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On universal sums involving polygonal numbers

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Abstract

For $m = 3, 4, 5, \ldots$, those

$$p_m(n) = m\binom{n}{2} + n \quad (n = 0, 1, 2, \ldots)$$

are called m-gonal numbers (or polygonal numbers of order m), and those $p_m(x)$ with $x \in \mathbb{Z}$ are called generalized m-gonal numbers. A confirmed claim of Fermat asserts that any natural number is the sum of m m-gonal numbers. In this talk we introduce various recent problems and results involving polygonal numbers and related ternary quadratic forms. For example, the speaker has proved that any natural number can be written as the sum of a triangular number, an even square and a generalized pentagonal number, and that each natural number is the sum of four generalized octagonal numbers. Some problems in this area look quite challenging, for example, the speaker conjectures that for any positive integers a, b, c with $(a, b, c) \neq (1, 1, 1), (2, 2, 2)$ every natural number can be represented by $\lfloor x^2/a \rfloor + \lfloor y^2/b \rfloor + \lfloor z^2/c \rfloor$ with x, y, z integers.

Part I. On sums of triangular numbers and squares

Triangular numbers and squares

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$.
Triangular numbers are those

$$T_n = \sum_{r=0}^{n} r = \frac{n(n+1)}{2} \quad (n \in \mathbb{N}).$$

$$0,\ 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ 45,\ 55,\ \ldots$$

Note that

$$T_{-n-1} = \frac{(-n-1)(-n)}{2} = T_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Squares $n^2 = (-n)^2$:

$$0,\ 1,\ 4,\ 9,\ 16,\ 25,\ 36,\ 49,\ 64,\ 81,\ 100,\ \ldots$$

Lagrange's theorem and the Gauss-Legendre theorem

Lagrange's Theorem Each $n \in \mathbb{N}$ can be written as the sum of four squares.

Gauss-Legendre Theorem. $n \in \mathbb{N}$ can be written as the sum of three squares if and only if n is not of the form $4^k(8l+7)$ with $k, l \in \mathbb{N}$.

Corollary (Gauss). Each $n \in \mathbb{N}$ can be written as $T_x + T_y + T_z$ with $x, y, z \in \mathbb{Z}$.

Proof. By the Gauss-Legendre theorem, there are $u, v, w \in \mathbb{Z}$ such that $8n + 3 = u^2 + v^2 + w^2$. As $u^2 + v^2 + w^2 \equiv 3 \pmod 4$, we must have $2 \nmid uvw$. So we may write $u = 2x + 1$, $v = 2y + 1$ and $w = 2z + 1$ with $x, y, z \in \mathbb{Z}$. Hence

$$n = \frac{u^2 - 1}{8} + \frac{v^2 - 1}{8} + \frac{w^2 - 1}{8} = T_x + T_y + T_z.$$

Liouville's and Ramanujan's observations

Liouville's Theorem (Liouville, 1862). Let $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b \leqslant c$. Then any $n \in \mathbb{N}$ can be written in the form $aT_x + bT_y + cT_z$ if and only if (a, b, c) is among

$$(1,1,1),\ (1,1,2),\ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4).$$

S. Ramanujan's Observation (confirmed by L.E. Dickson in 1927). There are totally 54 quadruples $(a, b, c, d) \in (\mathbb{Z}^+)^4$ with $a \leqslant b \leqslant c \leqslant d$ such that each $n \in \mathbb{N}$ can be written as $aw^2 + bx^2 + cy^2 + dz^2$ with $w, x, y, z \in \mathbb{Z}$.

$$r_{(a,b,c,d)}(n) = |\{(w,x,y,z) \in \mathbb{Z}^4 :\ aw^2 + bx^2 + cy^2 + dz^2 = n\}|$$

is known for the following 20 of the 54 quadruples

$$(1,1,1,1),(1,1,1,2),(1,1,2,2),(1,2,2,2),(1,1,1,3),$$
$$(1,1,2,3),(1,2,2,3),(1,1,3,3),(1,1,1,4),(1,1,2,4),$$
$$(1,2,2,4),(1,1,3,4),(1,2,4,4),(1,1,1,5),(1,1,2,6),$$
$$(1,2,2,6),(1,2,3,6),(1,2,4,6),(1,1,2,8),(1,2,4,8).$$

Mixed Sums of Squares and Triangular Numbers

Euler's Observation:

$$8n+1 = (2x)^2 + (2y)^2 + (2z+1)^2$$
$$\Longrightarrow n = \frac{x^2+y^2}{2} + T_z = \left(\frac{x+y}{2}\right)^2 + \left(\frac{x-y}{2}\right)^2 + T_z.$$

Lionnet's Assertion (proved by Lebesgue & Réalis in 1872). Any $n \in \mathbb{N}$ is the sum of two triangular numbers and a square.

B. W. Jone and G. Pall [Acta Math. 1939]. Every $n \in \mathbb{N}$ is the sum of a square, an *even* square and a triangular number.

Theorem (i) [Z. W. Sun, Acta Arith. 2007] Any $n \in \mathbb{N}$ is the sum of an *even* square and two triangular numbers.
(ii) (Conjectured by Z. W. Sun and proved by B. K. Oh and Sun [JNT, 2009]). Any positive integer n can be written as the sum of a square, an *odd* square and a triangular number.

Mixed Sums of Squares and Triangular Numbers

In 2005 Z. W. Sun [Acta Arith. 2007] investigated what kind of mixed sums $ax^2+by^2+cT_z$ or $ax^2+bT_y+cT_z$ (with $a,b,c\in\mathbb{Z}^+$) are universal (i.e., all natural numbers can be so represented). This project was completed via three papers: Z. W. Sun [Acta Arith. 2007], S. Guo, H. Pan & Z. W. Sun [Integers, 2007], and B. K. Oh & Sun [JNT, 2009].

List of all universal $ax^2+by^2+cT_z$ **or** $ax^2+bT_y+cT_z$:

$T_x+T_y+z^2,\ T_x+T_y+2z^2,\ T_x+T_y+4z^2,\ T_x+2T_y+z^2,$
$T_x+2T_y+2z^2,\ T_x+2T_y+3z^2,\ T_x+2T_y+4z^2, 2T_2T_y+z^2,$
$2T_x+4T_y+z^2, 2T_x+5T_y+z^2,\ T_x+3T_y+z^2,\ T_x+4T_y+z^2,$
$T_x+4T_y+2z^2,\ T_x+6T_y+z^2,\ T_x+8T_y+z^2,\ T_x+y^2+z^2,$
$T_x+y^2+2z^2,\ T_x+y^2+3z^2,\ T_x+y^2+4z^2,\ T_x+y^2+8z^2,$
$T_x+2y^2+2z^2,\ T_x+2y^2+4z^2, 2T_x+y^2+z^2, 2T_x+y^2+2y^2,$
$4T_x+y^2+2z^2.$

A problem of S. Ramanujan

In 1916 Ramanujan conjectured that

(1) *the only positive even numbers not of the form* $x^2 + y^2 + 10z^2$ *are those* $4^k(16l + 6)$ $(k, l \in \mathbb{N})$

and

(2) *sufficiently large odd numbers are of the form* $x^2 + y^2 + 10z^2$.

In 1927 L. E. Dickson [Bull. AMS] proved (1). In 1990 W. Duke and R. Schulze-Pillot [Invent. Math.] confirmed (2).

In 1997 K. Ono and K. Soundararajan [Invent. Math.] proved that under the GRH any odd number greater than 2719 has the form $x^2 + y^2 + 10z^2$.

Note that

$$\begin{aligned} &2n + 1 = x^2 + y^2 + 10z^2 \text{ for some } x, y, z \in \mathbb{Z} \\ \iff &2n + 1 = (2x)^2 + 10y^2 + (2z + 1)^2 \text{ for some } x, y, z \in \mathbb{Z} \\ \iff &n = 2x^2 + 5y^2 + 4T_z \text{ for some } x, y, z \in \mathbb{Z}. \end{aligned}$$

Almost universal sums of squares and triangular numbers

For a function $f(x_1,\ldots,x_k):\mathbb{Z}^k\to\mathbb{N}$, if all sufficiently large integers can be written as $f(x_1,\ldots,x_k)$ with $x_1,\ldots,x_k\in\mathbb{Z}$ then we say that $f(x_1,\ldots,x_k)$ is *almost universal over* $\mathbb{Z}$.

Ramanujan asked for determining those quadruples $(a,b,c,d)\in(\mathbb{Z}^+)^4$ such that $aw^2+bx^2+cy^2+dz^2$ is almost universal over $\mathbb{Z}$. This problem was essentially solved by H.D. Kloosterman [Acta Math. 49(1926)] who introduced the useful Kloosterman sum as a tool.

Ben Kane and Z.-W. Sun [Trnas. Amer. Math. Soc. 362(2010), 6425-6455] determined completely when $ax^2+by^2+cT_z$ (with $a,b,c\in\mathbb{Z}^+$) is almost universal over $\mathbb{Z}$. As a consequence, *$ax^2+y^2+T_z$ is almost universal over $\mathbb{Z}$ if and only if each odd prime divisor of a is congruent to 1 or 3 mod 8.*

Kane and Sun also investigated almost universal sums $aT_x+bT_y+cT_z$ and $ax^2+bT_y+cT_z$. The remaining conjectures were solved by W.K. Chan, B. K. Oh and A. Haensch.

Universal sums $ax^2+by^2+f(z)$ with $x,y,z\in\mathbb{N}$

Theorem 1 (Sun, arXiv:1502.03056). Let $a,b\in\mathbb{Z}^+$ with $a\leqslant b$, and let $f(z)=c\binom{z}{2}+dz$ with $c,d\in\mathbb{Z}^+$ and $d\nmid c$. Suppose that $ax^2+by^2+f(z)$ is universal over $\mathbb{N}$. Then $ax^2+by^2+f(z)$ is on the following list:

$$x^2+2y^2+z(z+3),\ x^2+by^2+\frac{z(z+3)}{2}\ (b=1,2,3),$$

$$x^2+2y^2+\frac{z(z+5)}{2},\ x^2+2y^2+\frac{z(3z+1)}{2},\ x^2+2y^2+\frac{z(3z+5)}{2}.$$

Theorem 2. All the sums

$$x^2+y^2+\frac{z(z+3)}{2},\ T_x+T_y+\frac{z(z+2k+1)}{2}\ (1\leqslant k\leqslant 4),$$

$$T_x+T_y+z(z+2),\ T_x+2T_y+\frac{z(z+2k+1)}{2}\ (1\leqslant k\leqslant 3),$$

$$T_x+y^2+z(z+2k)\ (k=1,2,3),\ T_x+(2y)^2+\frac{z(z+3)}{2},$$

$$T_x+y^2+\frac{z(z+2k+1)}{2}\ (k=1,2,3,4,5,6,7).$$

are universal over $\mathbb{N}$.

On $x(ax+b)+y(ay+c)+z(az+d)$ with $x,y,z\in\mathbb{Z}$

As $T_n = T_{-n-1}$ for all $n \in \mathbb{N}$, we see that

$$\{T_n:\ n\in\mathbb{N}\} = \{T_{2x} = x(2x+1):\ x\in\mathbb{Z}\}$$

and hence

$$\{x(2x+1)+y(2y+1)+z(2z+1):\ x,y,z\in\mathbb{Z}\} = \mathbb{N}.$$

Motivated by this, we obtain the following result.

Theorem (Sun, arXiv:1505.03679) Let $a,b,c,d\in\mathbb{N}$ with $a>2$ and $b\leqslant c\leqslant d\leqslant a$. Then

$$\{x(ax+b)+y(ay+c)+z(az+d):\ x,y,z\in\mathbb{Z}\},$$

if and only if the quadruple (a,b,c,d) is among

$$(3,0,1,2),\ (3,1,1,2),\ (3,1,2,2),\ (3,1,2,3),\ (4,1,2,3).$$

On $x(ax+1)+y(by+1)+z(cz+1)$ with $x, y, z \in \mathbb{Z}$

Theorem (Sun, arXiv:1505.03679) Let $a, b, c \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c$. If

$$\{x(ax+1)+y(by+1)+z(cz+1):\ x, y, z \in \mathbb{Z}\} = \mathbb{N}, \qquad (*)$$

then (a, b, c) is among the following 17 triples:

$$(1,1,2),\ (1,2,2),\ \underline{(1,2,3)},\ (1,2,4),\ (1,2,5),$$
$$(2,2,2),\ (2,2,3),\ (2,2,4),\ (2,2,5),\ (2,2,6),$$
$$(2,3,3),\ \underline{(2,3,4)},\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10).$$

Also, $(*)$ holds if (a, b, c) is among the 17 triples but not among

$$(2,2,6),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10). \qquad (**)$$

Conjecture (Sun). $(*)$ also holds if (a, b, c) is among $(**)$.

Part II. On universal sums of polygonal numbers

Polygonal Numbers

Polygonal numbers are nonnegative integers constructed geometrically from the regular polygons. For $m = 3, 4, 5, \ldots$, the m-gonal numbers are given by

$$p_m(n) = (m-2)\binom{n}{2} + n \quad (n = 0, 1, 2, \ldots).$$

Clearly

$$p_3(n) = T_n,\ p_4(n) = n^2,\ p_5(n) = \frac{3n^2 - n}{2},\ p_6(n) = 2n^2 - n = T_{2n-1}.$$

The larger m is, the more sparse m-gonal numbers are.

Euler's Discovery:

$$\frac{1}{\sum_{n=0}^{\infty} p(n)q^n} = \prod_{n=1}^{\infty}(1 - q^n) = \sum_{k=-\infty}^{\infty}(-1)^k q^{p_5(k)} \quad (|q| < 1),$$

where $p(n)$ $(n = 1, 2, 3, \ldots)$ is the number of ways to write n as the sum of positive integers (repetition allowed) and $p(0) := 1$.

Fermat's Assertion (1638). Any natural number n can be written as the sum of m m-gonal numbers.

Cauchy's Lemma

Those $p_5(n) = n(3n-1)/2$ ($n \in \mathbb{N}$) are called *pentagonal numbers*.
Those $p_6(n) = n(2n-1)$ ($n \in \mathbb{N}$) are called *hexagonal numbers*.

In 1813 Cauchy proved that for $m = 5, 6, \ldots$ every $n \in \mathbb{N}$ is the sum of m m-gonal numbers. The proof depends on the Gauss-Legendre theorem on sums of three squares and the following lemma.

Cauchy's Lemma. For odd integers $a, b \in \mathbb{Z}^+$ with $b^2 < 4a$ and $3a < b^2 + 2b + 4$, there exist $s, t, u, v \in \mathbb{N}$ such that $a = s^2 + t^2 + u^2 + v^2$ and $b = s + t + u + v$.

Cauchy's proof was simplified by M. B. Nathanson [Proc. AMS 99(1987)].

Diagonal Representations by Polygonal Numbers

$$n = \underline{p_3(x_1) + p_3(x_2) + p_3(x_3)}$$
$$n = \underline{p_4(x_1)} + p_4(x_2) + p_4(x_3) + p_4(x_4)$$
$$n = p_5(x_1) + \underline{p_5(x_2)} + p_5(x_3) + p_5(x_4) + p_5(x_5)$$
$$n = p_6(x_1) + p_6(x_2) + \underline{p_6(x_3)} + p_6(x_4) + p_6(x_5) + p_6(x_6)$$

Diagonal Representation:

$$n = \underline{p_4(x_1)} + \underline{p_5(x_2)} + \underline{p_6(x_3)}$$

Conjecture [Z. W. Sun, arxiv: 0905.0635, 2009]. Any $n \in \mathbb{N}$ can be written as the sum of a square, a pentagonal number and a hexagonal number. Also, we can write each $n \in \mathbb{N}$ as the sum of two squares and a pentagonal number, and as the sum of a triangular number, an even square and a pentagonal number.

Diagonal Representations by Polygonal Numbers

$$
\begin{aligned}
n =& \underline{p_{m+1}(x_1)} + p_{m+1}(x_2) + p_{m+1}(x_3) + \cdots + p_{m+1}(x_{m+1}) \\
n =& p_{m+2}(x_1) + \underline{p_{m+2}(x_2)} + p_{m+2}(x_3) + \cdots\cdots + p_{m+2}(x_{m+2}) \\
n =& p_{m+3}(x_1) + p_{m+3}(x_2) + \underline{p_{m+3}(x_3)} + \cdots\cdots\cdots + p_{m+3}(x_{m+3}) \\
& \cdots\cdots\cdots\cdots\cdots \\
n =& p_{2m}(x_1) + p_{2m}(x_2) + p_{2m}(x_3) + \cdots\cdots + \underline{p_{2m}(x_m)} + \cdots + p_{2m}(x_{2m})
\end{aligned}
$$

Diagonal Representation:

$$n = \underline{p_{m+1}(x_1)} + \underline{p_{m+2}(x_2)} + \underline{p_{m+3}(x_3)} + \cdots + \underline{p_{2m}(x_m)}$$

Conjecture on Diagonal Representations [Z. W. Sun, August 12, 2009]. Let $m \geqslant 3$ be an integer. Then any $n \in \mathbb{N}$ can be written in the form

$$p_{m+1}(x_1) + \cdots + p_{2m}(x_m) \quad \text{with } x_1, \ldots, x_m \in \mathbb{N}.$$

Strong Version of the Conjecture

Conjecture [Z. W. Sun, August 21, 2009]. Let $m \geqslant 3$ be an integer. Then any $n \in \mathbb{N}$ can be written in the form

$$p_{m+1}(x_1) + p_{m+2}(x_2) + p_{m+3}(x_3) + r$$

with $x_1, x_2, x_3 \in \mathbb{N}$ and $r \in \{0, \ldots, m-3\}$.

Remark. Clearly any $r \in \{0, \ldots, m-3\}$ can be written as $p_{m+4}(x_4) + \cdots + p_{2m}(x_m)$ with $x_4, \ldots, x_m \in \{0, 1\}$.

Verification of the Conjecture. (i) (Sun, 2009) $m = 3$ and $n \leqslant 10^6$; $4 \leqslant m \leqslant 10$ and $n \leqslant 10^6$; $10 < m \leqslant 40$ and $n \leqslant 10^5$.
(ii) (Mauro Fiorentini, 2015) $3 \leqslant m \leqslant 100$ and $n \leqslant 10^9$.

Prize: \$500 for the first rigorous proof of the full conjecture.

A Related Conjecture [Z. W. Sun, August 14, 2009]. For each $m = 3, 4, \ldots$ all sufficiently large integers have the form

$$p_{m+1}(x_1) + p_{m+2}(x_2) + p_{m+3}(x_3) \quad (x_1, x_2, x_3 \in \mathbb{N}).$$

Mixed Sums of Three Polygonal Numbers

Conjecture [Z. W. Sun, 2009]. Let $3 \leqslant i \leqslant j \leqslant k$ and $k \geqslant 5$. Then each $n \in \mathbb{N}$ can be written as the sum of an i-gonal number, a j-gonal number and a k-gonal number, if and only if (i, j, k) is among the following 31 triples:

$(3,3,5)$, $(3,3,6)$, $(3,3,7)$, $(3,3,8)$, $(3,3,10)$, $(3,3,12)$, $(3,3,17)$,
$(3,4,5)$, $(3,4,6)$, $(3,4,7)$, $(3,4,8)$, $(3,4,9)$, $(3,4,10)$, $(3,4,11)$,
$(3,4,12)$, $(3,4,13)$, $(3,4,15)$, $(3,4,17)$, $(3,4,18)$, $\underline{(3,4,27)}$,
$(3,5,5)$, $(3,5,6)$, $\underline{(3,5,7)}$, $(3,5,8)$, $(3,5,9)$, $(3,5,11)$, $(3,5,13)$,
$(3,7,8)$, $(3,7,10)$, $(4,4,5)$, $\underline{(4,5,6)}$.

Remark. Sun proved the 'only if' part. The 'if' part is difficult!

Sun [Sci. China Math. 58(2015)] also showed that there are only 95 candidates for universal sums of the form $ap_i + bp_j + cp_k$.

Generalized Polygonal Numbers

For $m \in \{3, 4, 5, \ldots\}$, those $p_m(x) = (m-2)\binom{x}{2} + x$ with $x \in \mathbb{Z}$ are called *generalized m-gonal numbers*.

Generalized hexagonal numbers are identical with triangular numbers, for,

$$p_6(x) = x(2x-1) = T_{2x-1} \text{ and } T_x = p_6\left(-\frac{x}{2}\right) = p_6\left(\frac{x+1}{2}\right).$$

Generalized pentagonal numbers are those $p_5(x) = x(3x-1)/2$ with $x \in \mathbb{Z}$. Note that $24p_5(x) + 1 = (6x-1)^2$.

That $p_5 + p_5 + p_5$ is universal over $\mathbb{Z}$ (equivalently, for any $n \in \mathbb{N}$ we can write $24n + 3 = x^2 + y^2 + z^2$ with x, y, z all relatively prime to 3), was first realized by R. K. Guy [Amer. Math. Monthly 1994]. To make Guy's proof complete, one needs the following Réalis identity

$$9(x^2+y^2+z^2) = (x-2y-2z)^2 + (y-2x-2z)^2 + (z-2x-2y)^2.$$

In 2004 G. Shimura [Amer. J. Math.] also obtained this via a very deep and abstract theory.

Universal $ap_k + bp_k + cp_k$ over $\mathbb{Z}$

Theorem [Z. W. Sun, arxiv: 0905.0635, 2009]. Suppose that $ap_k + bp_k + cp_k$ is universal over $\mathbb{Z}$, where $k \in \{4, 5, 7, 8, 9, \ldots\}$, $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b \leqslant c$. Then k is equal to 5 and (a, b, c) is among the following 20 triples:

$$(1,1,k)\ (k \in [1,10] \setminus \{7\}),$$
$$(1,2,2),\ (1,2,3),\ (1,2,4),\ (1,2,6),\ (1,2,8),$$
$$(1,3,3),\ (1,3,4),\ (1,3,6),\ (1,3,7),\ (1,3,8),\ (1,3,9).$$

Conjecture [Z. W. Sun, arxiv: 0905.0635, 2009]. The above 20 triples are indeed universal over $\mathbb{Z}$.

Theorem. (i) [Z. W. Sun, arxiv: 0905.0635, 2009] The sums

$$p_5 + p_5 + 2p_5,\ p_5 + p_5 + 4p_5,\ p_5 + 2p_5 + 2p_5,$$
$$p_5 + 2p_5 + 4p_5,\ p_5 + p_5 + 5p_5,\ p_5 + 3p_5 + 6p_5$$

are universal over $\mathbb{Z}$.

(ii) [G. Fan & Z. W. Sun, arxiv:0906.2450, 2009] $p_5 + 2p_5 + 6p_5$ and $p_5 + bp_5 + 3p_5\,(b = 1, 2, 3, 4, 9)$ are universal over $\mathbb{Z}$.

(iii) [Oh, arxiv:0911.1181] Remaining part of the conjecture holds.

Regular Ternary Quadratic Forms

A positive (definite) ternary quadratic form

$$Q(x, y, z) = ax^2 + by^2 + cz^2 + dyz + exz + fxy$$

with $a, b, c, d, e, f \in \mathbb{Z}$ is said to be *regular* if it represents an integer n (i.e., $Q(x, y, z) = n$ for some $x, y, z \in \mathbb{Z}$) if and only if it locally represents n (i.e., for any prime p the equation $Q(x, y, z) = n$ has integral solutions in the p-adic field $\mathbb{Q}_p$; in other words, for any $m \in \mathbb{Z}^+$ the congruence $Q(x, y, z) \equiv n \pmod{m}$ is solvable over $\mathbb{Z}$).

A full list of positive regular ternary quadratic forms was given by W. C. Jagy, I. Kaplansky and A. Schiemann [*There are 913 regular ternary forms*, Mathematika 44(1997), 332–341]. There are totally 102 regular forms $ax^2 + by^2 + cz^2$ with $1 \leqslant a \leqslant b \leqslant c$ and $\gcd(a, b, c) = 1$; for each of them those positive integers not represented by the form were described explicitly in Dickson's book published in 1939. However, we often meet irregular positive ternary quadratic forms when we investigate the universality of $ap_i + bp_j + cp_k$ over $\mathbb{Z}$.

Connection to Modular Forms

For a positive definite integral quadratic form $Q(x, y, z)$, we define

$$r_Q(n) := |\{(x, y, z) \in \mathbb{Z}^3 : \ Q(x, y, z) = n\}|.$$

The theta series

$$\theta_Q(z) = \sum_{n=0}^{\infty} r_Q(n) e^{2\pi i n z}$$

is a holomorphic function in the complex upper half-plane

$$\mathcal{H} = \{z \in \mathbb{C} : \ \mathrm{Im}(z) > 0\}.$$

Furthermore, there is a congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \ c \equiv 0 \pmod{N} \right\}$$

of $\mathrm{SL}_2(\mathbb{Z})$ and a Dirichlet character χ_Q mod N such that

$$\theta_Q\left(\frac{az+b}{cz+d}\right) = \chi_Q(d)(cz+d)^{3/2}\theta_Q(z)$$

$$\text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) \text{ and } z \in \mathcal{H}.$$

Some Results on Ternary Quadratic Forms

Theorem. [Z. W. Sun, Sci. Math. China 58(2015) (see also arxiv:0905.0635)] Let $n \in \mathbb{N}$. Then
(i)

$$6n + 1 = x^2 + 3y^2 + 24z^2 \quad \text{for some } x, y, z \in \mathbb{Z},$$

and consequently

$$n = T_x + (2y)^2 + p_5(z) \quad \text{for some } x, y, z \in \mathbb{Z}.$$

[*The proof needs several technique lemmas.*]

(ii)

$$12n + 4 = x^2 + 3y^2 + 3z^2 \quad \text{for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid x,$$

$$12n + 4, 12n + 8 = 3x^2 + y^2 + z^2 \quad \text{for some } x, y, z \in \mathbb{Z} \text{ with } 2 \nmid x.$$

[*The proof has the same flavor with Fermat's infinite descent.*]

A Lemma (Sun). For any $n \in \mathbb{N}$ we have

$$|\{(x, y) \in \mathbb{Z}^2 :\ x^2 + 3y^2 = 8n + 4 \text{ and } 2 \nmid xy\}| = \frac{2}{3}|\{(x, y) \in \mathbb{Z}^2 :\ x^2 + 3y^2 = 8n + 4\}|.$$

A result similar to Lagrange's theorem

By a theorem of Legendre, for each $m = 5, 7, 9, \ldots$, any integer $n \geqslant 28(m-2)^3$ can be expressed as the sum of four m-gonal numbers; in particular, any integer $n \geqslant 3500$ is the sum of four heptagonal numbers. It follows that

$$\{p_7(w) + p_7(x) + p_7(y) + p_7(z): \ w, x, y, z \in \mathbb{Z}\} = \mathbb{N}.$$

For any integer $m > 8$, clearly 5 cannot be written as the sum of four generalized m-gonal numbers.

Generalized octagonal numbers are those $p_8(x) = x(3x-2)$ with $x \in \mathbb{Z}$. Here is the list of such numbers up to 120:

$$0, \ 1, \ 5, \ 8, \ 16, \ 21, \ 33, \ 40, \ 56, \ 65, \ 85, \ 96, \ 120.$$

Note that $3p_8(x) + 1 = (3x-1)^2$.

Theorem (Z. W. Sun [JNT 162(2016)]). Each positive integer can be written as the sum of four generalized octagonal numbers one of which is odd.

A Lemma

Lemma 1. Any integer $n > 4$ can be written as the sum of four squares one of which is even and two of which are nonzero.

Proof. It is well-known that

$$r_4(m) = 8 \sum_{d|m \ \& \ 2\nmid d} d \quad \text{for all } m = 1, 2, 3, \ldots,$$

where

$$r_4(m) := |\{(w, x, y, z) \in \mathbb{Z}^4 : \ w^2 + x^2 + y^2 + z^4 = n\}|.$$

If $m > 1$ is an integer whose smallest prime divisor is p, then

$$r_4(m) \geqslant 8(1 + p) > 2^4$$

and hence m can be written as the sum of four squares (at least) two of which are nonzero.

By the above, we can write any integer $n > 4$ as the sum of four squares two of which are nonzero. If all the four squares are odd, then $n \equiv 4 \pmod 8$ and we can write $n/4 > 1$ in the form $w^2 + x^2 + y^2 + z^2$ with $w, x, y, z \in \mathbb{Z}$ and $wx \neq 0$, hence $n = (2w)^2 + (2x)^2 + (2y)^2 + (2z)^2$ with $2w \neq 0$ and $2x \neq 0$.

Another Lemma

Lemma 2. (i) Suppose that $x, y, z \in \mathbb{Z}$ are not all divisible by 3. Then there are $\bar{x}, \bar{y}, \bar{z} \in \mathbb{Z}$ not all divisible by 3 such that $\bar{x} \equiv x \pmod{2}$, $\bar{y} \equiv y \pmod{2}$, $\bar{z} \equiv z \pmod{2}$, and $9(x^2 + y^2 + z^2) = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$.
(ii) Suppose that x, y, z are integers with $x^2 + y^2 + z^2$ a positive multiple of 3. Then $x^2 + y^2 + z^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$ for some $\bar{x}, \bar{y}, \bar{z} \in \mathbb{Z}$ with $\bar{x} \equiv x \pmod{2}$, $\bar{y} \equiv y \pmod{2}$, $\bar{z} \equiv z \pmod{2}$, and $3 \nmid \bar{x}\bar{y}\bar{z}$.

Proof. If we apply (i) again and again we then obtain (ii).
Now we prove (i). As x, y, z are not all divisible by 3, there are $x' \in \{\pm x\}$, $y' \in \{\pm y\}$ and $z' \in \{\pm z\}$ such that $x' + y' + z' \not\equiv 0 \pmod{3}$. Let $\bar{x} = x' - 2y' - 2z'$, $\bar{y} = y' - 2x' - 2z'$ and $\bar{z} = z' - 2x' - 2y'$. Then

$$9((x')^2 + (y')^2 + (z')^2) = \bar{x}^2 + \bar{y}^2 + \bar{z}^2,$$

$$\bar{x} \equiv x \pmod{2},\ \bar{y} \equiv y \pmod{2},\ \bar{z} \equiv z \pmod{2},$$

$$\bar{x} \equiv \bar{y} \equiv \bar{z} \equiv x' + y' + z' \not\equiv 0 \pmod{3}.$$

Proof of the Theorem

For $w, x, y, z \in \mathbb{Z}$, we clearly have

$$n = w(3w-2) + x(3x-2) + y(3y-2) + z(3z-2)$$
$$\iff 3n+4 = (3w-1)^2 + (3x-1)^2 + (3y-1)^2 + (3z-1)^2.$$

If an integer m is not divisible by 3, then m or $-m$ can be written as $3x-1$ with $x \in \mathbb{Z}$. Also, $(3(1-2x)-1)^2 = 4(3x-1)^2$ for any $x \in \mathbb{Z}$. Thus, it suffices to show that $3n+4$ can be written as the sum of four squares none of which is divisible by 3 and one of which is even.

By Lemma 1, we may write $3n+4$ as $w^2+x^2+y^2+z^2$, where w, x, y, z are integers one of which is even and two of which are nonzero. Clearly, w, x, y, z cannot be all divisible by 3. Suppose that $3 \nmid w$. Note that x, y, z are not all zero and $x^2+y^2+z^2 \equiv 4 - w^2 \equiv 0 \pmod 3$. By Lemma 2(ii), $x^2+y^2+z^2 = \bar{x}^2+\bar{y}^2+\bar{z}^2$ for some $\bar{x}, \bar{y}, \bar{z} \in \mathbb{Z}$ with

$$\bar{x} \equiv x \pmod 2,\ \bar{y} \equiv y \pmod 2,\ \bar{z} \equiv z \pmod 2, \text{ and } 3 \nmid \bar{x}\bar{y}\bar{z}.$$

Note that $3n+4 = w^2+\bar{x}^2+\bar{y}^2+\bar{z}^2$ and $3 \nmid w\bar{x}\bar{y}\bar{z}$.

A conjecture

Definition. For $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, we let $r(n)$ denote the number of ways to write n as the sum of four unordered generalized octagonal numbers, and define $s(n)$ to be the number of ways to write n as the sum of four unordered generalized octagonal numbers not all even. Clearly, $r(n) \geqslant s(n)$ for all $n \in \mathbb{Z}^+$.

Conjecture (JNT 162(2016)]. Let n be a positive integer. If $r(n) = 1$, then we must have

$$3n + 4 \in \{7,\ 13,\ 19,\ 31,\ 43\} \cup E$$

where

$$E := \{2^{2k} :\ k = 2, 3, \ldots\} \cup \bigcup_{n \in \mathbb{N}} \{2^{2n+1}5,\ 2^{2n+1}11,\ 2^{2n+1}23\}.$$

If $s(n) = 1$, then we must have

$$3n+4 \in \{7,\ 13,\ 19,\ 31,\ 43,\ 4\times 7,\ 4\times 13,\ 4\times 19,\ 4\times 31,\ 4\times 43\} \cup E.$$

On universal $ap_8 + bp_8 + cp_8 + dp_8$

Theorem (Sun [JNT, 162(2016)]). (i) Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c \leqslant d$. Suppose that $ap_8 + bp_8 + cp_8 + dp_8$ is universal over $\mathbb{Z}$. Then we must have $a = 1$, and (b, c, d) is among the following 40 triples:

$$(1,1,1),\ (1,1,2),\ (1,1,3),\ (1,1,4),\ (1,2,2),\ (1,2,3),\ (1,2,4),$$
$$(1,2,5),\ (1,2,6),\ (1,2,7),\ (1,2,8),\ (1,2,9),\ (1,2,10),\ (1,2,11),$$
$$(1,2,12),\ (1,2,13),(1,3,3),\ (1,3,5),\ (1,3,6),\ (2,2,2),\ (2,2,3),$$
$$(2,2,4),\ (2,2,5),\ (2,2,6),(2,3,4),\ (2,3,5),\ (2,3,6),\ (2,3,7),$$
$$(2,3,8),\ (2,3,9),\ (2,4,4,),\ (2,4,5),\ (2,4,6),\ (2,4,7),\ (2,4,8),$$
$$(2,4,9),\ (2,4,10),\ (2,4,11),\ (2,4,12),\ (2,4,13).$$

(ii) $p_8 + bp_8 + cp_8 + dp_8$ is universal over $\mathbb{Z}$ if (b, c, d) is among the 40 triples but not among

$$(1,3,3),\ (1,3,6),\ (2,3,6),\ (2,3,7),\ (2,3,9). \qquad (*)$$

Conjecture (Sun [JNT, 162(2016)]) $p_8 + bp_8 + cp_8 + dp_8$ is universal over $\mathbb{Z}$ if (b, c, d) is among the five triples in $(*)$.

Three lemmas

Lemma 1 (Known result). A positive integer n can be written as the sum of four nonzero squares, if and only if it does not belong to the set

$$\{1, 3, 5, 9, 11, 17, 29, 41\} \cup \bigcup_{k\in\mathbb{N}} \{2\times 4^k, 6\times 4^k, 14\times 4^k\}.$$

Lemma 2. Let $w = x^2 + my^2$ be a positive integer with $m \in \{2, 5, 8\}$ and $x, y \in \mathbb{Z}$. Then $w = u^2 + mv^2$ for some integers u and v not all divisible by 3.

This is known in the case $m = 2$. The cases $m = 5$ and $m = 8$ were due to myself (cf. Sun [Sci. China Math. 2015]).

Lemma 3 (Z.-W. Sun). For any positive integer n, we can write $6n + 1$ as $x^2 + y^2 + 2z^2$, where x, y, z are integers with $2 \mid xy$ and $3 \nmid xyz$.

When $6n + 1$ is a square, in the proof I need a formula of S. Cooper and H. Y. Lam [JNT, 2013] on the number of solutions to the equation $x^2 + y^2 + 2z^2 = m^2$.

One more lemma

Lemma 4 Let $n \in \mathbb{N}$ and $r \in \{1, 3, 5, 7\}$. Let a, b, c, d be integers with

$$a \equiv 1 \pmod 2,\ b \equiv 2 \pmod 4,\ c \equiv 0 \pmod 4 \text{ and } d \equiv r \pmod 4.$$

(i) If $d \not\equiv r \pmod 8$, then for some $w \in \{a, b, c\}$ we have $n + dw^2 \neq 4^k(8m + r)$ for all $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

(ii) We have $n - dw^2 \notin S$ for some $w \in \{a, b, c\}$, where

$$S := \{8q - d :\ q \in \mathbb{Z}\} \cup \{4^k(8l + r) :\ k, l \in \mathbb{N}\}.$$

Part III. Other universal sums

Farhi's Conjecture

In 2013 Bakir Farhi [J. Integer Seq.] observed that

$$T_x = \frac{(2x+1)^2-1}{8} = \left\lfloor \frac{(2x+1)^2}{8} \right\rfloor$$

and hence any $n \in \mathbb{N}$ can be written as $\lfloor x^2/8 \rfloor + \lfloor y^2/8 \rfloor + \lfloor z^2/8 \rfloor$ with $x, y, z \in \mathbb{Z}$. Motivated by this, he investigated representations of $n \in \mathbb{N}$ by $\lfloor x^2/a \rfloor + \lfloor y^2/a \rfloor + \lfloor z^2/a \rfloor$, where $a \in \mathbb{Z}^+$.

Farhi proved that any $n \in \mathbb{N}$ with $n \not\equiv 2 \pmod{24}$ can be represented by $\lfloor x^2/3 \rfloor + \lfloor y^2/3 \rfloor + \lfloor z^2/3 \rfloor$ with $x, y, z \in \mathbb{Z}$. After this, S. Mezroui, A. Azizi and M. Ziane [J. Integer Seq. 2014] confirmed this in the remaining case $n \equiv 2 \pmod{24}$ via the known formula for $r(n) = |\{(x,y,z) \in \mathbb{Z}^3 :\ x^2+y^2+z^2 = n\}|$. Later Farhi provided an elementary proof of this.

Farhi's Conjecture (2014). For any integer $a \geqslant 3$, every $n \in \mathbb{N}$ can be represented as the sum of three elements of the set $\{\lfloor x^2/a \rfloor :\ x \in \mathbb{Z}\}$.

My general conjecture

It is known that for any $a, b, c \in \mathbb{Z}^+$ there are infinitely many positive integers not of the form $ax^2 + by^2 + cz^2$ with $x, y, z \in \mathbb{Z}$. In contrast, I have formulated the following general conjecture.

Conjecture (Z. W. Sun, 2015). Let a, b, c be positive integers with $a \leqslant b \leqslant c$.
(i) If $(a, b, c) \neq (1, 1, 1), (2, 2, 2)$, then

$$\left\{\left\lfloor \frac{x^2}{a} \right\rfloor + \left\lfloor \frac{y^2}{b} \right\rfloor + \left\lfloor \frac{z^2}{c} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

(ii) We have

$$\left\{\left\lfloor \frac{T_x}{a} \right\rfloor + \left\lfloor \frac{T_y}{b} \right\rfloor + \left\lfloor \frac{T_z}{c} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

Moreover, if the triple (a, b, c) is not among $(1, 1, 1)$, (1,1,3), (1,1,7), (1,3,3), (3,3,3), then

$$\left\{\left\lfloor \frac{x(x+1)}{a} \right\rfloor + \left\lfloor \frac{y(y+1)}{b} \right\rfloor + \left\lfloor \frac{z(z+1)}{c} \right\rfloor :\ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

Some results that I have proved

Theorem (Z. W. Sun, arXiv:1504.01608). (i) For each $m = 4, 6$, any $n \in \mathbb{N}$ can be written as $x^2 + (2y)^2 + \lfloor z^2/m \rfloor$ with $x, y, z \in \mathbb{Z}$. Also, any $n \in \mathbb{Z}^+$ can be expressed as $x^2 + y^2 + \lfloor z^2/5 \rfloor$ with $x, y, z \in \mathbb{Z}$ and $2 \nmid y$.
(ii) For any $\delta \in \{0, 1\}$, any $n \in \mathbb{Z}^+$ can be expressed as $x^2 + y^2 + \lfloor z^2/8 \rfloor$ with $x, y, z \in \mathbb{Z}$ and $y \equiv \delta \pmod 2$.
(iii) For each $m = 2, 3, 9, 21$, any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + \lfloor z^2/m \rfloor$ with $x, y, z \in \mathbb{Z}$. Also, for each $m = 3, 4, 6$ we have

$$\left\{x^2 + y^2 + \left\lfloor \frac{z(z+1)}{m} \right\rfloor : \ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

(iv) For each $m = 5, 6, 15$, we have

$$\left\{x^2 + \left\lfloor \frac{y^2}{m} \right\rfloor + \left\lfloor \frac{z^2}{m} \right\rfloor : x, y, z \in \mathbb{Z}\right\}$$
$$= \left\{\left\lfloor \frac{x^2}{m} \right\rfloor + \left\lfloor \frac{y^2}{m} \right\rfloor + \left\lfloor \frac{z^2}{m} \right\rfloor : x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

A lemma related to Huiwitz's result

Lemma 1. Suppose that $n \in \mathbb{Z}^+$ is not a power of two. Then there are $x, y, z \in \mathbb{Z}$ with $|x| < n$, $|y| < n$ and $|z| < n$ such that $x^2 + y^2 + z^2 = n^2$.

Proof. In 1907 Hurwitz showed that

$$\begin{aligned}&|\{(x,y,z) \in \mathbb{Z}^3 :\ x^2+y^2+z^2=n^2\}|\\ =&6\prod_{p>2}\left(\frac{p^{\mathrm{ord}_p(n)+1}-1}{p-1}+(-1)^{(p+1)/2}\frac{p^{\mathrm{ord}_p(m)}-1}{p-1}\right),\end{aligned}$$

where $\mathrm{ord}_p(n)$ is the order of n at the prime p. Note that

$$(\pm n)^2+0^2+0^2=0^2+(\pm n)^2+0^2=0^2+0^2+(\pm n)^2.$$

As n has an odd prime p, by Hurwitz's formula we have

$$|\{(x,y,z) \in \mathbb{Z}^3 :\ x^2+y^2+z^2=n^2\}| \geqslant 6\frac{p^{\mathrm{ord}_p(n)+1}-p^{\mathrm{ord}_p(n)}}{p-1} \geqslant 6p > 8$$

and hence there are $x, y, z \in \mathbb{Z}$ with $x^2, y^2, z^2 \neq n^2$ such that $x^2 + y^2 + z^2 = n^2$.

Two more lemmas

Lemma 2. Let m and n be integers with m^2+n^2 a positive multiple of 5. Then $m^2+n^2=x^2+y^2$ for some $x,y\in\mathbb{Z}$ with $5\nmid xy$.

Proof. Let a be the 5-adic valuation of $\gcd(m,n)$, and write $m=5^a m_0$ and $v=5^a n_0$ with $m_0,n_0\in\mathbb{Z}$ not all divisible by 5. Choose $\delta,\varepsilon\in\{\pm1\}$ such that $m_0'\not\equiv 2n_0' \pmod 5$, where $m_0'=\delta m_0$ and $n_0'=\varepsilon n_0$. Clearly, $5^2(m_0^2+n_0^2)=m_1^2+n_1^2$, where $m_1=3m_0'+4n_0'$ and $n_1=4m_0'-3n_0'$. Note that m_1 and n_1 are not all divisible by 5 since $m_1\not\equiv n_1 \pmod 5$. Continue this process, we finally write $m^2+n^2=5^{2a}(m_0^2+n_0^2)$ in the form x^2+y^2 with $x,y\in\mathbb{Z}$ not all divisible by 5. As $x^2+y^2=m^2+n^2\equiv 0 \pmod 5$, we must have $5\nmid xy$.

Lemma 3. Let $n>1$ be an integer with $n\equiv 1,6,9,14 \pmod{20}$ or $n\equiv 11,19 \pmod{40}$. Then we can write n as $5x^2+5y^2+z^2$ with $x,y,z\in\mathbb{Z}$ such that $x\not\equiv y \pmod 2$ if $n\equiv 1,6,9,14 \pmod{20}$, and $2\nmid y$ if $n\equiv 11,19 \pmod{40}$.

On $n = \lfloor x^2/5 \rfloor + \lfloor y^2/5 \rfloor + \lfloor z^2/5 \rfloor$

If $\{5n+5, 5n+6, 5n+9\} \subseteq E := \{4^k(8l+7) : \ k, l \in \mathbb{N}\}$, then $5n+6 \equiv 7 \pmod 8$ and hence $5n+9 \equiv 2 \pmod 8$ which leads a contradiction. Thus, for some $r \in \{0, 1, 4\}$ the number $5n+5+r$ is the sum of three squares. If $5(n+1)+r = m^2$ for some $m \in \mathbb{Z}^+$ which is not a power of two, then by Lemma 1 we have $5(n+1)+r = x^2+y^2+z^2$ for some $x, y, z \in \mathbb{Z}$ with $x^2, y^2, z^2 \neq 5(n+1)+r$. If $5(n+1)+r = (2^k)^2$ for some $k \in \mathbb{Z}^+$, then $r \in \{1, 4\}$, $5(n+1)+(5-r) = 4^k+5-2r \equiv 5-2r \equiv \pm 3 \pmod 8$ and hence $5(n+1)+(5-r) \notin E$. So, for a suitable choice of $r \in \{0, 1, 4\}$, we can write $5(n+1)+r = x^2+y^2+z^2$ with $x, y, z \in \mathbb{Z}$ and $x^2, y^2, z^2 \neq 5(n+1)+r$. Clearly, one of x^2, y^2, z^2, say z^2, is congruent to r modulo 5. Then x^2+y^2 is a positive multiple of 5. By Lemma 2, $x^2+y^2 = \bar{x}^2+\bar{y}^2$ for some $\bar{x}, \bar{y} \in \mathbb{Z}$ with $5 \nmid \bar{x}\bar{y}$. Without loss of generality we may assume that $\bar{x}^2 \equiv 1 \pmod 5$ and $\bar{y}^2 \equiv 4 \pmod 5$. Therefore,

$$n = \frac{\bar{x}^2-1}{5} + \frac{\bar{y}^2-4}{5} + \frac{z^2-r}{5} = \left\lfloor \frac{\bar{x}^2}{5} \right\rfloor + \left\lfloor \frac{\bar{y}^2}{5} \right\rfloor + \left\lfloor \frac{z^2}{5} \right\rfloor.$$

A conjecture involving the ceiling function

Conjecture (Sun, arXiv:1504.01608). Let $a, b, c \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c$.

(i) If the triple (a, b, c) is not among $(1,1,1), (1,1,2), (1,1,5)$, then for any $n \in \mathbb{N}$ there are $x, y, z \in \mathbb{Z}$ such that

$$n = \left\lceil \frac{x^2}{a} \right\rceil + \left\lceil \frac{y^2}{b} \right\rceil + \left\lceil \frac{z^2}{c} \right\rceil .$$

(ii) We have

$$\left\{ \left\lceil \frac{T_x}{a} \right\rceil + \left\lceil \frac{T_y}{b} \right\rceil + \left\lceil \frac{T_z}{c} \right\rceil : \ x, y, z \in \mathbb{Z} \right\} = \mathbb{N}.$$

Moreover, if the triple (a, b, c) is neither $(1,1,1)$ nor $(1,1,3)$, then for any $n \in \mathbb{N}$ there are $x, y, z \in \mathbb{Z}$ such that

$$n = \left\lceil \frac{x(x+1)}{a} \right\rceil + \left\lceil \frac{y(y+1)}{b} \right\rceil + \left\lceil \frac{z(z+1)}{c} \right\rceil .$$

Conjectures involving the floor function

Conjecture (Sun, 2015) Let α be a positive real number with $\alpha \neq 1$ and $\alpha \leqslant 1.5$. Define

$$S(\alpha) := \{x^2 + \lfloor \alpha x \rfloor : \ x \in \mathbb{Z}\}.$$

Then any positive integer can be written as the sum of three elements of $S(\alpha)$ one of which is odd.

Remark. Note that 2 cannot be written as the sum of three elements of $S(11/4)$, and 4 cannot be written as the sum of three elements of $S(8/5)$ one of which is odd.

Conjecture (Sun, 2015) Let $0 < \alpha \leqslant \beta \leqslant \gamma \leqslant 1.5$ such that two of α, β, γ are different from 1 or $\{\alpha, \beta, \gamma\} = \{1, 1/m\}$ for some $m = 2, 3, 4, \ldots$. Then any $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + \lfloor \alpha x \rfloor + \lfloor \beta y \rfloor + \lfloor \gamma z \rfloor$ with $x, y, z \in \mathbb{Z}$. In particular, if $a, b, c \in \mathbb{Z}^+$ are not all equal to one, then

$$\left\{x^2 + y^2 + z^2 + \left\lfloor \frac{x}{a} \right\rfloor + \left\lfloor \frac{y}{b} \right\rfloor + \left\lfloor \frac{z}{c} \right\rfloor : \ x, y, z \in \mathbb{Z}\right\} = \mathbb{N}.$$

Conjectures involving primes and the floor function

Conjecture (Z. W. Sun, 2015). Any integer $n > 1$ can be written as $p + \lfloor k(k+1)/4 \rfloor$, where p is a prime and k is a positive integer.

Conjecture (Z. W. Sun, 2015). Let $a, b \in \mathbb{Z}^+$ with $a + b > 2$. Then any integer $n > 2$ can be written as $\lfloor p/a \rfloor + \lfloor q/b \rfloor$ with p and q both prime.

Remark. In the case $a = b = 2$, this reduces to Goldbach's conjecture. In the case $\{a, b\} = \{1, 2\}$, this reduces to Lemoine's conjecture which states that any odd number greater than 5 can be written as $p + 2q$ with p and q both prime.

Conjecture (Z. W. Sun, 2015). Let

$$S = \left\{ \left\lfloor \frac{x}{9} \right\rfloor :\ x - 1 \text{ and } x + 1 \text{ are twin prime} \right\}$$
$$= \left\{ \left\lfloor \frac{x}{3} \right\rfloor :\ 3x - 1 \text{ and } 3x + 1 \text{ are twin prime} \right\}.$$

Then any $n \in \mathbb{Z}^+$ can be written as the sum of two distinct elements of S one of which is even.

Remark. This implies the Twin Prime Conjecture.

Some new conjectures

Conjecture (Z. W. Sun, Oct. 2, 2015). Any positive integer n can be written as $x^2+y^2+p(p\pm 1)/2$ with p prime and $x,y\in\mathbb{Z}$.
For example, 97 has a unique representation
$97=1^2+9^2+5(5+1)/2$ with 5 prime, and 538 has a unique representation $538=3^2+8^2+31(31-1)/2$ with 31 prime.
Let φ denote Euler's totient function. It is easy to see that all the numbers

$$\varphi(n^2)=n\varphi(n)\ (n=1,2,3,\ldots)$$

are pairwise distinct.
My following conjecture seems novel and curious.

Conjecture (Sun, Oct. 1, 2015). Any integer $n>1$ can be written as $x^2+y^2+\varphi(z^2)$, where x, y and z are integers with $0\leqslant x\leqslant y$ and $z>0$ such that y or z is prime.

For example, 13 has a unique representation $13=1^2+2^2+\varphi(4^2)$ with 2 prime, and 94415 has a unique representation $94415=115^2+178^2+\varphi(223^2)$ with 223 prime.

Some new conjectures

Recall that any positive integer can be represented as the sum of two squares and a positive triangular number. Below is a variant of this involving cubes.

Conjecture (Sun, Oct. 3, 2015). Any positive integer n can be written as the sum of a nonnegative cube, a square and a positive triangular number.

I have verified this for $n \leqslant 10^5$. For example, 306 has a unique representation: $306 = 1^3 + 13^2 + 16 \times 17/2$.

In contrast with Largrange's theorem, my following conjecture seems difficult.

Conjecture (Sun, Oct. 3, 2015). Any $n \in \mathbb{N}$ can be written as

$$w^2 + x^3 + y^4 + 2z^4 \quad \text{with } w, x, y, z \in \mathbb{N}.$$

For example, 1248 has a unique representation

$$1248 = 31^2 + 5^3 + 0^4 + 2 \times 3^4.$$

References

For main sources of my above conjectures, you may look at my preprints:

1. Z.-W. Sun, *On universal sums of polygonal numbers*, Sci. China Math. 58(2015), 1367–1396. arXiv:0905.0635

2. Z.-W. Sun, *A result similar to Lagrange's theorem*, J. Number Theory 16(2016), 190-211. arXiv:1503.03743

3. Z.-W. Sun, *Natural numbers represented by* $\lfloor x^2/a \rfloor + \lfloor y^2/b \rfloor + \lfloor z^2/c \rfloor$, arXiv:1504.01608, http://arxiv.org/abs/1504.01608.

3. Z.-W. Sun, *On* $x(ax+1)+y(by+1)+z(cz+1)$ *and* $x(ax+b)+y(ay+c)+z(az+d)$, arXiv:1505.03679, http://arxiv.org/abs/1505.03679.

Thank you!