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A SURVEY OF ZERO-SUM PROBLEMS ON ABELIAN GROUPS

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Abstract. Let G be a finite abelian group. A zero-sum problem on G asks for the smallest positive integer k such that for any sequence $a_1,\ldots,a_k$ of elements of G there exists a subsequence of required length the sum of whose terms vanishes. In this talk we will give a survey of problems and results in this field. In particular, we will talk about Olson's theorem on the Davenport constanst of an abelian p-group, Reiher's celebrated proof of the Kemnitz conjecture, and the speaker's extension of the Erdős-Ginzburg-Ziv theorem involving covers of $\mathbb{Z}$.

1. Basic concepts and results on zero-sums

Let G be an (additive) abelian group of order n. If $a_1,\ldots,a_n\in G$, then by the pigeonhole principle the following elements (of G)

$$s_0=0,\ s_1=a_1,\ s_2=a_1+a_2,\ \ldots,\ s_n=a_1+\ldots+a_n$$

cannot be pairwise distinct, thus for some $0\leqslant i<j\leqslant k$ we have

$$\sum_{i<r\leqslant j}a_r=s_j-s_i=0.$$

If $a_1,\ldots,a_{n^2-n+1}\in G$, then for some $a\in G$ the set $J=\{1\leqslant i\leqslant n^2-n+1\colon a_i=a\}$ has cardinality at least n, therefore for $I\subseteq J$ with $|I|=n$ we have $\sum_{i\in I}a_i=na=0$. (Recall that $|G|=n$ and so the additive order of a divides n.)

The *zero-sum problem* on a finite abelian group G asks for the least positive integer k such that for any $a_1,\ldots,a_k\in G$ there exists an $I\subseteq\{1,\ldots,k\}$ of required cardinality satisfying $\sum_{i\in I}a_i=0$. If we simply require $|I|>0$, then the smallest k is denoted by $D(G)$ and called *Davenport's constant*. If we require $|I|=|G|$, then the smallest k is denoted by $S(G)$. By the preceding passage, both $D(G)$ and $S(G)$ exist, moreover $D(G)\leqslant|G|$ and $|S(G)|\leqslant|G|^2-|G|+1$.

Let $\mathbb{Z}_n=\mathbb{Z}/n\mathbb{Z}=\{\bar{a}=a+n\mathbb{Z}\colon a\in\mathbb{Z}\}$ be the ring of residue classes modulo n. Any cyclic group of order n is isomorphic to the additive group of $\mathbb{Z}_n$. For $a_1=\ldots=a_{n-1}=\bar{1}$ and $\emptyset\neq I\subseteq\{1,\ldots,n-1\}$, we clearly have $\sum_{i\in I}a_i=|I|\bar{1}\neq\bar{0}$. So we cannot have $D(\mathbb{Z}_n)\leqslant n-1$. Therefore $D(\mathbb{Z}_n)=n=|\mathbb{Z}_n|$. If $a_1=\ldots=a_{n-1}=\bar{1}$ and $a_n=\ldots=a_{2n-2}=\bar{0}$, then there is no $I\subseteq\{1,\ldots,2n-2\}$ with $|I|=n$ such that $\sum_{i\in I}a_i=\bar{0}$. This together with the following EGZ theorem shows that $S(\mathbb{Z}_n)=2n-1$.

EGZ Theorem [P. Erdős, A. Ginzurg and A. Ziv, Bull. Research Council Israel, 10(1961)]**.** *Let n be any positive integer, and let $a_1,\ldots,a_{2n-1}\in\mathbb{Z}$. Then there is an $I\subseteq\{1,\ldots,2n-1\}$ with $|I|=n$ such that $\sum_{i\in I}a_i\equiv 0\pmod n$.*

Historically this is the first nontrivial result on zero-sum problems.

It is easy to reduce the EGZ theorem to the case when n is a prime number p. Note that $\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}$ is a finite field. The EGZ theorem can be deduced from the following classical result.

Chevalley-Warning Theorem. *Let F be a finite field of order $q=p^\alpha$ where p is a prime. Let $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)$ be polynomials over F with $\sum_{i=1}^m\deg f_i<n$. Let N denote the number of solutions of the following system*

$$\begin{cases} f_1(x_1,\ldots,x_n)=0,\\ \ldots\ldots\ldots\ldots\ldots\\ f_m(x_1,\ldots,x_n)=0\end{cases}$$

of equations over F. Then p divides N, in particular $N \neq 1$.

Now we explain why the EGZ theorem holds when n is a prime. Let p be a prime and F be a field of order p. Let $a_1, \ldots, a_{2p-1} \in F$,

$$f_1(x_1, \ldots, 2p-1) = \sum_{k=1}^{2p-1} x_k^{p-1} \text{ and } f_2(x_1, \ldots, x_{2p-1}) = \sum_{k=1}^{2p-1} a_k x_k^{p-1}.$$

Note that $\deg f_1 + \deg f_2 < 2p-1$ and $f_1(0, \ldots, 0) = f_2(0, \ldots, 0) = 0$. By the Chevalley-Warning theorem, there are $x_1, \ldots, x_{2p-1} \in F$ such that

$$f_1(x_1, \ldots, x_{2p-1}) = f_2(x_1, \ldots, x_{2p-1}) = 0 \text{ and } I = \{1 \leqslant k \leqslant 2p-1 : x_k \neq 0\} \neq \emptyset.$$

As $0 = f_1(x_1, \ldots, x_{2p-1}) = \sum_{i \in I} x_i^{p-1} = |I|1$, we must have $p \mid |I|$ and hence $|I| = p$ since $0 < |I| < 2p$. Now that $f_2(x_1, \ldots, x_{2p-1}) = 0$, we also have $\sum_{i \in I} a_i = 0$.

Observe that $S(\mathbb{Z}_n) = 2n - 1 = D(\mathbb{Z}_n) + n - 1$. In 1996 W. D. Gao [J. Number Theory, 58(1996)] proved that $S(G) = D(G) + |G| - 1$ for any finite abelian group G.

Davenport constants are named after H. Davenport who showed in 1966 that if K is an algebraic number field with ideal class group G, then $D(G)$ is the maximal number of prime ideals (counting multiplicity) in the decomposition of an irreducible integer in K. In 1969 Olson [J. Number Theory, 1969] used the knowledge of group rings to determine the Davenport constant of any abelian group of prime power order.

Olson's Theorem. *Let p be a prime and let G be an additive abelian p-group isomorphic to $\mathbb{Z}_{p^{\alpha_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{\alpha_r}}$. Let $a_1, \ldots, a_k \in G$ with $k \geqslant 1 + \sum_{s=1}^{r}(p^{\alpha_s} - 1)$. Then there exists a nonempty $I \subseteq [1, k]$ such that $\sum_{i \in I} a_i = 0$, moreover for any $a \in G$ we have the congruence*

$$\sum_{\substack{I \subseteq [1,k] \\ \sum_{i \in I} a_i = a}} (-1)^{|I|} \equiv 0 \pmod{p}.$$

Olson's theorem essentially says that if p is a prime and $\alpha_1,\ldots,\alpha_r\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ then

$$D(\mathbb{Z}_{p^{\alpha_1}}\oplus\ldots\oplus\mathbb{Z}_{p^{\alpha_r}})=1+\sum_{s=1}^r(p^{\alpha_s}-1).$$

For, the sequence of elements of $G=\mathbb{Z}_{p^{\alpha_1}}\oplus\ldots\oplus\mathbb{Z}_{p^{\alpha_r}}$ consisting of $p^{\alpha_1}-1$ copies of $\langle 1,0,\ldots,0\rangle$,..., $p^{\alpha_r}-1$ copies of $\langle 0,\ldots,0,1\rangle$, does not have a nonempty zero-sum subsequence.

Olson's theorem implies the EGZ theorem. In fact, if $a,a_1,\ldots,a_{2p-1}\in\mathbb{Z}_p$ where p is a prime, then by Olson's theorem for $\mathbb{Z}_p^2$ we have

$$\left|\left\{I\subseteq[1,2p-1]:\ |I|=p\text{ and }\sum_{i\in I}a_i=a\right\}\right|\equiv\begin{cases}1\ (\mathrm{mod}\ p) & \text{if } a=0,\\ 0\ (\mathrm{mod}\ p) & \text{if } a\neq 0.\end{cases}$$

In 1994 W. R. Alford, A. Granville and C. Pomerance [Ann. Math.] employed an upper bound for the Davenport constant of the unit group of the ring $\mathbb{Z}_n$ to prove that there are infinitely many Carmichael numbers which are those composites m such that $a^{m-1}\equiv 1\ (\mathrm{mod}\ m)$ for any $a\in\mathbb{Z}$ with $(a,m)=1$.

2. An analysis of Reiher's proof of the Kemnitz conjecture

What is the smallest integer $l=s(\mathbb{Z}_n^2)$ such that every sequence of l elements in $\mathbb{Z}_n^2=\mathbb{Z}_n\oplus\mathbb{Z}_n$ contains a zero-sum subsequence of length n?

In 1983 A. Kemnitz [Ars Combin.] posed the following conjecture.

Kemnitz's conjecture. *For any $a_1,\ldots,a_{4n-3}\in\mathbb{Z}_n\oplus\mathbb{Z}_n$, there exists an $I\subseteq\{1,\ldots,4n-3\}$ with $|I|=n$ such that $\sum_{i\in I}a_i=0$.*

Kemnitz's conjecture is equivalent to the equality $s(\mathbb{Z}_n^2)=4n-3$ because $4n-3$ is the above conjecture cannot be replaced by a smaller number. In fact, let

$$a_1=\ldots=a_{n-1}=(0,0),\qquad a_n=\ldots=a_{2n-2}=(0,1),$$

$$a_{2n-1}=\ldots=a_{3n-3}=(1,0),\ a_{3n-2}=\ldots=a_{4n-4}=(1,1),$$

then there is no $I \subseteq \{1, \ldots, 4n-4\}$ with $|I| = n$ such that $\sum_{i\in I} a_i = (0,0)$. The Kemnitz conjecture can be reduced to the case with n a prime.

In 1993 Alon and Dubiner showed that $s(\mathbb{Z}_n^2) \leqslant 6n-5$. In 2000 Rónyai [Combinatorica] was able to prove that $s(\mathbb{Z}_p^2) \leqslant 4p-2$ for every prime p; in 2001 W. D. Gao [J. Combin. Theory Ser. A] used Olson's group ring approach to deduce that $s(\mathbb{Z}_q^2) \leqslant 4q-2$ for any prime power q. All these results were obtained by various ingenious algebraic methods.

The following lemma plays an indispensable role in the study of the Kemnitz conjecture.

The Alon-Dubiner Lemma. *Let q be a prime power, and let $a_1, \ldots, a_{3q}$ be elements of $\mathbb{Z}_q^2$ with $a_1 + \ldots + a_{3q} = 0$. Then there is an $I \subseteq [1, 3q]$ with $|I| = q$ such that $\sum_{i\in I} a_i = 0$.*

Let us recall a useful formula due to Zhi-Wei Sun which was motivated by Sun's study of covers of $\mathbb{Z}$ and the polynomial method.

Theorem 2.1 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]**.** *Let R be a ring with identity, and let $f(x_1, \ldots, x_k)$ be a polynomial over R. If $J \subseteq [1, k]$ and $|J| \geqslant \deg f$, then we have the formula*

$$\sum_{I\subseteq J} (-1)^{|J|-|I|} f([1 \in I], \ldots, [k \in I]) = \left[\prod_{j\in J} x_j\right] f(x_1, \ldots, x_k)$$

where $[x_1^{i_1} \ldots x_k^{i_k}] f(x_1, \ldots, x_k)$ denotes the coefficient of the monomial $x_1^{i_1} \ldots x_k^{i_k}$ in the polynomial $f(x_1, \ldots, x_k)$, and we let $[i \in I]$ be 1 or 0 according to whether $i \in I$ or not.

The EGZ theorem and Olson's theorem **are easy consequences of the above formula!**

By using the powerful formula and the Alon-Dubiner lemma, Sun [Electron.

Res. Announc. Amer. Math. Soc. 9(2003); arXiv:math.NT/0305369] obtained the following result concerning the Kemnitz conjecture.

Theorem 2.2 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003), 51-60; arXiv:math.NT/0305369]. *Let p be a prime and let $h>0$ be an integer. Let $a_i,b_i\in\mathbb{Z}$ for $i=1,\ldots,4p^h-2$.*

(i) *Set $\mathcal{I}=\{I\subseteq[1,4p^h-2]\colon\sum_{i\in I}a_i\equiv\sum_{i\in I}b_i\equiv 0\pmod{p^h}\}$. Then*

$$|\{I\in\mathcal{I}\colon |I|=p^h\}|\equiv|\{I\in\mathcal{I}\colon|I|=3p^h\}|+2\pmod p.$$

(ii) *Suppose that*

$$\sum_{\substack{I,J\subseteq[1,4p^h-3]\\|I|=|J|=p^h-1\\I\cap J=\emptyset}}\Big(\prod_{i\in I}a_i\Big)\Big(\prod_{j\in J}b_j\Big)\not\equiv 2\pmod p.$$

Then there exists an $I\subseteq[1,4p^h-3]$ with $|I|=p^h$ such that $\sum_{i\in I}a_i\equiv\sum_{i\in I}b_i\equiv 0\pmod{p^h}$.

In 2003 C. Reiher released a preprint "*On Kemnitz's conjecture concerning lattice points in the plane*", in which he completely proved the Kemnitz conjecture which had been open for 20 years! This work represents one of the most important achievements in the theory of zero-sums.

Reiher's paper has 4 pages. Pages 1–3 are devoted to 5 sophisticated corollaries to the Chevalley-Warning theorem which are needed later. Actually this can be significantly simplified by using the first part of the above theorem with $a_{4p-2}=b_{4p-2}=0$.

A Consequence of Theorem 2.2(i). *Let p be a prime and let $h>0$ be an integer. Let $a_i,b_i\in\mathbb{Z}$ for $i=1,\ldots,4p^h-3$. Set $\mathcal{I}=\{I\subseteq[1,4p^h-3]\colon\sum_{i\in I}a_i\equiv\sum_{i\in I}b_i\equiv 0\pmod{p^h}\}$. Then*

$$\begin{aligned}&|\{I\in\mathcal{I}\colon|I|=p^h\}|+|\{I\in\mathcal{I}\colon|I|=p^h-1\}|\\\equiv&|\{I\in\mathcal{I}\colon|I|=3p^h\}|+|\{I\in\mathcal{I}\colon|I|=3p^h-1\}|+2\pmod p.\end{aligned}$$

On the last page of his paper, C. Reiher provided a key lemma which is obtained by a combinatorial method rather than an algebraic method.

Reiher's Lemma. *Let p be a prime and let $a_i, b_i \in \mathbb{Z}$ for $i = 1, \ldots, 4p-3$. Set*

$$\mathcal{I} = \Big\{ I \subseteq [1, 4p-3] \colon \sum_{i \in I} a_i \equiv \sum_{i \in I} b_i \equiv 0 \pmod{p} \Big\}.$$

Then, either $\{I \in \mathcal{I} \colon |I| = p\} \neq \emptyset$ or

$$|\{I \in \mathcal{I} \colon |I| = p-1\}| \equiv |\{I \in \mathcal{I} \colon |I| = 3p-1\}| \pmod{p}.$$

Sketch of the Proof. For $J \subseteq [1, 4p-3]$ and $n = 1, 2, \ldots$ let

$$(n, J) := \left| \left\{ I \subseteq J \colon |I| = n \ \& \ \sum_{i \in I} a_i \equiv \sum_{i \in I} b_i \equiv 0 \pmod{p} \right\} \right|.$$

It is easy to show that if $|J| \in \{3p-1, 3p-2\}$ then $(2p, J) \equiv (p, J) - 1 \pmod{p}$.

Now assume that $\{I \in \mathcal{I} \colon |I| = p\} = \emptyset$, i.e., $(p, J) = 0$ for any $J \subseteq [1, 4p-3]$. Let N denote the number of partitions $[1, 4p-3] = I_1 \cup I_2 \cup I_3$ satisfying

$$|I_1| = p-1, \quad |I_2| = p-2, \quad |I_3| = 2p$$

and furthermore

$$\sum_{i \in I_1} a_i \equiv \sum_{i \in I_1} b_i \equiv 0 \pmod{p}, \quad \sum_{i \in I_3} a_i \equiv \sum_{i \in I_3} b_i \equiv 0 \pmod{p}$$

(and hence $\sum_{i \in [1,4p-3] \setminus I_2} a_i \equiv \sum_{i \in [1,4p-3] \setminus I_2} b_i \equiv 0 \pmod{p}$). We count N in two ways. Observe that

$$N = \sum_{I_1} (2p, [1, 4p-3] \setminus I_1) \equiv \sum_{I_1} (-1) = -(p-1, [1, 4p-3]) \pmod{p}.$$

On the other hand,

$$N = \sum_{I_2} (2p, [1, 4p-3] \setminus I_2) \equiv \sum_{[1,4p-3] \setminus I_2} (-1) = -(3p-1, [1, 4p-3]) \pmod{p}.$$

So we have the congruence $(p-1, [1, 4p-3]) \equiv (3p-1, [1, 4p-3]) \pmod{p}$. $\square$

We remark that the prime power version of this lemma also holds.

Combining Reiher's Lemma, the Alon-Dubiner lemma and the above consequence of Theorem 2.2(i), we immediately obtain the following result of Reiher.

The Kemnitz-Reiher Theorem. *The Kemnitz conjecture is true.*

What does Reiher's solution teach us? When we apply a powerful algebraic method in combinatorics, we should also realize its disadvantage and should not forget combinatorial methods. **A combination of algebraic methods and combinatorial methods might be more powerful!**

3. Further extensions and some open problems

For $a\in\mathbb{Z}$ and $n\in\mathbb{Z}^+$, let $a(n)$ represent the residue class

$$a+n\mathbb{Z}=\{x\in\mathbb{Z}:\ x\equiv a\ (\mathrm{mod}\ n)\}.$$

A finite system $A=\{a_s(n_s)\}_{s=1}^k$ of residue classes is said to be an *m-cover* of $\mathbb{Z}$ (with $m\in\mathbb{Z}^+$) if $w_A(x)\geqslant m$ for all $x\in\mathbb{Z}$, where

$$w_A(x)=|\{1\leqslant s\leqslant k:\ x\in a_s(n_s)\}|.$$

In 2003 Z. W. Sun established connections between covers of $\mathbb{Z}$ and some classical theorems on zero-sums such as $D(\mathbb{Z}_m)=m$, the EGZ theorem, the Alon-Dubiner lemma and Olson's theorem. The following theorem in the case $n_1=\cdots=n_k=1$ reduces to known results on zero-sums.

Theorem 3.1 [Z. W. Sun, Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`]. *Let $A=\{a_s(n_s)\}_{s=1}^k$ and let q be a prime power.*

(i) *If A forms a q-cover of $\mathbb{Z}$, then for any $m_1,\ldots,m_k\in\mathbb{Z}$ there exists a nonempty $I\subseteq[1,k]$ such that $\sum_{s\in I}m_s/n_s\in q\mathbb{Z}$.*

(ii) *If $\{w_A(x):\ x\in\mathbb{Z}\}\subseteq\{2q-1,2q\}$, then for any $c_1,\ldots,c_k\in\mathbb{Z}_q$ there exists an $I\subseteq[1,k]$ such that $\sum_{s\in I}1/n_s=q$ and $\sum_{s\in I}c_s=0$.*

(iii) *If A is an exact $3q$-cover of $\mathbb{Z}$, then for any $c_1,\ldots,c_k\in\mathbb{Z}_q^2$ with $c_1+\ldots+c_k=0$, there exists an $I\subseteq[1,k]$ such that $\sum_{s\in I}1/n_s=q$ and $\sum_{s\in I}c_s=0$.*

(iv) *Let G be an additive abelian group of order q. Write $G \cong \mathbb{Z}_{p^{\alpha_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\alpha_r}}$ with p a prime, and set $d(G) = 1 + \sum_{i=1}^{r}(p^{\alpha_i} - 1)$. Suppose that A is a $d(G)$-cover of $\mathbb{Z}$. Then, for any $m_1, \ldots, m_k \in \mathbb{Z}$ and $c_1, \ldots, c_k \in G$, there is a nonempty $I \subseteq [1, k]$ such that $\sum_{s\in I} c_s = 0$ and $\sum_{s\in I} m_s/n_s \in \mathbb{Z}$.*

The following conjecture of Zhi-Wei Sun seems very difficult.

Sun's Conjecture. *If we replace the prime power q in Theorem 3.1 by a general positive integer, and use $D(G)$ instead of $d(G)$ in part (iii), then the new version of Theorem 3.1 still holds.*

It seems that we cannot have a similar extension of the Kemnitz-Reiher theorem.

The following weighted EGZ theorem was conjectured by Y. Caro in the early 1990s and proved by D. Grynkiewicz completely in 2005.

Theorem 3.2 [D. J. Grynkiewicz, Combinatorica, to appear]**.** *Let $w_1, \ldots, w_m$ be integers with $w_1 + \cdots + w_m \equiv 0 \pmod{n}$ where $n \in \mathbb{Z}^+$. Then, for any $a_1, \ldots, a_{m+n-1} \in \mathbb{Z}$ there are distinct $i_1, \ldots, i_m \in [1, m + n - 1]$ such that*

$$w_1 a_{i_1} + \cdots + w_m a_{i_m} \equiv 0 \pmod{n}.$$

When $m = n$ and $w_1 = \cdots = w_m = 1$, this reduces to the well-known EGZ theorem. Kneser's theorem on sumsets plays a central role in Grynkiewicz's proof of Theorem 3.2. Before Grynkiewicz's conclusive work, Theorem 3.2 was proved by N. Alon in the case $m = n = p$ with p a prime, by Y. O. Hamidoune [Discrete Math. 162(1996)] in the case where $w_1, \ldots, w_m$ are all relatively prime to n.

J. E. Olson gave an extension of the EGZ theorem to any finite group.

Theorem 3.3 [J. E. Olson, J. Number Theory 8(1976)]**.** *Let G be any additive group of order n. Then, for any $a_1, \ldots, a_{2n-1} \in G$ there are distinct indices $i_1, \ldots, i_n$ such that $a_{i_1} + \cdots + a_{i_n} = 0$.*

Olson even conjectured that we can require further that the indices $i_1,\ldots,i_n$ are in increasing order.

Let $1\leqslant k\leqslant n$, and let $f(n,k)$ be the least positive integer l such that if $a_1,\ldots,a_l\in\mathbb{Z}_n$ and $|\{a_1,\ldots,a_l\}|=k$ then $\sum_{i\in I}a_i=0$ for some $I\subseteq[1,l]$ with $|I|=n$. By the EGZ theorem, $f(n,k)\leqslant 2n-1$. To determine $f(n,k)$ is a subtle problem.

Theorem 3.4. *Let k and n be positive integers with $k\leqslant n$.*

(i) [Y. O. Hamidoune, O. Ordaz and A. Ortnno, Combin. Probab. Comput. 7(1998)] *We have $f(n,k)\leqslant 2n-k+1$.*

(ii) [L. Gallardo, G. Grekos and J. Pihko, Acta Arith. 89(1999)] *If $n\geqslant 5$ and $1+n/2<k<n$ then $f(n,k)=n+2$.*

(iii) [C. Wang, Acta Arith. 108(2003)] *If $k=2m+1\geqslant 3$ is odd and $n\geqslant\max\{4m^2-4,m(m+3)/2+2\}$, then $f(n,k)=2n-m^2-1$. If $k=2m\geqslant 3$ is even and $n\geqslant\max\{4m(m-1)-4,m(m+1)/2+1\}$, then $f(n,k)=2n-m(m-1)-1$.*

For the Davenport constant of a general finite abelian group, there are some known upper bounds.

Theorem 3.5. *Let G be a finite abelian group isomorphic to $\mathbb{Z}_{n_1}\oplus\mathbb{Z}_{n_2}\oplus\cdots\oplus\mathbb{Z}_{n_r}$ with $1<n_1\mid n_2\mid\cdots\mid n_r$. (Thus, r is the rank of G and n_r is the exponent of G.)*

(i) [R. Meshulam, Discrete Math. 84(1990)] $D(G)\leqslant n_r(1+\log(|G|/n_r))$.

(ii) [V. Dimitrov, 2003] $D(G)<(500r\log r)^r(1+\sum_{i=1}^r(n_i-1))$.

The Kemnitz-Reiher theorem indicates that $s(\mathbb{Z}_n^2)=4n-3$. What about $s(\mathbb{Z}_n^3)$? C. Elsholtz [Combinatorica 24(2004)] proved that $s(\mathbb{Z}_n^3)\geqslant 9n-8$ for any odd $n>1$. On the basis of this result, many zero-sum researchers (including the speaker) conjectured that $s(\mathbb{Z}_n^3)=9n-8$ for every odd integer $n=3,5,\ldots$. (This seems very difficult.) In 2005 Y. Edel, C. Elsholtz, A. Geroldinger, S. Kubertin and L.

Rackham showed that $S(\mathbb{Z}_n^4) \geqslant 20n-19$ for all odd $n>1$, and equality holds when n is a power of 3. C. Elsholtz [Combinatorica 24(2004)] proved that for $n=3,5,\ldots$ and $k \geqslant 3$ we have

$$s(\mathbb{Z}_n^k) \geqslant 1.125^{\lfloor k/3 \rfloor} 2^k (n-1)+1;$$

on the other hand, Alon and Dubier [Combinatorica 15(1995)] showed that there is an absolute constant $M>0$ such that

$$s(\mathbb{Z}_n^k) \leqslant (Mk\log_2 k)^k n \quad \text{ for all } k,n \in \mathbb{Z}^+.$$

In 2005 A. Bialostocki posed the following interesting conjecture.

Bialostocki's Conjecture. *Let $n>0$ be an even integer. If $a_1,\ldots,a_n,b_1,\ldots,b_n \in \mathbb{Z}_n$ and $a_1+\cdots+a_n=b_1+\cdots+b_n=0$, then there is a permutation $\sigma \in S_n$ such that $\sum_{i=1}^n a_i b_{\sigma(i)}=0$.*

The following conjecture is very challenging.

Olson's Conjecture. *For any $k,n \in \mathbb{Z}^+$ we have $D(\mathbb{Z}_n^k)=1+k(n-1)$.*

The case $k=1$ is trivial. J. E. Olson [J. Number Theory 1(1969)] confirmed the conjecture in the case where n is a prime power as well as the case $k=2$. N. Alon, S. Friedland and G. Kalai [J. Combin. Theory Ser. B 37(1984)] mentioned the conjecture explicitly in their paper.