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ON REPRESENTATIONS OF INTEGERS INVOLVING TRIANGULAR NUMBERS

Zhi-Wei Sun

Department of Mathematics
Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Abstract. Triangular numbers are integers of the form $T_x = x(x+1)/2$ ($x \in \mathbb{Z}$). We will talk about some recent problems and results on representations of integers involving triangular numbers. In particular, we will introduce the recent work of B. Kane and the speaker on mixed sums of squares and triangular numbers, and the speaker's conjectures on sums of primes and triangular numbers.

1. Universal mixed sums of squares and triangular numbers

As $\sum_{k=0}^n k = n(n+1)/2$, those

$$T_x = \frac{x(x+1)}{2} \qquad (x \in \mathbb{Z})$$

are called *triangular numbers*. Clearly

$$T_{-x} = T_{x-1} \quad \text{and} \quad 8T_x + 1 = (2x+1)^2.$$

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ we have the following known results.

(i) (Lagrange's theorem) $n = w^2 + x^2 + y^2 + z^2$ for some $w, x, y, z \in \mathbb{Z}$.

(ii) (Gauss-Legendre Theorem) $n = x^2 + y^2 + z^2$ for some $x, y, z \in \mathbb{Z}$ if and only if n is not of the form $4^k(8l + 7)$ with $k, l \in \mathbb{N}$.

(iii) (Fermat) $n = T_x + T_y + T_z$ for some $x, y, z \in \mathbb{Z}$, equivalently $8n+3$ is a sum of three squares of (odd) integers.

(iv) (Euler) $n = x^2 + y^2 + T_z$ for some $x, y, z \in \mathbb{Z}$. In fact, $8n + 1 = (2x)^2 + (2y)^2 + (2z + 1)^2$ for some $x, y, z \in \mathbb{Z}$ with $x \equiv y \pmod 2$; this yields the representation

$$n = \frac{x^2 + y^2}{2} + T_z = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + T_z$$

(v) (E. Lionnet, V. A. Lebesgue and M. S. Réalis, 1872) $n = x^2 + T_y + T_z$ for some $x, y, z \in \mathbb{Z}$.

(vi) (B. W. Jones and G. Pall [Acta Math., 1939]) We can write $8n+1$ in the form $8x^2 + 32y^2 + z^2$ with $x, y, z \in \mathbb{Z}$, i.e., n is a sum of a square, an *even* square and a triangular number.

(vii) (Z. W. Sun [Acta Arith., 2007]) n is a sum of an *even* square and two triangular numbers. If $n \neq 2T_m$ for any $m \in \mathbb{N}$, then n is also a sum of an odd square and two triangular numbers.

(viii) (Z. W. Sun [Acta Arith., 2007]) If n is not a triangular number, then it is a sum of an *odd* square, an *even* square and a triangular number.

(ix) (B.-K. Oh and Z. W. Sun, `arxiv:0804.3750`) Suppose $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then n is a sum of a square, an *odd* square and a triangular number. n cannot be written as a sum of an odd square and two triangular numbers if and only if it is of the form $2T_m$ ($m \in \mathbb{Z}^+$) with $2m+1$ having no prime divisor congruent to 3 mod 4.

(x) (B.-K. Oh and Z. W. Sun, `arxiv:0804.3750`) $p = 2n+1$ is a prime congruent to 3 modulo 4 if and only if T_n cannot be expressed as a sum of two odd squares and a triangular number, i.e., $p^2 = x^2 + 8(y^2 + z^2)$ for no odd integers x, y, z.

In the proofs of the results (vii) and (viii), Sun made use of Jacobi's triple product identity

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+aq^{2n-1})(1+a^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} a^n q^{n^2} \quad (|q|<1)$$

and its three by-products:

$$\varphi(-q) = \prod_{n=1}^{\infty}(1-q^{2n-1})^2(1-q^{2n}) \quad \text{(Gauss)},$$
$$\psi(q) = \prod_{n=1}^{\infty}\frac{1-q^{2n}}{1-q^{2n-1}} \quad \text{(Gauss)},$$
$$\prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)q^{T_n} \quad \text{(Jacobi)},$$

where

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{T_n}.$$

In the proofs of (ix) and (x), the following lemma plays an important role.

Hurwitz's Lemma. *Let $n > 0$ be an odd integer, and let $p_1, \ldots, p_r$ be all the distinct prime divisors of n congruent to 3 mod 4. Write $n = n_0 \prod_{0<i\leqslant r} p_i^{\alpha_i}$, where $n_0, \alpha_1, \ldots, \alpha_r$ are positive integers and n_0 has no prime divisors congruent to 3 mod 4. Then*

$$|\{(x,y,z)\in\mathbb{Z}^3:\ x^2+y^2+z^2=n^2\}| = 6n_0 \prod_{0<i\leqslant r}\left(p_i^{\alpha_i} + 2\frac{p_i^{\alpha_i}-1}{p_i-1}\right).$$

Let $a,b,c\in\mathbb{Z}^+$, and let $f(x,y,z)$ be any of the following three forms:

$$ax^2+by^2+cT_z,\ ax^2+bT_y+cT_z,\ aT_x+bT_y+cT_z.$$

We call f *universal* if each $n\in\mathbb{N}$ can be written in the form $f(x,y,z)$ with $x,y,z\in\mathbb{Z}$.

Liouville's Theorem. *Let $a,b,c\in\mathbb{Z}^+$ be positive integers with $a\leqslant b\leqslant c$. Then $aT_x+bT_y+cT_z$ is universal if and only if (a,b,c) is among the following vectors:*

$$(1,1,1),\ (1,1,2),\ \ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4).$$

It is known that for any $a,b,c\in\mathbb{Z}^+$ there are infinitely many positive integers not of the form $ax^2+by^2+cz^2$ with $x,y,z\in\mathbb{Z}$.

In 2005 Z. W. Sun initiated the determination of universal forms of the type $ax^2+by^2+cT_z$ or $ax^2+bT_y+cT_z$ with $a,b,c\in\mathbb{Z}^+$.

Theorem 1.3 [Z. W. Sun, Acta Arith. 127(2007), 103-113]**.**

(1) *Let a,b,c be positive integers with $a\leqslant b$. If $ax^2+by^2+cT_z$ is universal, then (a,b,c) is among the following vectors:*

$$(1,1,1),\ (1,1,2),\ (1,2,1),\ (1,2,2),\ (1,2,4),$$
$$(1,3,1),\ (1,4,1),\ (1,4,2),\ (1,8,1),\ (2,2,1).$$

(2) *Let a,b,c be positive integers with $b\geqslant c$. If $ax^2+bT_y+cT_z$ is universal, then (a,b,c) is among the following vectors:*

$$(1,1,1),\ (1,2,1),\ (1,2,2),\ (1,3,1),\ (1,4,1),\ (1,4,2),\ (1,5,2),$$
$$(1,6,1),\ (1,8,1),\ (2,1,1),\ (2,2,1),\ (2,4,1),\ (3,2,1),\ (4,1,1),\ (4,2,1).$$

(3) *The converses of (i) and (ii) can be reduced to the following conjecture:*

$$x^2 + 8T_y + T_z,\ x^2 + 3y^2 + T_z,\ x^2 + 3T_y + T_z,$$
$$x^2 + 6T_y + T_z,\ 3x^2 + 2T_y + T_z,\ 4x^2 + 2T_y + T_z$$

are all universal.

In 2007 S. Guo, H. Pan and Z. W. Sun [Integers 7(2007)] proved that the last 5 forms in Theorem 1.3(iii) are indeed universal; the following identity of Jacobi plays an important role in the proofs:

$$3(x^2 + y^2 + z^2) = (x + y + z)^2 + 2\left(\frac{x + y - 2z}{2}\right)^2 + 6\left(\frac{x - y}{2}\right)^2.$$

In 2008 B. K. Oh and Z. W. Sun showed that $x^2+8T_y+T_z$ is also universal. Thus the conjecture in Theorem 1.3(iii) is true and the determination of universal forms of the type $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$ has been completed.

2. On sums of primes and triangular numbers

Let us restate Sun's result (viii) in Section 1 as follows:

Each $n \in \mathbb{N}$ can be written in the form $p + T_z$ with $z \in \mathbb{Z}$ where p is either zero or a sum of an odd square and an even square.

Recall that any prime $p \equiv 1 \pmod 4$ is a sum of an odd square and an even square. So I was led to make the following conjecture in March 2008.

Conjecture 2.1 (Z. W. Sun, March 2008)**.**

(i) **Each natural number $n \neq 216$ can be written in the form $p + T_x$, where p is zero or a prime.**

(ii) *In general, for any $a, b \in \mathbb{N}$ and $r \in \mathbb{Z}$ with $\gcd(r, 2^b) = 1$, all sufficiently large integers can be written in the form $2^a p + T_x$ with $x \in \mathbb{Z}$, where p is either zero or a prime congruent to r mod 2^b.*

A key motivation for the second part of Conjecture 2.1 is that $\{T_x : x \in \mathbb{Z}\}$ contains a complete system of residues modulo any powers of two. I have shown that neither 2^a nor 2^b in Conjecture 2.1(ii) can be replaced by a positive integer which is not a power of two.

For $a \in \mathbb{N}$ we define $f(a)$ to be the largest integer not in the form $2^a p + T_x$, where p is zero or a prime, and x is an integer. Conjecture 2.1 and related computations suggest that

$$f(0) = 216,\ f(1) = 43473,\ f(2) = 849591 \text{ and } f(3) = 3527370.$$

Concerning the particular case $a = 0$ and $b \in \{2, 3\}$ of Conjecture 2.1, we have a more concrete conjecture which has been verified for $n \leqslant 2,000,000$:

(i) Each natural number $n > 88956$ can be written in the form $p + T_x$ with $x \in \mathbb{Z}^+$, where p is either zero or a prime congruent to 1 mod 4. Each natural number $n > 90441$ can be written in the form $p + T_x$ with $x \in \mathbb{Z}^+$, where p is either zero or a prime congruent to 3 mod 4.

(ii) For $r \in \{1, 3, 5, 7\}$, we can write any integer $n > N_r$ in the form $p + T_x$ with $x \in \mathbb{Z}$, where p is either zero or a prime congruent to r mod

8, and

$$N_1 = 1004160, \quad N_3 = 1142625, \quad N_5 = 779646, \quad N_7 = 893250.$$

In March 2008, I verified Conjecture 2.1(i) for $n \leqslant 17,000,000$ and then T. Noe continued the verification for $n \leqslant 2 \times 10^9$. It is interesting to compare Conjecture 2.1 with the Goldbach conjecture. Note that

$$|\{p \leqslant x:\ p \text{ is a prime}\}| \sim \frac{x}{\log x} \quad \text{and} \quad |\{T_n \leqslant x:\ n \in \mathbb{N}\}| \sim \sqrt{2x}.$$

The following result of Linnik (1960) is also remarkable: Any sufficiently large integer can be written as a sum of a prime and two squares (of integers).

Now we are in the 21st century, *not* the time of Fermat or Goldbach. In general, it seems quite difficult to formulate a new and simple conjecture concerning representations involving primes. So, when I formulated Conjecture 2.1 in March 2008, I felt very happy and excited!

Each prime $p \equiv 1 \pmod 4$ can be written in the form $x^2 + y^2$ with x even and y odd. Thus Conjecture 2.1 implies that for any $a = 0, 1, 2, \ldots$ all sufficiently large integers have the form $2^a(x^2 + y^2) + T_z$ with $x, y, z \in \mathbb{Z}$. If $p = x^2 + y^2$ with x even and y odd, then $2p = (x + y)^2 + (x - y)^2$ with $x \pm y$ odd. Thus our following conjecture is reasonable in view of Conjecture 2.1.

Conjecture 2.2 (Z. W. Sun, April 2008)**.** (i) *A natural number can be written as a sum of two* **even** *squares and a triangular number unless it* *is among the following list of* 19 *exceptions:*

$$2,\ 12,\ 13,\ 24,\ 27,\ 34,\ 54,\ 84,\ 112,\ 133,$$

$$162,\ 234,\ 237,\ 279,\ 342,\ 399,\ 652,\ 834,\ 864.$$

(ii) *Each natural number $n \notin E$ is either a triangular number, or a sum of a triangular number and two* **odd** *squares, where the exceptional set E consists of the following* 25 *numbers:*

$$4,\ 7,\ 9,\ 14,\ 22,\ 42,\ 43,\ 48,\ 52,\ 67,\ 69,\ 72,\ 87,\ 114,$$

$$144,\ 157,\ 159,\ 169,\ 357,\ 402,\ 489,\ 507,\ 939,\ 952,\ 1029.$$

We have verified Conjecture 2.2 for $n \leqslant 2 \times 10^6$.

As a complement to the Goldbach conjecture for *even* numbers, in 1894 E. Lemoine conjectured that any odd integer greater than 5 can be written in the form $p + 2q$ where p and q are primes.

I have also made a new conjecture for *odd* numbers.

Conjecture 2.3 (Z. W. Sun, May 2008)**.** (i) **Any odd integer** $n > 1$ **can be written in the form** $p + x(x+1)$ **with** p **a prime and and** x **an integer.**

(ii) *In general, for any $b \in \mathbb{N}$ and $r \in \{1, 3, 5, \dots\}$ all sufficiently large odd integers can be written in the form $p + x(x+1)$ with $x \in \mathbb{Z}$, where p is a prime congruent to r mod 2^b.*

Concerning Conjecture 2.3 in the cases $b = 2, 3$ we have the following concrete conjecture which has been verified for odd integers not exceeding 10^6.

(i) Let $n > 1$ be an odd integer. Then n can be written in the form $p + x(x + 1)$ with p a prime congruent to 1 mod 4 and x an integer, if and only if n is not among the following 30 **multiples of three**:

3, 9, 21, 27, 45, 51, 87, 105, 135, 141,

189, 225, 273, 321, 327, 471, 525, 627, 741, 861,

975, 1197, 1461, 1557, 1785, 2151, 12285, 13575, 20997, 49755.

Also, n can be written in the form $p + x(x + 1)$ with p a prime congruent to 3 mod 4 and x an integer, if and only if n is not among the following 15 **multiples of three**:

57, 111, 297, 357, 429, 615, 723, 765,

1185, 1407, 2925, 3597, 4857, 5385, 5397.

(ii) For each $r \in \{1, 3, 5, 7\}$, any odd integer $n > M_r$ can be written in the form $p + x(x + 1)$ with p a prime congruent to r mod 8 and x an integer, where

$$M_1 = 358245, \ M_3 = 172995, \ M_5 = 359907, \ M_7 = 444045.$$

Conjectures 2.1-2.3 has been made public via messages to the Number Theory Mailing List sent in March-May, 2008.

Let $m > 1$ be an integer, and let $a \in \mathbb{N}$ with $(2^a, m) = 1$. We define

$$S_m^{(a)} = \{n > m : \ (m, n) = 1 \ \& \ n \neq 2^a p + mT_x \text{ for any prime } p \text{ and integer } x\}$$

and simply write S_m for $S_m^{(0)}$. Clearly

$$S_m^{(a)} = \bigcup_{\substack{1\leqslant r\leqslant m\\ (r,m)=1}} \{r+mn:\ n\in S_m^{(a)}(r)\},$$

where

$$S_m^{(a)}(r) = \left\{n\in\mathbb{Z}^+:\ n\neq \frac{2^a p-r}{m}+T_x \text{ for any prime } p \text{ and integer } x\right\}.$$

(We also abbreviate $S_m^{(0)}(r)$ to $S_m(r)$.) What can we say about these exceptional sets? Are they finite?

A Result of Z. W. Sun (Sun, 2008)**.** (i) *Let $m>1$ be an odd integer, and let $a\in\mathbb{N}$. If r is a positive integer such that $2r$ is a quadratic residue modulo m, then there are infinitely many positive integers not of the form $(2^a p-r)/m^2+T_x$, where p is a prime and x is an integer. Therefore the set $S_{m^2}^{(a)}$ is infinite.*

(ii) *Let $m=2^\alpha m_0$ be a positive even integer with $\alpha, m_0\in\mathbb{Z}^+$ and $2\nmid m_0$. If $r\in\mathbb{Z}^+$ is a quadratic residue modulo m_0 with*

$$r\equiv 2^\alpha+1 \pmod{2^{\min\{\alpha+1,3\}}},$$

then there are infinitely many positive integers not of the form $(p-r)/(2m^2)+T_x$, where p is a prime and x is an integer. Thus S_{2m^2} is an infinite set.

In view of the above result and some computational results, Sun raised the following conjecture.

Conjecture 2.4 (Z. W. Sun, 2008). *Let $m > 1$ be an integer.*

(i) *Assume that m is odd. If m is not a square, then S_m, $S_m^{(1)}$, $S_m^{(2)}, \ldots$ are all finite. If $m = m_0^2$ with $m_0 \in \mathbb{Z}^+$, and r is a positive integer with $(r, m) = 1$ such that $2r$ is a quadratic non-residue mod m_0, then $S_m^{(a)}(r)$ is finite for every $a = 0, 1, 2, \ldots$.*

(ii) *Suppose that m is even. If m is not twice an even square, then the set S_m is finite. If $m = 2(2^\alpha m_0)^2$ with $\alpha, m_0 \in \mathbb{Z}^+$ and $2 \nmid m_0$, and r is a positive integer with $(r, m) = 1$ such that r is a quadratic non-residue modulo m_0 or $r \not\equiv 2^\alpha + 1 \pmod{2^{\min\{\alpha+1,3\}}}$, then the set $S_m(r)$ is finite.*

Example 2.1. (i) Among $1, \ldots, 15$ only 1 and 4 are quadratic residues modulo 15. For any $a \in \mathbb{N}$, both $S_{15^2}^{(a)}(2)$ and $S_{15^2}^{(a)}(8)$ are infinite by Sun's result, while

$$S_{225}^{(a)}(1),\ S_{225}^{(a)}(4),\ S_{225}^{(a)}(7),\ S_{225}^{(a)}(11),\ S_{225}^{(a)}(13),\ S_{225}^{(a)}(14)$$

should be finite as predicted by Conjecture 1.3(i).

(ii) Let r be a positive odd integer. By Sun's result, $S_{2\times 8^2}(r)$ is infinite if $r \equiv 1 \pmod 8$. On the other hand, by Conjecture 2.4, $S_{2\times 8^2}(r)$ should be finite when $r \not\equiv 1 \pmod 8$. When $(r, 18) = 1$, the set $S_{2\times 18^2}(r)$ is infinite if $r \equiv 7 \pmod{12}$ (i.e., r is quadratic residue mod 3^2 with $r \equiv 3 \pmod 4$) by the result of Sun, and it is finite otherwise by Conjecture 2.4(ii)). Similarly, when $(r, 20) = 1$, the set $S_{2\times 20^2}(r)$ is infinite if $r \equiv 21, 29 \pmod{40}$ (i.e., r is quadratic residue mod 5 with $r \equiv 5 \pmod 8$)) (by Sun's result), and it is finite otherwise (by Conjecture 2.4(ii)).

In view of Conjecture 2.4(ii), the sets S_2, S_6, S_{12} and $S_{288}(19)$ should be finite. Our computations up to 10^6 suggest further that

$$S_2 = S_6 = S_{12} = S_{288}(19) = \emptyset.$$

Recall that

$$S_2 = \{2n+1 :\ n \in \mathbb{Z}^+ \text{ and } 2n+1 \neq p+2T_x \text{ for any prime } p \text{ and integer } x\}.$$

So $S_2 = \emptyset$ is an equivalent version of Conjecture 2.3(i).

Conjectures in this section, together with some related results, are contained in my preprint *"On sums of primes and triangular numbers"* (`arXiv:0803:3737`), which will appear in the new journal *Journal of Combinatorics and Number Theory.*

3. On almost universal mixed sums of squares and triangular numbers

In 1917 S. Ramanujan found all those $a, b, c, d \in \mathbb{Z}^+$ such that every natural number can be represented by $ax^2+by^2+cz^2+dw^2$ with $x, y, z, w \in \mathbb{Z}$. He also asked for determining all those $a, b, c, d \in \mathbb{Z}^+$ such that $ax^2 + by^2 + cz^2 + dw^2$ represents all sufficiently large integers; this problem was essentially solved by H. D. Kloosterman [Acta Math. 49(1926)] with help of the useful Kloosterman sum, this represents a major breakthrough in the field of quadratic forms.

For ternary quadratic forms, things become more sophisticated and very challenging. It is known that for any $a, b, c \in \mathbb{Z}^+$ the set $\{ax^2 + by^2 + cz^2 :$

$x, y, z \in \mathbb{Z}\}$ cannot have asymptotic density 1. Ramanujan ever wrote that the positive odd integers not of the form $x^2 + y^2 + 10z^2$ are

3, 7, 21, 31, 33, 43, 67, 79, 87, 133, 217, 219, 223, 253, 307, 391.

Note that

$$\begin{aligned} & 2n+1 = x^2 + y^2 + 10z^2 \text{ for some } x, y, z \in \mathbb{Z} \\ \iff & 2n+1 = (2x)^2 + 10y^2 + (2z+1)^2 \text{ for some } x, y, z \in \mathbb{Z} \\ \iff & n = 2x^2 + 5y^2 + 4T_z \text{ for some } x, y, z \in \mathbb{Z}. \end{aligned}$$

W. Duke and R. Schulze-Pillot [Invent. Math. 99(1990)] were able to show that sufficiently large odd integers can be written in the form $x^2+y^2+10z^2$, equivalently all sufficiently large integers can be represented by the form $2x^2+5y^2+4T_z$. Ken Ono and K. Soundararajan [Invent. Math. 130(1997)] showed that the generalized Riemann hypothesis implies that the only positive odd integers not in the form $x^2 + y^2 + 10z^2$ are those listed by Ramanujan together with 679 and 2719, in other words those natural numbers not in the form $2x^2 + 5y^2 + 4T_z$ are as follows:

1, 3, 10, 15, 16, 21, 33, 39, 43, 66, 108, 109, 111, 126, 153, 195, 339, 1359.

Conjecture 3.1 (Z. W. Sun [Acta Arith. 2007]). *Every $n \in \mathbb{N}$ can be written in the form $x^2 + 2y^2 + 3T_z$ (with $x, y, z \in \mathbb{Z}$) except $n = 23$, in the form $x^2 + 5y^2 + 2T_z$ (or the equivalent form $5x^2 + T_y + T_z$) except $n = 19$, in the form $x^2 + 6y^2 + T_z$ except $n = 47$, and in the form $2x^2 + 4y^2 + T_z$ except $n = 20$.*

The speaker has verified the conjecture for $n \leqslant 10,000$.

Recall that any prime $p \equiv 1 \pmod 4$ can be written in the form x^2+y^2 with $x, y \in \mathbb{Z}$, and any prime $p \equiv 1, 3 \pmod 8$ can be written in the form x^2+2y^2 with $x, y \in \mathbb{Z}$. Let $m, n \in \mathbb{N}$ with $n-m=2k+\delta$ and $\delta \in \{0,1\}$. If Conjecture 2.1 holds, then sufficiently large integers can be written in the form

$$2^{m+2k}(x^2+2^\delta y^2)+T_z = 2^m(2^k x)^2 + 2^n y^2 + T_z.$$

This, together with some computation, led me to make the following conjecture.

Conjecture 3.2 (Z. W. Sun, April 2008). *Let $m, n \in \mathbb{N}$. Then sufficiently large integers can be written in any of the following forms:*

$$2^m x^2 + 2^n y^2 + T_z,\ x^2 + 2^n 3y^2 + T_z,\ x^2 + 2^n 3T_y + T_z,\ 2^n 3x^2 + 2T_y + T_z,$$

$$2^m x^2 + 2^n T_y + T_z,\ 2^m T_x + 2^n T_y + T_z,\ 2^n 3T_x + 2T_y + T_z,\ 2^n 5T_x + T_y + T_z.$$

A Result of B. Kane. (J. Combin. Number Theory, to appear)

(i) *Conjecture 3.1 holds under the generalized Riemann hypothesis. Without GRH, sufficiently large integers can be written in any of the forms $x^2+2y^2+3T_z$, $x^2+5y^2+2T_z$, $5x^2+T_y+T_z$, $x^2+6y^2+T_z$, $2x^2+4y^2+T_z$.*

(ii) *Conjecture 3.2 is valid for the first four forms, but there are counterexamples for the remaining four forms.*

Kane's proof involves many deep tools such as modular forms, class numbers and L-functions.

Let $a,b,c\in\mathbb{Z}^+$. For $f(x,y,z)=ax^2+by^2+cT_z$ or $ax^2+bT_y+cT_z$ or $aT_x+bT_y+cT_z$, we define the exceptional set

$$E(f)=\{n\in\mathbb{N}:\ f(x,y,z)=n \text{ for no } x,y,z\in\mathbb{Z}\}.$$

If $E(f)$ is finite, then we call f *almost universal.* If $E(f)$ has asymptotic density zero, then we say that f is *asymptotically universal.*

Any positive integer a can be uniquely written in the form $2^{v_2(a)}a'$ with $v_2(a)\in\mathbb{N}$ and $2\nmid a'$; $v_2(a)$ is called the 2-adic order of a and a' is called the odd part of a.

For $a\in\mathbb{Z}$ and $m\in\mathbb{Z}^+$, by $a\ R\ m$ we mean that a is quadratic residue modulo m, i.e., a is relatively prime to m and the equation $x^2\equiv a\pmod m$ is solvable over $\mathbb{Z}$. For an integer a and a positive odd integer m, it is well known that $a\ R\ m$ if and only if the Legendre symbol $(\frac{a}{p})$ equals 1 for every prime divisor p of m.

In a recent preprint `arXiv:0808.2761`, B. Kane and Z. W. Sun obtained the following theorem on asymptotically universal forms.

Theorem 3.1 B. Kane and Z. W. Sun, 2008)**.** *Let $a,b,c\in\mathbb{Z}^+$ with $\gcd(a,b,c)=1$.*

(i) *The form $ax^2+by^2+cT_z$ is asymptotically universal if and only if*

$$-2bc\ R\ a',\ -2ac\ R\ b',\ -ab\ R\ c',\ \text{and either } 4\nmid c \text{ or } (4\|c\ \&\ 2\|ab).$$

(ii) *The form $ax^2+bT_y+cT_z$ is asymptotically universal if and only if*

$$-bc\ R\ a',\ -2ac\ R\ b',\ -2ab\ R\ c',\ \text{and either } 4\nmid b \text{ or } 4\nmid c.$$

(iii) *The form* $aT_x+bT_y+cT_z$ *is asymptotically universal if and only if*

$$-bc\ R\ a',\ -ac\ R\ b',\ and -ab\ R\ c'.$$

Via computation we note that many asymptotically universal forms are almost universal. Here are some of our observations:

$$E(T_x+4T_y+5T_z)=\{2\},\ E(x^2+T_y+11T_z)=\{8\},$$
$$E(2x^2+5T_y+T_z)=E(6x^2+2T_y+T_z)=E(9x^2+2T_y+T_z)=\{4\}.$$

$$E(2x^2+3T_y+2T_z)=\{1,16\},\ E(3x^2+5T_y+T_z)=\{2,7\},$$
$$E(4x^2+4T_y+T_z)=\{2,108\},\ E(5x^2+4T_y+T_z)=\{2,16,31\},$$
$$E(x^2+4T_y+3T_z)=\{2,6,80\},\ E(2T_x+3T_y+4T_z)=\{1,8,31\}.$$

$$E(2x^2+5y^2+T_z)=\{4,27\},\ E(11T_x+2T_y+T_z)=\{4,25\},$$
$$E(9T_x+2T_y+T_z)=\{4,46\},\ E(T_x+2T_y+6T_z)=\{4,50\}.$$
$$E(x^2+T_y+9T_z)=\{8,47\},\ E(x^2+T_y+12T_z)=\{8,20\},$$
$$E(x^2+2T_y+10T_z)=E(T_x+T_y+10T_z)=\{5,8\},$$
$$E(11x^2+y^2+T_z)=\{8,\ 34,\ 348\}.$$

Any $n\in\mathbb{Z}^+$ can be uniquely written in the form a^2q with $a,q\in\mathbb{Z}^+$ and q squarefree, we use $\mathcal{SF}(n)$ to denote $q=\prod_{p\mid n,\,2\nmid v_p(n)}p$, the squarefree part of n.

Now we turn to the determination of all almost universal forms of the type $ax^2+by^2+cT_z$.

Theorem 3.2 (B. Kane and Z. W. Sun, 2008)**.** *Let* $a, b, c \in \mathbb{Z}^+$ *with* $\gcd(a,b,c) = 1$ *and* $v_2(a) \geq v_2(b)$*. Suppose that* $ax^2 + by^2 + cT_z$ *is asymptotically universal. Then it is NOT almost universal if and only if we have the following* $(1) - (3)$*.*

(1) $2|a$, $4 \nmid c$, $a' \equiv b' \pmod{2^{3-v_2(c)}}$, *and*

$$\begin{cases} 4 \nmid b \Rightarrow v_2(a) \equiv c \pmod 2 \\ 2 \nmid bc \Rightarrow 8 \mid a \ \&\ 8 \mid (b-c). \end{cases}$$

(2) *All prime divisors of* $\mathcal{SF}(a'b'c')$ *are congruent to* 1 *modulo* 4 *if* $v_2(a) \equiv v_2(b) \pmod 2$*, and congruent to* 1 *or* 3 *modulo* 8 *otherwise.*

(3) *The equation* $2^{3-v_2(c)}(ax^2 + by^2) + c'z^2 = \mathcal{SF}(a'b'c')$ *has no integral solutions.*

We also have similar results for the forms $ax^2 + bT_y + cT_z$ and $aT_x + bT_y + cT_z$. For details see Theorems 1.12 and 1.16 of Kane and Sun's paper (`arXiv:0808.2761`).

Here are some examples which are asymptotically universal but not almost universal.

$$5x^2 + 4y^2 + 2T_z,\ 4x^2 + y^2 + 50T_z,\ 8x^2 + y^2 + 9T_z,$$

$$2x^2 + 2y^2 + 25T_z,\ 76x^2 + 216y^2 + T_z,\ 58x^2 + 250y^2 + T_z;$$

$$8x^2 + 8T_y + T_z,\ 8x^2 + 16T_y + T_z,\ T_x + 4T_y + 4T_z.$$

Let m be any positive integer. By Theorem 3.1 and similar results for $ax^2 + bT_y + cT_z$ and $aT_x + bT_y + cT_z$, we have the following consequences.

Corollary 3.1. *Let* $a, b \in \mathbb{Z}^+$ *with* b *odd.*

(i) *We have*

$$\begin{aligned} &ax^2+2y^2+4bT_z \text{ is almost universal}\\ \iff& ax^2+2by^2+4T_z \text{ is almost universal}\\ \iff& 2\nmid a,\ -2a\ R\ b \text{ and } -b\ R\ a. \end{aligned}$$

In particular,

$$\begin{aligned} &\text{all sufficiently large odd numbers are represented by } 2ax^2+y^2+z^2\\ \iff& ax^2+2y^2+4T_z\sim ax^2+2T_y+2T_z \text{ is almost universal}\\ \iff& \text{all prime divisors of } a \text{ are congruent to 1 mod 4}. \end{aligned}$$

(ii) *If* $\mathcal{SF}(a')$ *or* $\mathcal{SF}(b)$ *has a prime divisor* $p\equiv 3 \pmod 4$ *(which happens when* a' *or* b *is congruent to* 3 *mod* 4*), then*

$$\begin{aligned} &ax^2+by^2+2T_z \text{ is almost universal}\\ \iff& ax^2+y^2+2bT_z \text{ is almost universal}\\ \iff& -a\ R\ b \text{ and } -b\ R\ a' \end{aligned}$$

and

$$\begin{aligned} &ax^2+2y^2+bT_z \text{ is almost universal}\\ \iff& ax^2+2by^2+T_z \text{ is almost universal}\\ \iff& -2a\ R\ b \text{ and } -b\ R\ a'. \end{aligned}$$

Corollary 3.2. *Let $m \in \mathbb{Z}^+$. Then*

$$\begin{aligned} &x^2 + my^2 + T_z \text{ is almost universal} \\ \iff &mx^2 + 2T_y + T_z \text{ is almost universal} \\ \iff &\text{all odd prime divisors of } m \text{ are congruent to 1 or 3 mod 8,} \end{aligned}$$

and

$$\begin{aligned} &x^2 + y^2 + 2mT_z \text{ is almost universal} \\ \iff &mT_x + 2T_y + 2T_z \text{ is almost universal} \\ \iff &mT_x + 2y^2 + 4T_z \text{ is almost universal} \\ \iff &\text{all prime divisors of } m \text{ are congruent to 1 mod 4.} \end{aligned}$$

Also,

$$\begin{aligned} &2x^2 + 2y^2 + mT_z \text{ is almost universal} \\ \iff &m \text{ is squarefree, and all prime divisors of } m \text{ are congruent to 1 mod 4.} \end{aligned}$$

Corollary 3.3. *Let $m \in \mathbb{Z}^+$.*

(i) *When $v_2(m) \neq 3$, the form $mT_x + T_y + T_z$ is almost universal if and only if all odd prime divisors of m are congruent to 1 modulo 4 and $v_2(m) \neq 5, 7, \ldots$.*

(ii) *The form $mT_x + 2T_y + T_z$ is almost universal if each odd prime divisor of m is congruent to 1 or 3 mod 8, and either $m' \equiv 1 \pmod 8$ or $v_2(m) \neq 4, 6, \ldots$. We also have the converse when $v_2(m) \neq 4$.*

In our joint paper, to get the desired results Kane and I use modular forms and the theory of ternary quadratic forms.

Let $Q(x,y,z)=ax^2+by^2+cT_z$ with $a,b,c\in\mathbb{Z}^+$, and define

$$r_Q(n):=|\{(x,y,z)\in\mathbb{Z}^3:\ Q(x,y,z)=n\}|.$$

Observe that

$$r_Q(n)=|\{(x,y,z)\in\mathbb{Z}^3:\ 8ax^2+8by^2+c(2z+1)^2=8n+c\}|.$$

Set

$$Q_1(x,y,z)=8ax^2+8by^2+cz^2 \text{ and } Q_2(x,y,z)=8ax^2+8by^2+4cz^2.$$

Then Q_1 and Q_2 are ternary quadratic forms and

$$r_Q(n)=r_{Q_1}(8n+c)-r_{Q_2}(8n+c).$$

The special linear group $\mathrm{SL}_2(\mathbb{Z})$ is given by

$$\mathrm{SL}_2(\mathbb{Z})=\left\{M=\begin{pmatrix}a&b\\c&d\end{pmatrix}:\ a,b,c,d\in\mathbb{Z}\ \&\ \det M=ad-bc=1\right\}.$$

For any positive integer N,

$$\Gamma_0(N)=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{SL}_2(\mathbb{Z}):\ c\equiv 0\ (\mathrm{mod}\ N)\right\}$$

is a subgroup of $\mathrm{SL}_2(\mathbb{Z})$ with index $N\prod_{p\mid N}(1+1/p)$. The group $\Gamma_0(N)$ acts on the set

$$\mathcal{H}=\{z\in\mathbb{C}:\ \mathrm{Im}(z)>0\}\quad\text{(the complex upper half-plane).}$$

if we define

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z := \frac{az+b}{cz+d} \quad \text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ and } z \in \mathcal{H}.$$

For a positive definite integral quadratic form $Q(x, y, z)$, the theta series

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) e^{2\pi i n z}$$

is a holomorphic function in the complex upper half-plane $\mathcal{H}$. Furthermore, there are a congruence subgroup $\Gamma_0(N)$ of $\mathrm{SL}_2(\mathbb{Z})$ and a Dirichlet character χ_Q modulo N such that

$$\theta_Q\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = \chi_Q(d)(cz+d)^{3/2}\theta_Q(z)$$
$$\text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ and } z \in \mathcal{H}.$$

Thus θ_Q is a modular form of half-integral weight $3/2$.

The theta series θ_Q splits naturally into three parts:

$$\theta_Q = \theta_{gen(Q)} + \left(\theta_{spn(Q)} - \theta_{gen(Q)}\right) + \left(\theta_Q - \theta_{spn(Q)}\right),$$

where the n-th coefficients of $\theta_{gen(Q)}$ and $\theta_{spn(Q)}$ are the weighted average of the number of representations of n by the genus and spinor genus of Q, respectively. Furthermore, $\theta_{gen(Q)}$ is an Eisenstein series, $\theta_{spn(Q)} - \theta_{gen(Q)}$ is a cusp form in the space of lifts of one dimensional theta series, and $\theta_Q - \theta_{spn(Q)}$ is a cusp form in the orthogonal complement of the space of one dimensional theta series. Moreover, the coefficients of $\theta_{spn(Q)} - \theta_{gen(Q)}$ are supported at finitely many square classes and the growth of the coefficients of the Eisenstein series overwhelms the growth of the coefficients of the cusp forms in the orthogonal complement of lifts of one dimensional theta series.

We need to know which square classes of coefficients $t\mathbb{Z}^2$ are supported by $\theta_{spn(Q)} - \theta_{gen(Q)}$. Kneser [Math. Z. 11(1961)] gave a necessary condition and Schulze-Pillot [J. Number Theory 12(1980)] extended this to give necessary and sufficient conditions. A. Earnest, J. S. Hsia and D. Hung [J. London Math. Soc. 50(1994)] made the sophisticated necessary and sufficient conditions of Schulze-Pillot in explicit and convenient form. This work will be useful for our purpose.

To avoid too many definitions and technical things, we will not talk more. For the detailed proofs of Th. 3.2 and related results, the reader may consult my paper with Kane which is available from the website `http://arxiv.org/abs/0808.2761`.

THANK YOU VERY MUCH!