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On Generalized Central Trinomial Coefficients

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Abstract

For $b, c \in \mathbb{Z}$ and $n = 0, 1, \ldots$, we define $T_n(b, c)$ to be the coefficient of x^n in the expansion of $(x^2 + bx + c)^n$ and call it a generalized central trinomial coefficient. Since $T_n(2, 1) = \binom{2n}{n}$, and $T_n(1, 1)$ is the central trinomial coefficient T_n in combinatorics, we may view $T_n(b, c)$ $(n = 0, 1, \ldots)$ as a natural common extension of the central binomial coefficients and the central trinomial coefficients. In recent years the speaker found many surprising arithmetic properties of $T_n(b, c)$. In this talk we will tell the story of recent results and conjectures on p-adic congruences or series for $1/\pi$ involving $T_n(b, c)$.

Main References:

1. Zhi-Wei Sun, *Congruences involving generalized central trinomial coefficients*, Sci. Math. China 57(2014), 1375–1400.
2. Zhi-Wei Sun, *On sums related to central binomial and trinomial coefficients*, in: M. B. Nathanson (ed.), Combinatorial and Additive Number Theory: CANT 2011 and 2012, Springer Proc. in Math. & Stat., Vol. 101, Springer, New York, 2014, pp. 257-312.

2005, Lyon

During Jan.–March, 2005, I visited Prof. Jiang Zeng at Univ. of Lyon-I. Dr. Victor Junwei Guo was then a postdoctor there.

Dr. Guo told me his following conjecture:

Given $l, m \in \mathbb{N}$ one has

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l}\binom{2k}{k-2l+m} = \begin{cases} \binom{2m/3}{m/3}\binom{m/3}{l-m/3} & \text{if } 3 \mid m, \\ 0 & \text{otherwise;} \end{cases}$$

in other words,

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{l}\binom{2k}{k-2l+m} = [3 \mid m]\binom{l}{\lceil m/3\rceil}\binom{2\lceil m/3\rceil}{l},$$

where $\lceil\cdot\rceil$ is the ceiling function, and for an assertion A we adopt the notation

$$[A] = \begin{cases} 1 & \text{if } A \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

2005, California

During May 2005–May 2006, I visited Univ. of California at Irvine, my PhD student Hao Pan wrote that he had an idea to prove Guo's conjectural identity at I told him soon after I came back from Lyon. Later we wrote a joint paper to prove a further extension of the identity via the generating-function method:

Theorem (H. Pan and Z.-W. Sun) Provided that $l, m, n \in \mathbb{N}$, we have

$$\sum_{k=0}^{l}(-1)^{m-k}\binom{l}{k}\binom{m-k}{n}\binom{2k}{k-2l+m} = \sum_{k=0}^{l}\binom{l}{k}\binom{2k}{n}\binom{n-l}{m+n-3k-l}.$$

This theorem in the case $l = n$ yields Guo's conjectural identity.

2006, California

The paper joint with Hao Pan was submitted to *Discrete Math.* One of the referees suggests that we should find some applications of the proved identity. To meet this purpose, we deduced some congruences modulo primes from the identity.

Theorem (H. Pan and Z.-W. Sun) Let p be a prime and $d \in \{0, \ldots, p\}$. Then

$$\sum_{k=0}^{p-1} \binom{2k}{k+d} \equiv \left(\frac{p-d}{3}\right) \pmod{p},$$

where the Legendre symbol $(\frac{a}{3})$ coincides with the unique integer in $\{0, \pm 1\}$ satisfying $a \equiv (\frac{a}{3}) \pmod 3$. Also,

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k+d}}{k} \equiv \begin{cases} d^{-1}(-1+2(-1)^d+3[3 \mid p-d]) \pmod{p} & \text{if } d \neq 0, \\ -[p=3] \pmod{p} & \text{if } d=0. \end{cases}$$

2007, Nanjing

In 2007 Roberto Tauraso at Roma told me that he was trying to prove Adamchuk's conjecture that for any prime $p > 3$ we have

$$\sum_{k=1}^{p+(\frac{p+1}{3})}\binom{2k}{k} \equiv 0 \pmod{p^2}, \text{ i.e., } \sum_{k=0}^{p-1}\binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

But he met certain difficulty that he could not overcome. In August 2007, I successfully overcome the difficulty by using the closed form of the combinatorial sum $\sum_{k\equiv r \pmod 3}\binom{n}{k}$ first studied by E. Lehmer in 1938.

Joint work with Tauraso

Theorem (Z.-W. Sun and R. Tauraso [Int. J. Number Theory 7(2011)]). Let p be any prime and let a be any positive integer. For $d = 0, 1, \ldots, p^a - 1$, we have

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+d} \equiv \left(\frac{p^a-d}{3}\right) - p[p=3]\left(\frac{d}{3}\right) + 2p^a \sum_{0<k<d}\frac{(-1)^{k-1}}{k}\left(\frac{d-k}{3}\right) \pmod{p^2}.$$

In particular,

$$\sum_{k=0}^{p^a-1}\binom{2k}{k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p^a-1}\binom{2k}{k+1} \equiv \left(\frac{p^a-1}{3}\right) - p[p=3] \pmod{p^2}.$$

Some general supercongruences involving one binomial coefficient

Let p be an odd prime and let $m \in \mathbb{Z}$ with $m \not\equiv 0 \pmod{p}$. Using linear recurrent sequences, Z.-W. Sun determined

(1) $\sum_{k=0}^{p-1} \binom{2k}{k}/m^k \bmod p^2$ (Sci. China Math. 2010)

(2) $\sum_{k=0}^{(p-1)/2} \binom{2k}{k}/m^k \bmod p^2$ (Taiwan. J. Math. 2013)

(3) $\sum_{k=0}^{p-1} \binom{p-1}{k}\binom{2k}{k}/m^k \bmod p^2$ (Colloq. Math. 2012)

(4) $\sum_{k=0}^{p-1} \binom{3k}{k}/m^k \bmod p$ (preprint, 2009)

For example,

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{m^k} \equiv \left(\frac{m^2-4m}{p}\right) + u_{p-(\frac{m^2-4m}{p})} \pmod{p^2},$$

where $\{u_n\}_{n\geqslant 0}$ is the Lucas sequence given by

$$u_0 = 0,\ u_1 = 1,\ \text{and}\ u_{n+1} = (m-2)u_n - u_{n-1}\ (n = 1, 2, 3, \ldots).$$

In particular, $\sum_{k=0}^{p-1} \binom{2k}{k}/2^k \equiv (\frac{-1}{p}) \pmod{p^2}$.

On $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k$ modulo p^2

A Conjecture of Rodriguez-Villegas proved by E. Mortenson. If p is an odd prime, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)=(-1)^{(p-1)/2}\pmod{p^2}.$$

Remark. (a) By Stirling's formula,

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\quad\text{as } n\to+\infty.$$

It follows that

$$\binom{2k}{k}^2\sim\frac{16^k}{k\pi}.$$

(b) Mortenson'proof involves Gauss and Jacobi sums and the p-adic Gamma function. In fact, now there are elementary proofs.

An elementary proof of $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k \equiv (\frac{-1}{p}) \pmod{p^2}$

Let $p = 2n+1$ be a prime. As observed by van Hammer, for $k = 0, \ldots, n$ we have

$$\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{n}{k}\binom{-n-1}{k}$$
$$=\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k}$$
$$\equiv\binom{-1/2}{k}^2 = \left(\frac{\binom{2k}{k}}{(-4)^k}\right)^2 = \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}.$$

Thus Zhi-Hong Sun and Roberto Tauraso deduced that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k$$
$$=\sum_{k=0}^{n}\binom{-n-1}{k}\binom{n}{n-k} = \binom{-1}{n} = (-1)^n \pmod{p^2}.$$

On $\sum_{k=0}^{p-1}\binom{2k}{k}^2/m^k \bmod p^2$

Theorem (conjectured by Z. W. Sun [JNT, 2011] and proved by Z. H. Sun [Proc. AMS 2011]). If $p\equiv 1 \pmod 4$ is a prime and $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-1)/4}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}$$
$$\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

If $p\equiv 3 \pmod 4$ is a prime, then $\sum_{k=0}^{p-1}\binom{2k}{k}^2/32^k\equiv 0 \pmod{p^2}$.

Open Conjecture (Z. W. Sun [J. Number Theory 131(2011)]). Let $p\equiv 3 \pmod 4$ be a prime. Then

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}^2}{(-8)^k}\equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\pmod{p^3}.$$

A result of van Hamme and a conjecture of van Hamme

Theorem (van Hamme, 1997) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

This result was also conjectured by Rodriguez-Villegas in 2003 and re-proved by E. Mortenson via modular forms.

Motivated by the known identity

$$\sum_{k=0}^{\infty}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}=\frac{2}{\pi}$$

(which was first proved by G. Bauer in 1859 and rediscovered by S. Ramanujan in 1914), L. van Hamme formulated the following conjecture in 1997.

van Hamme's Conjecture (1997). For any odd prime p we have

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\equiv(-1)^{(p-1)/2}p \pmod{p^3}.$$

A refinement of van Hamme's Conjecture

The above conjecture of van Hamme was confirmed by E. Mortenson [Proc. Amer. Math. Soc. 136(2008)]. Later I got a further refinement of this congruence.

Theorem [Z.-W. Sun, Illinois J. Math. 56(2012)] For any odd prime p, we have

$$\begin{aligned}\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k} \equiv& \sum_{k=0}^{(p-1)/2}(4k+1)\frac{\binom{2k}{k}^3}{(-64)^k}\\ \equiv&(-1)^{(p-1)/2}p+p^3E_{p-3} \pmod{p^4},\end{aligned}$$

where $E_0, E_1, \ldots,$ are the Euler numbers defined by

$$E_0 = 1, \text{ and } \sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}E_{n-k} = 0 \ (n = 1, 2, 3, \ldots).$$

2010, India

I visited India during Jan.-Feb. 2010. On Jan. 23 I suddenly realized that I should combine the congruences for $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ and $\sum_{k=0}^{p-1}k\binom{2k}{k}^3/m^k$ mod p^2. After I found $\sum_{k=0}^{p-1}\binom{2k}{k}^3/4096^k$ mod p^2 and conjectured the congruence

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right)-p^3E_{p-3}\pmod{p^4},$$

I got to know that van Hamme had the conjecture

$$\sum_{k=0}^{p-1}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv 5p\left(\frac{-1}{p}\right)\pmod{p^3}$$

motivated by Ramanujan's identity

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi}.$$

Thus I became interested in Ramanujan-type series and wrote to several mathematicians to get Hamme's paper.

Ramanujan-type series for $1/\pi$

General forms of Ramanujan-type series:

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^3}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},$$

$$\sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k}, \quad \sum_{k=0}^{\infty}(ak+b)\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}.$$

There are totally 36 known Ramanujan-type series for $1/\pi$ with a, b, m rational.

D. V. Chudnovsky and G. V. Chudnovsky (1987):

$$\sum_{k=0}^{\infty}\frac{545140134k+13591409}{(-640320)^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}=\frac{3\times 53360^2}{2\pi\sqrt{10005}}.$$

Remark. This yielded the record for the calculation of π during 1989-1994.

My Philosophy about Series for $1/\pi$

Given a *regular* identity of the form

$$\sum_{k=0}^{\infty}(bk+c)\frac{a_k}{m^k}=\frac{C}{\pi},$$

where $a_k, b, c, m \in \mathbb{Z}$, bm is nonzero and C^2 is rational, we must have

$$\sum_{k=0}^{n-1}(bk+c)a_k m^{n-1-k}\equiv 0 \pmod{n}$$

for any positive integer n. Furthermore, there exist an integer m' and a squarefree positive integer d with the class number of $\mathbb{Q}(\sqrt{-d})$ in $\{1,2,2^2,2^3,\ldots\}$ (and with $C/\sqrt{d}$ often rational) such that either $d>1$ and for any prime $p>3$ not dividing dm we have

$$\sum_{k=0}^{p-1}\frac{a_k}{m^k}\equiv\begin{cases}(\frac{m'}{p})(x^2-2p) \pmod{p^2} & \text{if } 4p=x^2+dy^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-d}{p})=-1,\end{cases}$$

or $d=1$, $\gcd(15,m)>1$, and for any prime $p\equiv 3 \pmod{4}$ with $p\nmid 3m$ we have $\sum_{k=0}^{p-1}a_k/m^k\equiv 0 \pmod{p^2}$.

Central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](1+x+x^2)^n \text{ (the coefficient of } x^n \text{ in } (1+x+x^2)^n)$$

$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Motivated by the work of Sun and Tauraso on $\sum_{k=0}^{p-1}\binom{2k}{k}$ modulo p^2, H.-Q. Cao and H. Pan obtained the following result.

Theorem (Hui-Qin Cao and Hao Pan, 2010) For any prime $p > 3$,

$$\sum_{k=0}^{p-1}(-1)^k T_k \equiv p\frac{3(\frac{p}{3})-1}{2} \pmod{p^2}, \sum_{k=0}^{p-1}\frac{T_k}{3^k} \equiv p\frac{(\frac{p}{3})+1}{2} \pmod{p^2}.$$

Theorem (H. Q. Cao and Z.-W. Sun, 2010). For any prime $p > 3$,

$$T_{p-1} \equiv \left(\frac{p}{3}\right)3^{p-1} \pmod{p^2}.$$

2010, Xuzhou

In 2010 I attended the 4th National Conf. on Combin. and Graph Theory held at Xuzhou. During the conference, I obtained the following result.

Theorem (Z.-W. Sun, 2010) For any odd prime p, we have

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod{p}.$$

Those

$$D_n := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \ (n = 0, 1, 2, \ldots)$$

are called central Delannoy numbers. In combinatorics, D_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$.

Theorem (Z.-W. Sun, 2010) For any odd prime p we have

$$\sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p}.$$

Conjecture on central trinomial coefficients

Conjecture (Sun, 2010) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8k+5)T_k^2 \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{T_k H_k}{3^k} \equiv \frac{3+(\frac{p}{3})}{2} - p\left(1+\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

where H_k denotes the harmonic number $\sum_{0<j\leqslant k} 1/j$.

Mod p^2 congruences for Motzkin numbers

The nth Motzkin number

$$M_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

is the number of paths from $(0,0)$ to $(n,0)$ which never dip below the line $y = 0$ and are made up only of the allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$.

Conjecture (Sun, 2010). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} M_k T_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) + \frac{p}{6}\left(1 - 9\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Generalized central trinomial coefficients and generalized Motzkin numbers

Given $b, c \in \mathbb{Z}$, the *generalized central trinomial coefficients*

$$T_n(b,c) := [x^n](x^2+bx+c)^n = [x^0](b+x+cx^{-1})^n$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} b^{n-2k}c^k$$

and we introduce the *generalized Motzkin numbers*

$$M_n(b,c) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} \frac{b^{n-2k}c^k}{k+1}$$

$(n = 0, 1, 2, \ldots)$. Note that

$$T_n = T_n(1,1),\ M_n = M_n(1,1),\ T_n(2,1) = [x^n](x+1)^{2n} = \binom{2n}{n},$$

and

$$M_n(2,1) = \sum_{k=0}^{n} \binom{n}{2k} C_k 2^{n-2k} = C_{n+1}.$$

Generating functions

Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. H. S. Wilf observed that

$$\sum_{n=0}^{\infty} T_n(b,c)x^n = \frac{1}{\sqrt{1-2bx+dx^2}}$$

which implies the recursion

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c) \ \ (n \in \mathbb{Z}^+).$$

By the Zeilberger algorithm we have

$$(n+3)M_{n+1}(b,c) = (2n+3)bM_n(b,c) - ndM_{n-1}(b,c) \ \ (n \in \mathbb{Z}^+),$$

and hence

$$2cx^2 \sum_{n=0}^{\infty} M_n(b,c)x^n = 1 - bx - \sqrt{1-2bx+dx^2}.$$

Relations between $T_n(b,c)$ and Legendre polynomials

For the Legendre polynomials $P_n(t)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{t-1}{2}\right)^k$, it is known that

$$\sum_{n=0}^{\infty} P_n(t)x^n = \frac{1}{\sqrt{1-2tx+x^2}}.$$

Thus, if $d=b^2-4c\neq 0$ then

$$\sum_{n=0}^{\infty} T_n(b,c)\left(\frac{x}{\sqrt{d}}\right)^n = \frac{1}{\sqrt{1-2bx/\sqrt{d}+d(x/\sqrt{d})^2}} = \sum_{n=0}^{\infty} P_n\left(\frac{b}{\sqrt{d}}\right)x^n$$

and hence

$$T_n(b,c) = (\sqrt{d})^n P_n\left(\frac{b}{\sqrt{d}}\right).$$

It follows that

$$T_n(2x+1,x^2+x)=P_n(2x+1) \quad \text{for all } x\in\mathbb{Z};$$

in particular, $D_n=P_n(3)=T_n(3,2)$.

2010, Stanford Univ.

Theorem (Sun, 2010). Let b and c be integers.

(i) Let p be an odd prime not dividing $m \in \mathbb{Z}$. Then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)}{m^k} \equiv \left(\frac{(m-b)^2-4c}{p}\right) \pmod p$$

and

$$2c\sum_{k=0}^{p-1} \frac{M_k(b,c)}{m^k} \equiv (m-b)^2-((m-b)^2-4c)\left(\frac{(m-b)^2-4c}{p}\right) \pmod p.$$

(ii) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} \equiv 0 \pmod n$$

and

$$6\sum_{k=0}^{n-1} kT_k(b,c^2)(b-2c)^{n-1-k} \equiv 0 \pmod n.$$

Congruences modulo n

Theorem (Sun, 2010). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$.
(i) For any $n \in \mathbb{Z}^+$, we have

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} \equiv 0 \pmod{n},$$

and furthermore

$$b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} = nT_n(b,c)T_{n-1}(b,c).$$

(ii) Suppose that $b^2 - 4c = 1$, i.e., there is an $m \in \mathbb{Z}$ such that $b = 2m+1$, $c = m^2 + m$ and hence $T_k(b,c) = D_k(m)$. Then

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k(b,c) = \sum_{k=1}^{n}\binom{n}{k}\binom{n+k-1}{k-1}\left(\frac{b-1}{2}\right)^{k-1} \in \mathbb{Z}$$

for all $n \in \mathbb{Z}^+$.

On $\sum_{k=0}^{p-1} T_k(b,c)^2/m^k \bmod p$

Theorem (Sun, 2010) Let $b, c \in \mathbb{Z}$ with $d = b^2 - 4c$ and let p be an odd prime.

(i) If $p \nmid d$, then we have

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)^2}{d^k} \equiv \left(\frac{cd}{p}\right) \pmod{p}.$$

If $b \not\equiv 2c \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv \left(\frac{-c^2}{p}\right) \pmod{p}.$$

(ii) Assume $p \nmid c$. If $p \nmid d$, then
$\sum_{k=0}^{p-1} T_k(b,c)M_k(b,c)/d^k \equiv 0 \pmod{p}$.
If $D = b^2 - 4c^2 \not\equiv 0 \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2)M_k(b,c^2)}{(b-2c)^{2k}} \equiv \frac{4b}{b+2c}\left(\frac{D}{p}\right) \pmod{p}.$$

A Corollary

Define

$$D_n(x) = P_n(2x+1) = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k \ (n = 0, 1, 2, \ldots).$$

Note that $D_n(1)$ is the central Delannoy number D_n.

Since $D_k(x) = T_k(2x+1, x^2+x)$ and $(2x+1)^2 - 4(x^2+x) = 1$, we have

Corollary Let p be an odd prime. For any integer x we have

$$\sum_{k=0}^{p-1} D_k(x)^2 \equiv \left(\frac{x(x+1)}{p}\right) \pmod{p}.$$

In particular,

$$\sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p}.$$

Congruences modulo n

Conjecture (Sun, 2010) Let $b, c \in \mathbb{Z}$. For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8ck + 4c + b)T_k(b, c^2)^2(b - 2c)^{2(n-1-k)} \equiv 0 \pmod{n}.$$

If p is an odd prime not dividing $b(b - 2c)$, then

$$\sum_{k=0}^{p-1}(8ck + 4c + b)\frac{T_k(b, c^2)^2}{(b - 2c)^{2k}} \equiv p(b + 2c)\left(\frac{b^2 - 4c^2}{p}\right) \pmod{p^2}.$$

2011, Taiwan

Theorem (Sun, 2011). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)\,T_k(b,c)^2 d^{n-1-k} = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_k c^k d^{n-1-k}.$$

If c is nonzero and p is an odd prime not dividing d, then

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} \equiv 1 + \frac{b^2}{c}\cdot\frac{(\frac{d}{p})-1}{2} \pmod{p}.$$

Corollary. For each $n = 1, 2, 3, \ldots$ we have

$$\sum_{k=0}^{n-1}(2k+1)D_k^2 \equiv 0 \pmod{n^2}.$$

Congruences involving $D_k(x)^3$ or $D_k(x)^4$

Recall that

$$D_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k = P_n(2x+1).$$

Conjecture (Sun, 2010). Let x be any integer. If p is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1} (2k+1) D_k(x)^3 \equiv p \left(\frac{-4x-3}{p}\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1} (2k+1) D_k(x)^4 \equiv p \pmod{p^2}.$$

This was recently proved by Victor J. W. Guo [arXiv:1412.7724].

Conjectures involving $D_k(x)^3$

I also had conjectures on $\sum_{k=0}^{p-1}(-1)^k D_k(x)$ mod p^2 with $x = 2, -4, 1/2, -1/4, 1/8, -1/16$. Here is an example.

Conjecture [Z.-W. Sun, J. Number Theory 131(2011)] Let $p > 3$ be a prime. Then

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(\frac{1}{2}\right)^3$$

$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1, 7 \pmod{24} \text{ and } p = x^2 + 6y^2, \\ 8x^2 - 2p \pmod{p^2} & \text{if } p \equiv 5, 11 \pmod{24} \text{ and } p = 2x^2 + 3y^2, \\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p}) = -1. \end{cases}$$

A theorem on $\sum_{k=0}^{p-1} T_k(b,c)^2/m^k \bmod p^2$

Theorem (Sun, 2011) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{T_k(6,-3)^2}{48^k}\equiv\left(\frac{-1}{p}\right)+\frac{p^2}{3}E_{p-3}\pmod{p^3},$$
$$\sum_{k=0}^{p-1}\frac{T_k(2,-1)^2}{8^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{T_k(2,-3)^2}{16^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},$$

where $E_0,E_1,E_2,\ldots$ are Euler numbers.

A Lemma. Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. For any $n\in\mathbb{N}$ we have

$$T_n(b,c)^2=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2c^kd^{n-k}.$$

Conjectural congruences involving powers of $T_k(b,c)$

Conjecture (Sun, 2010). Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{4^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{36^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},$$

and

$$\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{8^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{(-64)^k}$$

$$\equiv\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{(-64)^k}\equiv\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{512^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\text{ and }p=x^2+6y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\text{ and }p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1.\end{cases}$$

Conjectural congruences involving powers of $T_k(b,c)$

Conjecture (Sun, 2011). Let $p>3$ be a prime. Then

$$\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{8^{3k}}\equiv\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{16^{3k}}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1 \pmod 4\ \&\ p=x^2+4y^2,\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4;\end{cases}$$

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{(-8)^{3k}}\equiv\left(\frac{6}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{12^{3k}}$$

$$\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,3 \pmod 8\ \&\ p=x^2+2y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-2}{p})=-1, \text{ i.e., } p\equiv 5,7 \pmod 8.\end{cases}$$

Conjectural congruences involving powers of $T_k(b, c)$

Also,

$$\sum_{k=0}^{p-1}(7k+4)\frac{T_k(10,49)^3}{(-8)^{3k}} \equiv \frac{p}{14}\left(\frac{2}{p}\right)\left(65-9\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(7k+3)\frac{T_k(10,49)^3}{12^{3k}} \equiv \frac{3p}{28}\left(13+15\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

For each $n = 1, 2, 3, \ldots$ we have

$$\sum_{k=0}^{n-1}(7k+4)\,T_k(10,49)^3(-8^3)^{n-1-k} \equiv 0 \pmod{4n},$$

$$\sum_{k=0}^{n-1}(7k+3)\,T_k(10,49)^3(12^3)^{n-1-k} \equiv 0 \pmod{n}.$$

Asymptotic Behavior of $T_n(b,c)$

By the Laplace-Heine formula, for $x \notin [-1,1]$ we have

$$P_n(x) \sim \frac{(x+\sqrt{x^2-1})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{x^2-1}} \quad \text{as } n \to +\infty.$$

It follows that if $b>0$ and $c>0$ then

$$T_n(b,c) \sim f_n(b,c) := \frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}.$$

as $n \to +\infty$. Note that $T_n(-b,c)=(-1)^n T_n(b,c)$.

Conjecture (Sun, 2011): For $b>0$ and $c>0$, we have

$$T_n(b,c) = f_n(b,c)\left(1+\frac{b-4\sqrt{c}}{16n\sqrt{c}}+O\left(\frac{1}{n^2}\right)\right)$$

as $n \to +\infty$. If $b \in \mathbb{R}$ and $c<0$, then

$$\lim_{n\to\infty} \sqrt[n]{|T_n(b,c)|} = \sqrt{b^2-4c}.$$

Remark. The conjecture was later proved by S. Wagner.

The birth of the first $1/\pi$-series involving $T_n(b,c)$

Conjecture (Z.-W. Sun, Dec., 2010) Let p be an odd prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-256)^k}T_k(1,16)$$

$$\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6\pmod 7.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2}{(-256)^k}T_k(1,16)\equiv 7p\left(\frac{-1}{p}\right)\pmod{p^2}.$$

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Conjecture (Z. W. Sun, Jan. 2 (2:00 am), 2011). We have

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2T_k(1,16)=\frac{24}{\pi}.$$

My conjectural series involving $T_k(b,c)$ for $1/\pi$

In Jan.-Feb. 2011, I introduced 40 series for $1/\pi$ of the following five types with a,b,c,d,m integers and $mbcd(b^2-4c)$ nonzero.

`Type I.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}^2T_k(b,c)/m^k$.

`Type II.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}\binom{3k}{k}T_k(b,c)/m^k$.

`Type III.` $\sum_{k=0}^\infty(a+dk)\binom{4k}{2k}\binom{2k}{k}T_k(b,c)/m^k$.

`Type IV.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}^2T_{2k}(b,c)/m^k$.

`Type V.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}\binom{3k}{k}T_{3k}(b,c)/m^k$.

In August I added 8 new series for $1/\pi$ of type III.

In October I found 10 conjectural series for $1/\pi$ of two new types:

`Type VI.` $\sum_{k=0}^\infty(a+dk)T_k^3(b,c)/m^k$.

`Type VII.` $\sum_{k=0}^\infty(a+dk)\binom{2k}{k}T_k^2(b,c)/m^k$.

Recall that a series $\sum_{k=0}^\infty a_k$ is said to *converge at a geometric rate with ratio r* if

$$\lim_{k\to+\infty}\frac{a_{k+1}}{a_k}=r\in(0,1).$$

My conjectural series of type I

$$\sum_{k=0}^{\infty}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2T_k(1,16)=\frac{24}{\pi},$$
$$\sum_{k=0}^{\infty}\frac{30k+7}{(-1024)^k}\binom{2k}{k}^2T_k(34,1)=\frac{12}{\pi},$$
$$\sum_{k=0}^{\infty}\frac{30k-1}{4096^k}\binom{2k}{k}^2T_k(194,1)=\frac{80}{\pi},$$
$$\sum_{k=0}^{\infty}\frac{42k+5}{4096^k}\binom{2k}{k}^2T_k(62,1)=\frac{16\sqrt{3}}{\pi}.$$

Remark. The first identity was found by me soon after I waked up

in the deep night of Jan. 1, 2011. This began my discovery of many new series for $1/\pi$. Note that $T_k(1,16)\sim\frac{3}{4}\cdot\frac{9^k}{\sqrt{k\pi}}$.

My conjectural series of type II

I have 12 conjectural series of type II. Here are five of them.

$$\sum_{k=0}^{\infty}\frac{15k+2}{972^k}\binom{2k}{k}\binom{3k}{k}T_k(18,6)=\frac{45\sqrt{3}}{4\pi},$$

$$\sum_{k=0}^{\infty}\frac{91k+12}{10^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(10,1)=\frac{75\sqrt{3}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{6930k+559}{102^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(102,1)=\frac{1445\sqrt{6}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{210k-7157}{198^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(287298,1)=\frac{114345\sqrt{3}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{63k+11}{(-13500)^k}\binom{2k}{k}\binom{3k}{k}T_k(40,1458)=\frac{25}{12\pi}(3\sqrt{3}+4\sqrt{6}).$$

Remark. The 4th series converges very slow (with geometric ratio 71825/71874), even 2000 terms could not contribute one digit. Prof. G. Almkvist wondered how I could find the identity.

Some of my conjectural series of type III

$$\sum_{k=0}^{\infty}\frac{85k+2}{66^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(52,1)=\frac{33\sqrt{33}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{28k+5}{(-96^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(110,1)=\frac{3\sqrt{6}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{3080k-58871}{39216^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(23990402,1)=\frac{17974\sqrt{2451}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{80k+9}{264^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(257,256)=\frac{11\sqrt{66}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi}.$$

Remark. Some mathematicians (including my twin brother Z. H. Sun) wondered how I could find the last identity involving $14\sqrt{210}+21\sqrt{42}$.

My conjectural series of type IV

I have 18 conjectural series of type IV. Here are five of them.

$$\sum_{k=0}^{\infty}\frac{340k+59}{(-480^2)^k}\binom{2k}{k}^2T_{2k}(62,1)=\frac{120}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{13940k+1559}{(-5760^2)^k}\binom{2k}{k}^2T_{2k}(322,1)=\frac{4320}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{14280k+899}{39200^{2k}}\binom{2k}{k}^2T_{2k}(198,1)=\frac{1155\sqrt{6}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{57720k+3967}{439280^{2k}}\binom{2k}{k}^2T_{2k}(5778,1)=\frac{2890\sqrt{19}}{\pi},$$

$$\sum_{k=0}^{\infty}\frac{1615k-314}{243360^{2k}}\binom{2k}{k}^2T_{2k}(54758,1)=\frac{1989\sqrt{95}}{4\pi}.$$

Remark. I conjectured that my list of the 18 series of type IV with $c=1$ is complete! Prof. G. Almkvist asked me why I thought so.

My conjectural series of type V

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}.$$

$u_k=\binom{3k}{k}T_{3k}(62,1)$ satisfies a very complicated recursion:

$$\begin{aligned}
&(n+2)^2(2n+1)(2n+3)(8652n^2+11536n+3525)u_{n+2}\\
=&372(2n+1)(6n+7)(25021584n^4+116767392n^3\\
&+188134216n^2+121113048n+25958565)u_{n+1}\\
&-127401984000(3n+1)^2(3n+2)^2(8652n^2+28840n+23713)u_n\\
&-9(n+2)P(n)62^n\binom{2n+2}{n}\binom{3n+2}{n}\binom{3n+2}{2n},
\end{aligned}$$

where

$$\begin{aligned}
P(n):=&31420906020n^5+136307337012n^4+127456779135n^3\\
&-126369328953n^2-174985958380n+705000.
\end{aligned}$$

Latest Progress

Quite recently, some important progress on my conjectural series for $1/\pi$ of types I-V was made by H. H. Chan, J. Wan and W. Zudilin, see the papers

1. H. H. Chan, J. Wan and W. Zudilin, *Legendre polynomials and Ramanujan-type series for* $1/\pi$, Israel J. Math. 194(2013), 183–207.

2. J. Wan and W. Zudilin, *Generating functions of Legendre polynomials: a tribute to Fred Brafman*, J. Approx. Theory 164(2012), 488–503.

Their work depends heavily on Brafman's identity [Proc. AMS 2(1951), 942-949] and its extensions.

Brafman's Identity. We have

$$\sum_{n=0}^{\infty}\frac{(s)_n(1-s)_n}{n!^2}P_n(x)z^n$$
$$={}_2F_1\left(s,1-s;1\ \middle|\ \frac{1-\rho-z}{2}\right){}_2F_1\left(s,1-s;1\ \middle|\ \frac{1-\rho+z}{2}\right),$$

Comments

The above two papers contain detailed proofs of 7 conjectural series of mine, but I cannot find all the details for proofs of other conjectural series of type I-V In my opinion, the crucial parts related to modular equations of higher degrees in such proofs should be in detail so that anyone can check the proofs if he (or she) wishes.

In October 2011 I introduced conjectural series of types VI and VII which could not be proved by using Brafman's identity.

Besides the 61 series involving $T_k(b, c)$, I have totally 178 conjectural series for $1/\pi$. For the full list, see my article *List of conjectural series for powers of* π *and other constants* `http://arxiv.org/abs/1102.5649`

All my conjectural series come from **combinations of philosophy, intuition, inspiration, experience and computation!**

My conjectural series of type VI

$$\sum_{k=0}^{\infty}\frac{66k+17}{(2^{11}3^3)^k}T_k^3(10,11^2)=\frac{540\sqrt{2}}{11\pi},$$

$$\sum_{k=0}^{\infty}\frac{126k+31}{(-80)^{3k}}T_k^3(22,21^2)=\frac{880\sqrt{5}}{21\pi},$$

$$\sum_{k=0}^{\infty}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)=\frac{432}{95\pi}(195\sqrt{14}+94\sqrt{2}).$$

I would like to offer \$300 as the prize for the person who can provide first rigorous proofs of all the above three identities. The last one was inspired by my following conjecture for primes $p>3$.

$$\sum_{k=0}^{p-1}\frac{3990k+1147}{(-288)^{3k}}T_k^3(62,95^2)$$
$$\equiv\frac{p}{19}\left(17563\left(\frac{-14}{p}\right)+4230\left(\frac{-2}{p}\right)\right)\pmod{p^2}.$$

My conjectural series of type VII

I have 7 conjectural series of type VII, here are five of them.

$$\sum_{k=0}^{\infty}\frac{221k+28}{450^k}\binom{2k}{k}T_k^2(6,2)=\frac{2700}{7\pi},$$

$$\sum_{k=0}^{\infty}\frac{24k+5}{28^{2k}}\binom{2k}{k}T_k^2(4,9)=\frac{49}{9\pi}(\sqrt{3}+\sqrt{6}),$$

$$\sum_{k=0}^{\infty}\frac{3696k+445}{46^{2k}}\binom{2k}{k}T_k^2(7,1)=\frac{1587\sqrt{7}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{450296k+53323}{(-5177196)^k}\binom{2k}{k}T_k^2(171,-171)=\frac{113535\sqrt{7}}{2\pi},$$

$$\sum_{k=0}^{\infty}\frac{2800512k+435257}{434^{2k}}\binom{2k}{k}T_k^2(73,576)=\frac{10406669}{2\sqrt{6}\,\pi}.$$

Remark. W. Zudilin [Bull. Austral. Math. Soc., 89(2014)] proved some of these identities with the 2nd and the 5th formulas remain open.

Final words on series

I have totally 234 conjectural series for $1/\pi$ or other important constants. For the full list and related references, see my article

List of conjectural series for powers of π *and other constants*

`http://arxiv.org/abs/1102.5649`

Thank you!