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Conjectures and Results on Generalized Trinomial Coefficients and Motzkin Numbers

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Abstract

Given integers b and c, the generalized trinomial coefficient $T_n(b, c)$ denotes the coefficient of x^n in the expansion of $(x^2 + bx + c)^n$. In this talk we will introduce the speaker's various conjectures and results on congruences involving generalized trinomial coefficients and related Motzkin numbers.

On $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k$ modulo p^2

A Conjecture of Rodriguez-Villegas proved by E. Mortenson. If p is an odd prime, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)=(-1)^{(p-1)/2}\pmod{p^2}.$$

Remark. (a) By Stirling's formula,

$$n!\sim\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\ \text{ as } n\to+\infty.$$

It follows that

$$\binom{2k}{k}^2\sim\frac{16^k}{k\pi}.$$

(b) Mortenson'proof involves Gauss and Jacobi sums and the p-adic Gamma function. In fact, now there are elementary proofs.

An elementary proof of $\sum_{k=0}^{p-1}\binom{2k}{k}^2/16^k \equiv (\frac{-1}{p}) \pmod{p^2}$

Let $p = 2n+1$ be a prime. As observed by van Hammer, for $k = 0, \ldots, n$ we have

$$\begin{aligned}\binom{n}{k}\binom{n+k}{k}(-1)^k &= \binom{n}{k}\binom{-n-1}{k} \\ =&\binom{(p-1)/2}{k}\binom{(-p-1)/2}{k} \\ \equiv&\binom{-1/2}{k}^2 = \left(\frac{\binom{2k}{k}}{(-4)^k}\right)^2 = \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}.\end{aligned}$$

Thus Zhi-Hong Sun and Roberto Tauraso deduced that

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv& \sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k \\ =&\sum_{k=0}^{n}\binom{-n-1}{k}\binom{n}{n-k} = \binom{-1}{n} = (-1)^n \pmod{p^2}.\end{aligned}$$

On $\sum_{k=0}^{p-1}\binom{2k}{k}^2/m^k \bmod p^2$

Theorem (conjectured by Z. W. Sun [JNT, 2011] and proved by Z. H. Sun [Proc. AMS 2011]). If $p\equiv 1 \pmod 4$ is a prime and $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-1)/4}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}$$
$$\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right) \pmod{p^2}.$$

If $p\equiv 3 \pmod 4$ is a prime, then $\sum_{k=0}^{p-1}\binom{2k}{k}^2/32^k\equiv 0 \pmod{p^2}$.

Open Conjecture (Z. W. Sun [J. Number Theory 131(2011)]). Let $p\equiv 3 \pmod 4$ be a prime. Then

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}^2}{(-8)^k}\equiv 0 \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k} \pmod{p^3}.$$

Delannoy polynomials and Legendre polynomials

Those

$$D_n := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \ (n = 0, 1, 2, \ldots)$$

are called central Delannoy numbers. In combinatorics, D_n is the number of lattice paths from $(0, 0)$ to (n, n) with steps $(1, 0), (0, 1)$ and $(1, 1)$.

We define

$$D_n(x) := \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k.$$

The *Legendre polynomial* of degree n is given by

$$P_n(x) := \frac{1}{n!2^n} \cdot \frac{d^n}{dx^n}(x^2 - 1)^n = D_n\left(\frac{x-1}{2}\right).$$

Note that if $p = 2n + 1$ is a prime then

$$D_n(x) \equiv \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k} x^k \pmod{p^2}.$$

On central trinomial coefficients

The nth central trinomial coefficient:

$$T_n := [x^n](1+x+x^2)^n \text{ (the coefficient of } x^n \text{ in } (1+x+x^2)^n)$$
$$= \sum_{k=0}^{n} \binom{n}{k}\binom{n-k}{k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k}.$$

In combinatorics, T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Theorem (i) (H. Q. Cao and Sun, 2010). For any prime $p > 3$ we have

$$T_{p-1} \equiv \left(\frac{p}{3}\right) 3^{p-1} \pmod{p^2}.$$

(ii) (Z. W. Sun, 2010) For any odd prime p we have

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod{p}.$$

Conjecture on central trinomial coefficients

Conjecture (Sun, 2010) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8k+5)T_k^2 \equiv 0 \pmod{n}.$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(8k+5)T_k^2 \equiv 3p\left(\frac{p}{3}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{T_k H_k}{3^k} \equiv \frac{3+(\frac{p}{3})}{2} - p\left(1+\left(\frac{p}{3}\right)\right) \quad (\text{mod } p^2),$$

where H_k denotes the harmonic number $\sum_{0<j\leqslant k} 1/j$.

Mod p^2 congruences for Motzkin numbers

The nth Motzkin number

$$M_n := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k$$

is the number of paths from $(0,0)$ to $(n,0)$ which never dip below the line $y=0$ and are made up only of the allowed steps $(1,0)$, $(1,1)$ and $(1,-1)$.

Conjecture (Sun, 2010). Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1} M_k^2 \equiv (2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} kM_k^2 \equiv (9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1} M_k T_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) + \frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

Generalized central trinomial coefficients and generalized Motzkin numbers

Given $b, c \in \mathbb{Z}$, the *generalized central trinomial coefficients*

$$T_n(b,c) := [x^n](x^2+bx+c)^n = [x^0](b+x+cx^{-1})^n$$
$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k}\binom{2k}{k} b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} b^{n-2k}c^k$$

and we introduce the *generalized Motzkin numbers*

$$M_n(b,c) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} C_k b^{n-2k}c^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k}\binom{n}{k} \frac{b^{n-2k}c^k}{k+1}$$

$(n = 0, 1, 2, \ldots)$. Note that

$$T_n = T_n(1,1),\ M_n = M_n(1,1),\ T_n(2,1) = [x^n](x+1)^{2n} = \binom{2n}{n},$$

and

$$M_n(2,1) = \sum_{k=0}^{n} \binom{n}{2k} C_k 2^{n-2k} = C_{n+1}.$$

Generating functions

Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. H. S. Wilf observed that

$$\sum_{n=0}^{\infty} T_n(b,c)x^n = \frac{1}{\sqrt{1-2bx+dx^2}}$$

which implies the recursion

$$(n+1)T_{n+1}(b,c) = (2n+1)bT_n(b,c) - ndT_{n-1}(b,c) \ \ (n \in \mathbb{Z}^+).$$

By the Zeilberger algorithm we have

$$(n+3)M_{n+1}(b,c) = (2n+3)bM_n(b,c) - ndM_{n-1}(b,c) \ \ (n \in \mathbb{Z}^+),$$

and hence

$$2cx^2\sum_{n=0}^{\infty} M_n(b,c)x^n = 1 - bx - \sqrt{1-2bx+dx^2}.$$

Relations between $T_n(b, c)$ and Legendre polynomials

For Legendre polynomials, it is known that

$$\sum_{n=0}^{\infty} P_n(t)x^n = \frac{1}{\sqrt{1-2tx+x^2}}.$$

Thus, if $d = b^2 - 4c \neq 0$ then

$$\sum_{n=0}^{\infty} T_n(b,c)\left(\frac{x}{\sqrt{d}}\right)^n = \frac{1}{\sqrt{1-2bx/\sqrt{d}+d(x/\sqrt{d})^2}} = \sum_{n=0}^{\infty} P_n(b)x^n$$

and hence

$$T_n(b,c) = (\sqrt{d})^n P_n\left(\frac{b}{\sqrt{d}}\right).$$

It follows that

$$T_n(2x+1, x^2+x) = P_n(2x+1) = D_n(x) \quad \text{for all } x \in \mathbb{Z};$$

in particular, $D_n = T_n(3, 2)$.

On $\sum_{k=0}^{p-1} T_k(b,c)/m^k$ and $\sum_{k=0}^{p-1} M_k(b,c)/m^k$ mod p

Theorem (Sun, 2010). Let b and c be integers.

(i) Let p be an odd prime not dividing $m \in \mathbb{Z}$. Then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)}{m^k} \equiv \left(\frac{(m-b)^2-4c}{p}\right) \pmod{p}$$

and

$$2c\sum_{k=0}^{p-1} \frac{M_k(b,c)}{m^k} \equiv (m-b)^2-((m-b)^2-4c)\left(\frac{(m-b)^2-4c}{p}\right) \pmod{p}.$$

(ii) For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} T_k(b,c^2)(b-2c)^{n-1-k} \equiv 0 \pmod{n}$$

and

$$6\sum_{k=0}^{n-1} kT_k(b,c^2)(b-2c)^{n-1-k} \equiv 0 \pmod{n}.$$

Congruences modulo n

Theorem (Sun, 2010). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$.
(i) For any $n \in \mathbb{Z}^+$, we have

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} \equiv 0 \pmod{n},$$

and furthermore

$$b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k} = nT_n(b,c)T_{n-1}(b,c).$$

(ii) Suppose that $b^2 - 4c = 1$, i.e., there is an $m \in \mathbb{Z}$ such that $b = 2m+1$, $c = m^2 + m$ and hence $T_k(b,c) = D_k(m)$. Then

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k(b,c) = \sum_{k=1}^{n}\binom{n}{k}\binom{n+k-1}{k-1}\left(\frac{b-1}{2}\right)^{k-1} \in \mathbb{Z}$$

for all $n \in \mathbb{Z}^+$.

On $\sum_{k=0}^{p-1} T_k(b,c)^2/m^k \bmod p$

Theorem (Sun, 2010) Let $b, c \in \mathbb{Z}$ with $d = b^2 - 4c$ and let p be an odd prime.

(i) If $p \nmid d$, then we have

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)^2}{d^k} \equiv \left(\frac{cd}{p}\right) \pmod{p}.$$

If $b \not\equiv 2c \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv \left(\frac{-c^2}{p}\right) \pmod{p}.$$

(ii) Assume $p \nmid c$. If $p \nmid d$, then $\sum_{k=0}^{p-1} T_k(b,c)M_k(b,c)/d^k \equiv 0 \pmod{p}$.
If $D = b^2 - 4c^2 \not\equiv 0 \pmod{p}$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2)M_k(b,c^2)}{(b-2c)^{2k}} \equiv \frac{4b}{b+2c}\left(\frac{D}{p}\right) \pmod{p}.$$

A Corollary

Since $D_k(x) = T_k(2x+1, x^2+x)$ and $(2x+1)^2 - 4(x^2+x) = 1$, we have

Corollary Let p be an odd prime. For any integer x we have

$$\sum_{k=0}^{p-1} D_k(x)^2 \equiv \left(\frac{x(x+1)}{p}\right) \pmod{p}.$$

In particular,

$$\sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod{p}.$$

Congruences modulo n^2

Theorem (Sun, 2011). Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$. For any $n \in \mathbb{Z}^+$, we have

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2 d^{n-1-k} = \sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_k c^k d^{n-1-k}.$$

If c is nonzero and p is an odd prime not dividing d, then

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} \equiv 1 + \frac{b^2}{c}\cdot\frac{(\frac{d}{p})-1}{2} \pmod{p}.$$

Corollary. For each $n = 1, 2, 3, \ldots$ we have

$$\sum_{k=0}^{n-1}(2k+1)D_k^2 \equiv 0 \pmod{n^2}.$$

Congruences modulo n^2

Conjecture (Sun, 2010) Let $b, c \in \mathbb{Z}$. For any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1}(8ck+4c+b)T_k(b,c^2)^2(b-2c)^{2(n-1-k)} \equiv 0 \pmod{n}.$$

If p is an odd prime not dividing $b(b-2c)$, then

$$\sum_{k=0}^{p-1}(8ck+4c+b)\frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv p(b+2c)\left(\frac{b^2-4c^2}{p}\right) \pmod{p^2}.$$

A conjecture involving $D_k(x)$

Conjecture (Sun, 2010). Let x be any integer. If p is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^3 \equiv p\left(\frac{-4x-3}{p}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^4 \equiv p \pmod{p^2}.$$

A theorem on $\sum_{k=0}^{p-1} T_k(b,c)^2/m^k \bmod p^2$

Theorem (Sun, 2011) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{T_k(6,-3)^2}{48^k}\equiv\left(\frac{-1}{p}\right)+\frac{p^2}{3}E_{p-3}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{T_k(2,-1)^2}{8^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{T_k(2,-3)^2}{16^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},$$

where $E_0, E_1, E_2, \ldots$ are Euler numbers.

Lemma 1. Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. For any $n\in\mathbb{N}$ we have

$$T_n(b,c)^2=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2c^kd^{n-k}.$$

Another Lemma

Lemma 2 (Sun [JNT 131(2011)]) Let $p > 3$ be a prime. Then

$$\sum_{0\leqslant k<p/2}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv 0 \pmod{p^2},$$

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv \frac{p}{3}E_{p-3} \pmod{p^2}.$$

This lemma was motivated by the identity

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)16^k}=\frac{\pi}{3}.$$

We prove the first congruence via the combinatorial identity

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{(-1)^k}{2k+1}=\begin{cases}(-1)^n/(2n+1) & \text{if } 3\nmid 2n+1,\\ 2(-1)^{n-1}/(2n+1) & \text{if } 3\mid 2n+1.\end{cases}$$

Conjectural congruences involving powers of $T_k(b,c)$

Conjecture (Sun, 2010). Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{4^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{36^k}\equiv\left(\frac{-1}{p}\right)\pmod{p^2},$$

and

$$\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{8^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{(-64)^k}$$

$$\equiv\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{(-64)^k}\equiv\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{512^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\text{ and }p=x^2+6y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\text{ and }p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1.\end{cases}$$

Conjectural congruences involving powers of $T_k(b,c)$

Conjecture (Sun, 2011). Let $p>3$ be a prime. Then

$$\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{8^{3k}}\equiv\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{16^{3k}}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1\pmod 4\ \&\ p=x^2+4y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4;\end{cases}$$

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{(-8)^{3k}}\equiv\left(\frac{6}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{12^{3k}}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,3\pmod 8\ \&\ p=x^2+2y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-2}{p})=-1,\ \text{i.e., } p\equiv 5,7\pmod 8.\end{cases}$$

Conjectural congruences involving powers of $T_k(b,c)$

Also,

$$\sum_{k=0}^{p-1}(7k+4)\frac{T_k(10,49)^3}{(-8)^{3k}} \equiv \frac{p}{14}\left(\frac{2}{p}\right)\left(65-9\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}(7k+3)\frac{T_k(10,49)^3}{12^{3k}} \equiv \frac{3p}{28}\left(13+15\left(\frac{p}{3}\right)\right) \pmod{p^2}.$$

For each $n=1,2,3,\ldots$ we have

$$\sum_{k=0}^{n-1}(7k+4)\,T_k(10,49)^3(-8^3)^{n-1-k} \equiv 0 \pmod{4n},$$

$$\sum_{k=0}^{n-1}(7k+3)\,T_k(10,49)^3(12^3)^{n-1-k} \equiv 0 \pmod{n}.$$

More conjectures on Congruences

For more conjectures of mine on congruences, see

Z. W. Sun, *Open Conjectures on Congruences*, `arXiv:0911.5665`

which contains **100 unsolved conjectures** raised by me.

You are welcome to solve my conjectures!

Thank you!