k=1}^{p-1}\frac{B_k}{(-1)^k} \equiv 1 \equiv (-1)^n D_n \pmod{p}.$$

Prove part (i) by induction

Now suppose that $p \nmid n$. Observe that

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{B_k}{(-n)^k} &= \sum_{k=1}^{p-1} \frac{\sum_{l=0}^{k-1} \binom{k-1}{l} B_l}{(-n)^k} = \sum_{l=0}^{p-2} \frac{B_l}{(-n)^l} \sum_{k=l+1}^{p-1} \frac{\binom{k-1}{l}}{(-n)^{k-l}} \\
&= \sum_{l=0}^{p-2} \frac{B_l}{(-n)^{l+1}} \sum_{r=1}^{p-1-l} \frac{\binom{l+r-1}{r-1}}{(-n)^{r-1}} \\
&= \sum_{l=0}^{p-2} \frac{B_l}{(-n)^{l+1}} \sum_{r=1}^{p-1-l} \frac{\binom{-l-1}{r-1}}{n^{r-1}} \\
&\equiv \sum_{l=0}^{p-2} \frac{B_l}{(-n)^{l+1}} \sum_{r=1}^{p-1-l} \binom{p-1-l}{r-1} n^{-(r-1)} \\
&\equiv \sum_{l=0}^{p-1} \frac{B_l}{(-n)^{l+1}} \left(\left(1 + \frac{1}{n}\right)^{p-1-l} - \frac{1}{n^{p-1-l}} \right) \pmod{p}.
\end{aligned}$$

Prove part (i) by induction

Thus, applying Fermat's little theorem we get

$$-n\sum_{k=1}^{p-1}\frac{B_k}{(-n)^k}\equiv\sum_{l=1}^{p-1}\frac{B_l}{(-n-1)^l}-\sum_{l=1}^{p-1}\frac{B_l}{(-1)^l}\pmod{p}.$$

Therefore

$$\begin{aligned}\sum_{l=1}^{p-1}\frac{B_l}{(-n-1)^l}&\equiv -n\sum_{k=1}^{p-1}\frac{B_k}{(-n)^k}+\sum_{l=1}^{p-1}\frac{B_l}{(-1)^l}\\&\equiv -n(-1)^{n-1}D_{n-1}+1=(-1)^nD_n\pmod{p}.\end{aligned}$$

This concludes the induction step.

A further extension

The Touchard polynomial $T_n(x)$ of degree n is given by

$$T_n(x) = \sum_{k=0}^{n} S(n,k)x^k.$$

Note that $T_n(1) = B_n$. Similar to the recursion for Bell numbers, we have the recursion

$$T_{n+1}(x) = x\sum_{k=0}^{n} \binom{n}{k} T_k(x).$$

Theorem (Sun & Zagier, Bull. Austral. Math. Soc., 84(2011)). For every positive integer m, we have

$$(-x)^m \sum_{0<n<p} \frac{T_n(x)}{(-m)^n} \equiv -x^p \sum_{k=0}^{m-1} \frac{(m-1)!}{k!}(-x)^k \pmod{p}$$

for any prime p not dividing m.

Consequences

Let p be a prime. The theorem implies the congruence

$$\sum_{0<n<p} \frac{T_n(x)}{(-m)^n} \equiv \frac{1}{(-x)^{m-1}} \sum_{l=0}^{m-1} \frac{(m-1)!}{l!}(-x)^l \pmod{p}$$

for any p-adic integer $x \not\equiv 0 \pmod{p}$, in particular

$$\sum_{0<n<p} \frac{T_n(x)}{(-2)^n} \equiv \frac{x-1}{x} \pmod{p} \qquad \text{for } p \neq 2,$$

$$\sum_{0<n<p} \frac{T_n(x)}{(-3)^n} \equiv \frac{x^2-2x+2}{x^2} \pmod{p} \qquad \text{for } p \neq 3\,,$$

$$\sum_{0<n<p} \frac{T_n(x)}{(-4)^n} \equiv \frac{x^3-3x^2+6x-6}{x^3} \pmod{p} \qquad \text{for } p \neq 2\,.$$

There are some generalizations of the Sun-Zagier result, see, e.g., I. Mező and J. L. Ramírez [J. Number Theory 177(2017)].

An open problem

Theorem (Q.-H. Hou, H. Wen and Z.-W. Sun [Publ. Math. Debrecen 85(2014)]). The sequence $(\sqrt[n]{D_n})_{n\geqslant 2}$ is strictly increasing, and the sequence $(\sqrt[n+1]{D_{n+1}}/\sqrt[n]{D_n})_{n\geqslant 3}$ is strictly decreasing.

Conjecture (Z.-W. Sun, 2012). The sequence

$$\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n} \quad (n = 1, 2, 3, \ldots)$$

is strictly decreasing to the limit 1.

Progress: (1) Using the log-convexity of $(B_n)_{n\geqslant 1}$ proved by K. Engel in 1994, Yi Wang and Bao-Xuan Zhu [Sci. China Math. 57(2014)] showed that $(\sqrt[n]{B_n})_{n\geqslant 1}$ is strictly increasing.

(2) In December 2012, Hao Pan and Zheng Li (unpublished) independently showed that if N is large enough then

$$\sqrt[n+1]{B_{n+1}}/\sqrt[n]{B_n} \quad (n = N, N+1, \ldots)$$

is strictly decreasing.

Part II. On Delannoy numbers and Schroder numbers

Central Delannoy numbers

For $m, n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, the Delannoy number

$$D_{m,n} := \sum_{k\in\mathbb{N}} \binom{m}{k}\binom{n}{k}2^k$$

in combinatorics counts lattice paths from $(0,0)$ to (m,n) in which only east $(1,0)$, north $(0,1)$, and northeast $(1,1)$ steps are allowed
The n-th central Delannoy number $D_n = D_{n,n}$ has another well-known expression:

$$D_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}.$$

Theorem (Z.-W. Sun [Sci. China Math. 57(2014)]).
$n^2 \mid \sum_{k=0}^{n-1}(2k+1)D_k^2$ for all $n \in \mathbb{Z}^+$, and

$$\sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} \pmod{p}$$

for any odd prime p.

The polynomials $D_n(x)$ $(n = 0, 1, 2, \ldots)$

Define

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k (x+1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k \ (n = 0, 1, \ldots).$$

Note that $D_n(1)$ is the central Delannoy number D_n,

Conjecture (Sun [Sci. China Math. 57(2014)]). Let x be any integer. If p is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^3 \equiv p \left(\frac{-4x-3}{p}\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^4 \equiv p \pmod{p^2}.$$

This was finally proved by Victor J. W. Guo [Integral Transforms Spec. Funct. 26(2015)].

Catalan numbers and large Schröder numbers

The Catalan numbers are given by

$$C_k = \frac{1}{k+1}\binom{2k}{k} = \binom{2k}{k} - \binom{2k}{k+1} \in \mathbb{Z} \quad (k = 0, 1, 2, \ldots).$$

In combinatorics, the (large) Schröder numbers are given by

$$S_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\frac{1}{k+1} = \sum_{k=0}^{n} \binom{n+k}{2k} C_k \quad (n \in \mathbb{N}).$$

Both Catalan numbers and Schröder numbers have many combinatorial interpretations. For example, S_n is the number of lattice paths from the point $(0,0)$ to (n,n) with only allowed steps $(1,0)$, $(0,1)$ and $(1,1)$ which never rise above the line $y = x$.

Little Schröder numbers

For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, the n-th little Schröder number is given by

$$s_n := \sum_{k=1}^{n} N(n,k) 2^{k-1}$$

with the Narayana number $N(n, k)$ defined by

$$N(n,k) := \frac{1}{n}\binom{n}{k}\binom{n}{k-1} \in \mathbb{Z}.$$

Combinatorial Interpretation: s_n is the number of ways to insert parentheses into an expression of $n+1$ terms with two or more items within a parenthesis.

Relation to the Large Schröder Numbers:

$$S_n = 2s_n \quad \text{for all } n = 1, 2, 3, \ldots.$$

Congruences involving Schröder numbers

Theorem (i) (Sun [J. Number Theory 131(2011)]) For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{S_k}{6^k} \equiv 0 \pmod{p}.$$

(ii) (Sun [Acta Arith. 156(2012)]) If $p \equiv 1 \pmod 4$ is a prime and we write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$, then

$$S_{(p-1)/2} \equiv 2 \left(\frac{2}{p}\right) \left(2x - \frac{p}{x}\right) \pmod{p^2}.$$

Conjecture (Sun [JNT 131(2011)]) Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{p-1} D_k S_k \equiv -2p \sum_{k=1}^{p-1} \frac{(-1)^k + 3}{k} \pmod{p^4}.$$

This challenging conjecture was recently proved by Ji-Cai Liu [J. Number Theory 2016].

The polynomials $s_n(x)$ and $S_n(x)$

Note that $S_n = S_n(1)$, where

$$S_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \frac{x^k}{k+1} = \sum_{k=0}^{n} \binom{n+k}{2k} C_k x^k.$$

Motivated by the identity

$$s_n = \sum_{k=1}^{n} N(n,k) 2^{n-k},$$

we introduce the polynomial

$$s_n(x) := \sum_{k=1}^{n} N(n,k) x^{k-1} (x+1)^{n-k} \ (n = 1, 2, 3, \ldots).$$

Relations among $D_n(x), s_n(x)$ and $S_n(x)$

For any $n \in \mathbb{Z}^+$, we have

$$D_{n+1}(x) - D_{n-1}(x) = 2x(2n+1)S_n(x)$$

and

$$(x+1)s_n(x) = S_n(x).$$

This can be easily proved by induction on n.

When $x = 1$, this gives

$$D_{n+1} - D_{n-1} = 2(2n+1)S_n \quad \text{and} \quad 2s_n = S_n.$$

Result on $\sum_{k=0}^{n-1} D_k(x)s_{k+1}(x)$

Theorem (Z.-W. Sun [J. Number Theory 183(2018)]) (i)

$$\frac{1}{n}\sum_{k=0}^{n-1} D_k(x)s_{k+1}(x) = W_n(x(x+1)) \text{ for all } n=1,2,\ldots,$$

where

$$W_n(x)=\sum_{k=1}^{n} w(n,k)C_{k-1}x^{k-1}\in\mathbb{Z}[x]$$

with

$$w(n,k)=\frac{1}{k}\binom{n-1}{k-1}\binom{n+k}{k-1}\in\mathbb{Z}.$$

In particular, $n \mid \sum_{k=0}^{n-1} D_k s_{k+1}$ for all $n=1,2,3,\ldots$.

(ii) For any odd prime p, we have

$$\sum_{k=0}^{p-1} D_k s_{k+1}\equiv 2p^2\left(1-\frac{3}{p}(2^{p-1}-1)\right) \pmod{p^3}.$$

Two Lemmas

Lemma 1 (Sun [Acta Arith. 156(2012)]). Let $n \in \mathbb{Z}^+$. Then

$$n(n+1)S_n(x)^2 = \sum_{k=1}^{n} \binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}x^{k-1}(x+1)^{k+1}$$

and

$$\frac{D_{n-1}(x)+D_{n+1}(x)}{2}S_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k}^2 \frac{2k+1}{(k+1)^2}x^k(x+1)^{k+1}.$$

Lemma 2. For any $m, n \in \mathbb{Z}^+$ with $m \leqslant n$, we have the identity

$$\sum_{k=m}^{n} \binom{k+m}{2m}\left(2m+1-m(m+1)\frac{2k+1}{k(k+1)}\right) = \frac{(n-m)(n+m+1)}{n+1}\binom{n+m}{2m}.$$

Part III. On central trinomial coefficients and Motzkin numbers

Central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](1+x+x^2)^n \text{ (the coefficient of } x^n \text{ in } (1+x+x^2)^n)$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Theorem (i) (Z.-W. Sun [Sci. China Math. 57(2014)]) For any odd prime p, we have

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod{p}.$$

(ii) (H. Q. Cao and Sun [Colloq. Math. 139(2015)]). For any prime $p > 3$, we have

$$T_{p-1} \equiv \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}.$$

On central trinomial coefficients

Theorem (i) (Sun, arXiv:1610.03384) For any prime $p > 3$, we have

$$T_p \equiv 1 \pmod{p^2},$$

moreover

$$T_p \equiv 1 + \frac{p^2}{6}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3},$$

where $B_n(x)$ denotes the Bernoulli polynomial of degree n.

Conjecture (Sun, arXiv:1610.03384). $T_n \equiv 1 \pmod{n^2}$ for *no* composite number $n > 1$.

I have verified this for n up to 8×10^5. This conjecture, if true, provides an interesting characterization of primes via central trinomial coefficients. Note that T_n with $n \in \mathbb{N}$ can be computed efficiently since

$$(n+1)T_{n+1} = (2n+1)T_n + 3nT_{n-1} \quad \text{for } n = 1, 2, 3, \ldots.$$

Mod p^2 congruences for Motzkin numbers

The nth Motzkin number

$$M_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

is the number of paths from $(0,0)$ to $(n,0)$ which never dip below the line $y=0$ and are made up only of the allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$.

Conjecture (Z.-W. Sun [Sci. China Math. 57(2014)]). Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} M_k T_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) + \frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Generalized central trinomial coefficients and generalized Motzkin numbers

Given $b, c \in \mathbb{Z}$, the *generalized central trinomial coefficients*

$$T_n(b,c) := [x^n](x^2+bx+c)^n = [x^0](b+x+cx^{-1})^n$$
$$= \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k}\binom{n}{k} b^{n-2k}c^k$$

and the *generalized Motzkin numbers*

$$M_n(b,c) := \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k} C_k b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n-k}{k}\binom{n}{k} \frac{b^{n-2k}c^k}{k+1}$$

$(n = 0, 1, 2, \ldots)$. Note that

$$T_n = T_n(1,1),\ M_n = M_n(1,1),\ T_n(2,1) = [x^n](x+1)^{2n} = \binom{2n}{n},$$

and

$$M_n(2,1) = \sum_{k=0}^{n} \binom{n}{2k} C_k 2^{n-2k} = C_{n+1}.$$

Recurrences and Related Legendre Polynomials

Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. By the Zeilberger algorithm,

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c) \quad (n \in \mathbb{Z}^+),$$

$$(n+3)M_{n+1}(b,c) = (2n+3)bM_n(b,c) - ndM_{n-1}(b,c) \quad (n \in \mathbb{Z}^+).$$

It is known that

$$T_n(b,c) = (\sqrt{d})^n P_n\left(\frac{b}{\sqrt{d}}\right),$$

where the *Legendre polynomial* $P_n(t)$ is given by

$$P_n(t) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{t-1}{2}\right)^k = D_n\left(\frac{t-1}{2}\right).$$

It follows that

$$T_n(2x+1, x^2+x) = P_n(2x+1) = D_n(x) \quad \text{for all } x \in \mathbb{Z};$$

in particular, $D_n = P_n(3) = T_n(3,2)$.

Congruences modulo n^2

Theorem (Sun [Sci. China Math. 57(2014)]). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)\,T_k(b,c)^2 d^{n-1-k} = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_k c^k d^{n-1-k}.$$

If c is nonzero and p is an odd prime not dividing d, then

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} \equiv 1 + \frac{b^2}{c}\cdot\frac{(\frac{d}{p})-1}{2} \pmod{p}.$$

A Lemma. For any $n \in \mathbb{N}$ and $b, c \in \mathbb{Z}$ we have

$$T_k(b,c)^2 = \sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2 c^k (b^2-4c)^{n-k}.$$

Arithmetic means involving $T_k(b, c^2)^2$

Theorem (Conjectured by Sun in 2010 and confirmed by Mu and Sun in 2016) Let $b, c \in \mathbb{Z}$. For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8ck+4c+b)T_k(b,c^2)^2(b-2c)^{2(n-1-k)} \equiv 0 \pmod{n}.$$

If p is an odd prime not dividing $b(b-2c)$, then

$$\sum_{k=0}^{p-1}(8ck+4c+b)\frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv p(b+2c)\left(\frac{b^2-4c^2}{p}\right) \pmod{p^2}.$$

Corollary (Conjectured by Sun in 2010 and confirmed by Mu and Sun in 2016).

$$\sum_{k=0}^{n-1}(8k+5)T_k^2 \equiv 0 \pmod{n} \quad \text{for all } n \in \mathbb{Z}^+,$$

$$\sum_{k=0}^{p-1}(8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2} \text{ for any prime } p.$$

Mu and Sun's telescoping approach

To study some challenging conjectures of Sun on congruences for

$$S_n=\sum_{k=0}^{n-1}\sum_{l=0}^{k}F(k,l)\quad (n\in\mathbb{Z}^+),$$

where $F(k,l)$ is a bivariate hypergeometric term of k and l, Yan-Ping Mu and Z.-W. Sun [IJNT, in press; arXiv:1601.03954] search for two hypergeometric terms $G_1(k,l)$ and $G_2(k,l)$ with

$$F(k,l)=\Delta_k(G_1(k,l))+\Delta_l(G_2(k,l)),$$

where

$$G_1(k,l)=R_1(k,l)F(k,l)\quad\text{and}\quad G_2(k,l)=R_2(k,l)F(k,l)$$

with $R_1(k,l)$ and $R_2(k,l)$ rational functions, and

$$\Delta_k(G_1(k,l))=G_1(k+1,l)-G_1(k,l),$$
$$\Delta_l(G_2(k,l))=G_2(k,l+1)-G_2(k,l).$$

The resulting functions $G_1(k,l)$ and $G_2(k,l)$ we obtain are essentially well defined for $0\leqslant l\leqslant k\leqslant n-1$.

Mu and Sun's telescoping approach

Once we have $G_1(k,l)$ and $G_2(k,l)$ in hand, the sum S_n can be transformed to a single sum

$$S_n = \sum_{l=0}^{n-1} \big(G_1(n,l) - G_1(l,l)\big) + \sum_{k=0}^{n-1} \big(G_2(k,k+1) - G_2(k,0)\big)$$

We can use the `Maple` package *DoubleSum* given by Chen-Hou-Mu [J. Comput. Appl. Math. 196(2006)] or the `Mathematica` package *HolonomicFunctions* given by C. Koutschan [Math. Comput. Sci. 4(2010)], together with the package *DoubleSum* or the package *MultiSum* [Appl. Algebra Engrg. Comm. Comput. 13(2002)], to compute a suitable pair $(G_1(k,l), G_2(k,l))$.

Once we get a single sum for S_n, it would be be convenient to deduce Sun's conjectural congruences for S_n. Using this powerful method, Mu and Sun confirm several sophisticated open conjectures of Sun.

Three conjectural series involving $T_k(b,c)^3$

Conjecture (Z.-W. Sun, 2011).

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

I would like to offer \$300 as the prize for the person who can provide first rigorous proofs of all the above three identities. The last one was inspired by my following conjecture for primes $p>3$.

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)$$
$$\equiv\frac{p}{19}\left(17563\left(\frac{-14}{p}\right)+4230\left(\frac{-2}{p}\right)\right)\pmod{p^2}.$$

A conjecture involving $T_k(b,c)M_k(b,c)$

Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. Conjecture 5.5 of Sun [Sci. China Math. 57(2014)] asserts that

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k(b,c)M_k(b,c)d^{n-1-k} \in \mathbb{Z} \quad \text{for all } n = 1, 2, 3, \ldots.$$

How to prove this ? Recall that $T_k(3,2) = D_k$. In 2016 the speaker realized that $M_k(3,2)$ coincides with the little Schröder number s_{k+1}. Thus he was led to show

$$\frac{1}{n}\sum_{k=0}^{n-1} D_k s_{k+1} = \frac{1}{2n}\sum_{k=0}^{n-1} D_k S_{k+1} \in \mathbb{Z}$$

for all $n = 1, 2, 3, \ldots$.

Recall that in 2011 the speaker made a conjecture on $\sum_{k=0}^{p-1} D_k S_k$ modulo p^4 with p an odd prime, which was proved by Ji-Cai Liu in 2016.

A general theorem

Theorem (Sun [J. Number Theory 183(2018)]). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k(b,c)M_k(b,c)d^{n-1-k} = \sum_{k=1}^{n} w(n,k)C_{k-1}c^{k-1}d^{n-k} \in \mathbb{Z}.$$

Moreover, for any odd prime p not dividing cd, we have

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)M_k(b,c)}{d^k} \equiv \frac{pb^2}{2c}\left(\left(\frac{d}{p}\right) - 1\right) \pmod{p^2}.$$

Corollary. For any positive integer n, we have

$$\frac{1}{n}\sum_{k=0}^{n-1} T_k M_k(-3)^{n-1-k} = \sum_{k=1}^{n} w(n,k)C_{k-1}(-3)^{n-k} \in \mathbb{Z}.$$

Moreover, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} \frac{T_k M_k}{(-3)^k} \equiv \frac{p}{2}\left(\left(\frac{p}{3}\right) - 1\right) \pmod{p^2}.$$

A key lemma

Lemma. Let $b, c \in \mathbb{Z}$ with $d = b^2 - 4c \neq 0$. For any $n \in \mathbb{N}$, we have

$$T_n(b,c) = (\sqrt{d})^n D_n\left(\frac{b/\sqrt{d}-1}{2}\right)$$

and

$$M_n(b,c) = (\sqrt{d})^n s_{n+1}\left(\frac{b/\sqrt{d}-1}{2}\right),$$

therefore

$$\frac{T_n(b,c)M_n(b,c)}{d^n} = D_n(x)s_{n+1}(x)$$

with $x = (b/\sqrt{d}-1)/2$.

Part IV. Some open conjectures

A conjecture with \$48 prize

Conjecture (Z.-W. Sun, 2014) (i) For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv \frac{5}{12} p^2 B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

(ii) We have

$$\sum_{k=1}^{\infty} \frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}} = \frac{15}{2} K,$$

where

$$K := L\left(2, \left(\frac{-3}{\cdot}\right)\right) = \sum_{k=1}^{\infty} \frac{\left(\frac{k}{3}\right)}{k^2} = 0.781302412896\ldots.$$

Remark. I [Sci. China Math. 54(2011)] also conjectured for any prime $p > 3$ that

$$\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{48^k} \equiv \begin{cases} 2x - p/(2x) \pmod{p^2} & \text{if } p = x^2 + 3y^2\ (3 \mid x-1), \\ 3p/(2\binom{(p+1)/2}{(p+1)/6}) \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

On Apéry numbers

In 1978 Apéry proved that $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$ is irrational! Those *Apéry numbers*

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2$$

play important roles in Apéry's proof.

Conjecture (Z.-W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1}A_k\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p=x^2+2y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 5,7\pmod 8.\end{cases}$$

Remark. I [JNT, 2011] proved the mod p version of the conjectural congruence. The conjecture still remains open!

On Domb numbers

Domb numbers are given by

$$d_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \binom{2(n-k)}{n-k} \ (n = 0, 1, 2, \ldots).$$

Conjecture (Z.-W. Sun, 2013). For any positive integer n we have

$$\frac{1}{4n} \sum_{k=0}^{n-1} (5k+4) d_k \in \mathbb{Z}.$$

Theorem (Mu and Sun, 2016). For any integer $n > 1$, we have

$$\frac{1}{2n^3(n-1)} \sum_{k=0}^{n-1} (3k^2+k) d_k 16^{n-1-k} \in \mathbb{Z}.$$

Remark. This result was originally conjectured by Z.-W. Sun in [Sci. China Math. 54(2011)].

On $F_n = \sum_{k=0}^{n} \binom{n}{k}^3 (-8)^k$

Define

$$F_n = \sum_{k=0}^{n} \binom{n}{k}^3 (-8)^k \quad (n = 0, 1, 2, \ldots).$$

Theorem (Conjectured by Sun in 2011, and proved by Mu and Sun in 2016). For any positive integer n, the number

$$\frac{1}{n} \sum_{k=0}^{n-1} (6k+5)(-1)^k F_k$$

is always an odd integer.

Conjecture (Sun [Adv. Appl. Math. 51(2013)]).

$$\sum_{k=0}^{p-1} (-1)^k F_k \equiv \left(\frac{p}{3}\right) \pmod{p^2} \quad \text{for any prime } p > 3.$$

I [Adv. Appl. Math. 51(2013)] showed $\sum_{k=0}^{p-1} (-1)^k f_k \equiv \left(\frac{p}{3}\right)$ (mod p^2) for each prime $p > 3$, where $f_k := \sum_{l=0}^{k} \binom{k}{l}^3$.

Congruences involving dual sequences

For a sequence $(a_n)_{n\geqslant 0}$ of numbers, its *dual sequence* $(a_n^*)_{n\geqslant 0}$ is given by $a_n^* = \sum_{k=0}^{n}\binom{n}{k}(-1)^k a_k$. It is well known that $(a_n^*)^* = a_n$ for $n = 0, 1, 2, \ldots$.

Observe that

$$\sum_{k=0}^{n}\binom{n}{k}\frac{\binom{2k}{k}}{(-2)^k} = \begin{cases}\binom{2m}{m}/4^m & \text{if } n = 2m \text{ for some } m \in \mathbb{N},\\ 0 & \text{if } n \text{ is odd.}\end{cases}$$

As conjectured by Rodriguez-Villegas and proved by E. Mortenson [JNT 99(2013)], $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k \equiv \left(\frac{-1}{p}\right) \pmod{p^2}$ for any prime $p > 2$.

Conjecture (Sun [Finite Fields Appl. 46(2017)]). Let $p > 3$ be a prime. Then

$$\sum_{n=0}^{p-1}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\binom{2k}{k}}{2^k}\right)^2 \equiv \sum_{n=0}^{p-1}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{\binom{2k}{k}}{(-6)^k}\right)^2 \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

A conjecture involving $t_n(x)$

For $n \in \mathbb{N}$ define

$$s_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{x}{k}\binom{x+k}{k} = \sum_{k=0}^{n} \binom{n}{k}(-1)^k\binom{x}{k}\binom{-1-x}{k}.$$

During their study of special values of spectral zeta functions, K. Kimoto and M. Wakayama conjectured $\sum_{n=0}^{p-1} s_n(-\frac{1}{2})^2 \equiv \left(\frac{-1}{p}\right)$ (mod p^3) for any odd prime p, which was confirmed by L. Long, R. Osburn and H. Swisher [Proc. AMS 144(2016)].

Conjecture (Sun [Finite Fields Appl. 46(2017)]). For any $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, the number

$$\frac{1}{n}\sum_{k=0}^{n-1}(8k+5)t_k(x)^2$$

is always an integer congruent to 1 modulo 4, where $t_k(x) := \sum_{l=0}^{k} \binom{k}{l}\binom{x}{l}\binom{x+l}{l}2^l$.

Thank you!

My 1-3-5 Conjecture (with \$1350 prize for its solution): Any $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.

Examples:

$$7 = 1^2 + 1^1 + 1^2 + 2^2 \quad \text{with } 1 + 3 \times 1 + 5 \times 1 = 3^2,$$

and

$$43 = 1^2 + 5^2 + 4^2 + 1^2 \quad \text{with } 1 + 3 \times 5 + 5 \times 4 = 6^2.$$