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On universal sums $x(ax+b)/2+y(cy+d)/2+z(ez+f)/2$

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Abstract

As conjectured by Fermat and proved by Gauss, each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written $x(x+1)/2 + y(y+1)/2 + z(z+1)/2$ with $x, y, z \in \mathbb{N}$. Let a, b, c, d, e, f be integers with $a \geqslant c \geqslant e > 0$, $b > -a$ and $b \equiv a \pmod 2$, $d > -c$ and $d \equiv c \pmod 2$, $f > -e$ and $f \equiv e \pmod 2$. We find all candidates (a, b, c, d, e, f) for which each $n \in \mathbb{N}$ could be written $x(ax+b)/2 + y(cy+d)/2 + z(ez+f)/2$ with $x, y, z \in \mathbb{N}$, and show that some of the tuples indeed meet our purpose. When $b \in [0, a)$, $d \in [0, c)$ and $f \in [0, e)$, we investigate the universal tuples (a, b, c, d, e, f) over $\mathbb{Z}$ for which any $n \in \mathbb{N}$ can be written as $x(ax+b)/2 + y(cy+d)/2 + z(ez+f)/2$ with $x, y, z \in \mathbb{Z}$. For example, we show that any $n \in \mathbb{N}$ can be written as $x(x+1)/2 + y(3y+1)/2 + z(5z+1)/2$ with $x, y, z \in \mathbb{Z}$, and conjecture that each $n \in \mathbb{N}$ can be written as $x(x+1)/2 + y(3y+1)/2 + z(5z+1)/2$ with $x, y, z \in \mathbb{N}$.

Part I. Review of Backgrouds

Sums of four squares

Lagrange's Four-square Theorem (1770). Each $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$ can be written as the sum of four squares.

S. Ramanujan's Observation (confirmed by L.E. Dickson in 1927). There are totally 54 quadruples $(a, b, c, d) \in (\mathbb{Z}^+)^4$ with $a \leqslant b \leqslant c \leqslant d$ such that each $n \in \mathbb{N}$ can be written as $aw^2 + bx^2 + cy^2 + dz^2$ with $w, x, y, z \in \mathbb{Z}$. The 54 quadruples are

$(1,1,1,1)$, $(1,1,1,2)$, $(1,1,2,2)$, $(1,2,2,2)$, $(1,1,1,3)$, $(1,1,2,3)$,
$(1,2,2,3)$, $(1,1,3,3)$, $(1,2,3,3)$, $(1,1,1,4)$, $(1,1,2,4)$, $(1,2,2,4)$,
$(1,1,3,4)$, $(1,2,3,4)$, $(1,2,4,4)$, $(1,1,1,5)$, $(1,1,2,5)$, $(1,2,2,5)$,
$(1,1,3,5)$, $(1,2,3,5)$, $(1,2,4,5)$, $(1,1,1,6)$, $(1,1,2,6)$, $(1,2,2,6)$,
$(1,1,3,6)$, $(1,2,3,6)$, $(1,2,4,6)$, $(1,2,5,6)$, $(1,1,1,7)$, $(1,1,2,7)$,
$(1,2,2,7)$, $(1,2,3,7)$, $(1,2,4,7)$, $(1,2,5,7)$, $(1,1,2,8)$, $(1,2,3,8)$,
$(1,2,4,8)$, $(1,2,5,8)$, $(1,1,2,9)$, $(1,2,3,9)$, $(1,2,4,9)$, $(1,1,5,9)$,
$(1,1,2,10)$, $(1,2,3,10)$, $(1,2,4,10)$, $(1,2,5,10)$, $(1,1,2,11)$, $(1,2,4,11)$,
$1,1,2,12)$, $(1,2,4,12)$, $(1,1,2,13)$, $(1,2,4,13)$, $(1,1,2,14)$, $(1,2,4,14)$.

Universal sums of four mixed powers

If any $n \in \mathbb{N}$ can be written as $f(x_1, \ldots, x_n)$ with $x_1, \ldots, x_n$ in $\mathbb{N}$ (or $\mathbb{Z}$), then we say that f is *universal over* $\mathbb{N}$ (or $\mathbb{Z}$).

Theorem (Z.-W. Sun [JNT 175(2017)]) For any $a \in \{1, 4\}$ and $k \in \{4, 5, 6\}$, $aw^k + x^2 + y^2 + z^2$ is universal over $\mathbb{N}$.

Theorem (Z.-W. Sun [Nanjing Univ. J. Math. Biquarterly 34(2017)]) Let $a, b, c, d \in \mathbb{Z}^+$ with $a \leqslant b \leqslant c \leqslant d$, and let $h, i, j, k \in \{2, 3, \ldots\}$ with at most one of h, i, j, k equal to two. Suppose that $h \leqslant i$ if $a = b$, $i \leqslant j$ if $b = c$, and $j \leqslant k$ if $c = d$. If $f(w, x, y, z) = aw^h + bx^i + cy^j + dz^k$ is universal over $\mathbb{N}$, then $f(w, x, y, z)$ must be among the following 9 polynomials

$$w^2 + x^3 + y^4 + 2z^3,\ w^2 + x^3 + y^4 + 2z^4,\ w^2 + x^3 + 2y^3 + 3z^3,$$

$$w^2 + x^3 + 2y^3 + 3z^4,\ w^2 + x^3 + 2y^3 + 4z^3,\ w^2 + x^3 + 2y^3 + 5z^3,$$

$$w^2 + x^3 + 2y^3 + 6z^3,\ w^2 + x^3 + 2y^3 + 6z^4,\ w^3 + x^4 + 2y^2 + 4z^3.$$

Conjecture (Sun [Nanjing Univ. J. Math. Biquarterly 34(2017)]) All the 9 polynomials are universal over $\mathbb{N}$.

Triangular numbers

Triangular numbers are those

$$T_n = \sum_{r=0}^{n} r = \frac{n(n+1)}{2} \quad (n \in \mathbb{N}).$$

Note that

$$T_{-n-1} = \frac{(-n-1)(-n)}{2} = T_n \quad \text{for all} \quad n \in \mathbb{N}.$$

Theorem (conjectured by Fermat and proved by Gauss). Each $n \in \mathbb{N}$ can be written as $T_x + T_y + T_z$ with $x, y, z \in \mathbb{N}$.

Liouville's Theorem (Liouville, 1862). Let $a, b, c \in \mathbb{Z}^+$ and $a \leqslant b \leqslant c$. Then any $n \in \mathbb{N}$ can be written in the form $aT_x + bT_y + cT_z$ if and only if (a, b, c) is among

$$(1,1,1),\ (1,1,2),\ (1,1,4),\ (1,1,5),\ (1,2,2),\ (1,2,3),\ (1,2,4).$$

Mixed Sums of Squares and Triangular Numbers

Euler's Observation:

$$8n+1=(2x)^2+(2y)^2+(2z+1)^2$$
$$\Longrightarrow n=\frac{x^2+y^2}{2}+T_z=\left(\frac{x+y}{2}\right)^2+\left(\frac{x-y}{2}\right)^2+T_z.$$

Lionnet's Assertion (proved by Lebesgue & Réalis in 1872). Any $n\in\mathbb{N}$ is the sum of two triangular numbers and a square.

B. W. Jone and G. Pall [Acta Math. 1939]. Every $n\in\mathbb{N}$ is the sum of a square, an *even* square and a triangular number.

Theorem. (i) (Z. W. Sun, Acta Arith. 2007) Any $n\in\mathbb{N}$ is the sum of an *even* square and two triangular numbers.

(ii) (Conjectured by Z. W. Sun and proved by B. K. Oh and Sun [JNT, 2009]) Any positive integer n can be written as the sum of a square, an *odd* square and a triangular number.

Mixed Sums of Squares and Triangular Numbers

In 2005 Z. W. Sun [Acta Arith. 2007] investigated what kind of mixed sums $ax^2 + by^2 + cT_z$ or $ax^2 + bT_y + cT_z$ (with $a, b, c \in \mathbb{Z}^+$) are universal (i.e., all natural numbers can be so represented). This project was completed via three papers: Z. W. Sun [Acta Arith. 2007], S. Guo, H. Pan & Z. W. Sun [Integers, 2007], and B. K. Oh & Sun [JNT, 2009].

List of all universal $ax^2 + by^2 + cT_z$ **or** $ax^2 + bT_y + cT_z$:

$T_x + T_y + z^2,\ T_x + T_y + 2z^2,\ T_x + T_y + 4z^2,\ T_x + 2T_y + z^2,$
$T_x + 2T_y + 2z^2,\ T_x + 2T_y + 3z^2,\ T_x + 2T_y + 4z^2,\ 2T_x + T_y + z^2,$
$2T_x + 4T_y + z^2,\ 2T_x + 5T_y + z^2,\ T_x + 3T_y + z^2,\ T_x + 4T_y + z^2,$
$T_x + 4T_y + 2z^2,\ T_x + 6T_y + z^2,\ T_x + 8T_y + z^2,\ T_x + y^2 + z^2,$
$T_x + y^2 + 2z^2,\ T_x + y^2 + 3z^2,\ T_x + y^2 + 4z^2,\ T_x + y^2 + 8z^2,$
$T_x + 2y^2 + 2z^2,\ T_x + 2y^2 + 4z^2,\ 2T_x + y^2 + z^2,\ 2T_x + y^2 + 2y^2,$
$4T_x + y^2 + 2z^2.$

Polygonal Numbers

Polygonal numbers are nonnegative integers constructed geometrically from the regular polygons. For $m = 3, 4, 5, \ldots$, the m-gonal numbers are given by

$$p_m(n) = (m-2)\binom{n}{2} + n \quad (n = 0, 1, 2, \ldots).$$

Clearly

$$p_3(n) = T_n,\ p_4(n) = n^2,\ p_5(n) = \frac{3n^2 - n}{2},\ p_6(n) = 2n^2 - n = T_{2n-1}.$$

The larger m is, the more sparse m-gonal numbers are.

Fermat's Assertion (1638). Any natural number n can be written as the sum of m m-gonal numbers.

Confirmation: $m = 4$ (Lagrange 1770), $m = 3$ (Gauss 1796), $m \geqslant 5$ (Cauchy 1813).

Mixed Sums of Three Polygonal Numbers

Conjecture [Z. W. Sun, 2009]. Let $3 \leqslant i \leqslant j \leqslant k$ and $k \geqslant 5$. Then each $n \in \mathbb{N}$ can be written as the sum of an i-gonal number, a j-gonal number and a k-gonal number, if and only if (i, j, k) is among the following 31 triples:

$(3,3,5)$, $(3,3,6)$, $(3,3,7)$, $(3,3,8)$, $(3,3,10)$, $(3,3,12)$, $(3,3,17)$,
$(3,4,5)$, $(3,4,6)$, $(3,4,7)$, $(3,4,8)$, $(3,4,9)$, $(3,4,10)$, $(3,4,11)$,
$(3,4,12)$, $(3,4,13)$, $(3,4,15)$, $(3,4,17)$, $(3,4,18)$, $\underline{(3,4,27)}$,
$(3,5,5)$, $(3,5,6)$, $\underline{(3,5,7)}$, $(3,5,8)$, $(3,5,9)$, $(3,5,11)$, $(3,5,13)$,
$(3,7,8)$, $(3,7,10)$, $(4,4,5)$, $\underline{(4,5,6)}$.

Remark. Sun proved the 'only if' part. The 'if' part is difficult!

Sun [Sci. China Math. 58(2015)] also showed that there are only 95 candidates for universal sums over $\mathbb{N}$ of the form $ap_i(x) + bp_j(y) + cp_k(z)$.

On $x(ax+b)+y(ay+c)+z(az+d)$ with $x,y,z\in\mathbb{Z}$

If any $n\in\mathbb{N}$ can be written as $f(x_1,\ldots,x_n)$ with $x_1,\ldots,x_n\in\mathbb{Z}$, then we say that f is *universal over* $\mathbb{Z}$.

As $T_n=T_{-n-1}$ for all $n\in\mathbb{N}$, we see that

$$\{T_n:\ n\in\mathbb{N}\}=\{T_{2x}=x(2x+1):\ x\in\mathbb{Z}\}$$

and hence $x(2x+1)+y(2y+1)+z(2z+1)$ is universal over $\mathbb{Z}$.

Theorem (Z.-W. Sun [JNT 171(2017)]) Let $a,b,c,d\in\mathbb{N}$ with $a>2$ and $b\leqslant c\leqslant d\leqslant a$. Then $x(ax+b)+y(ay+c)+z(az+d)$ is universal over $\mathbb{Z}$ if and only if the quadruple (a,b,c,d) is among

$$(3,0,1,2),\ (3,1,1,2),\ (3,1,2,2),\ (3,1,2,3),\ (4,1,2,3).$$

On $x(ax+1)+y(by+1)+z(cz+1)$ with $x,y,z\in\mathbb{Z}$

Theorem (Z.-W. Sun [JNT 171(2017)]) (i) Let $a,b,c\in\mathbb{Z}^+$ with $a\leqslant b\leqslant c$. If $f_{a,b,c}(x,y,z):=x(ax+1)+y(by+1)+z(cz+1)$ is universal over $\mathbb{Z}$, then (a,b,c) is among the following 17 triples:

$$(1,1,2),\ (1,2,2),\ (1,2,3),\ (1,2,4),\ (1,2,5),$$
$$(2,2,2),\ (2,2,3),\ (2,2,4),\ (2,2,5),\ (2,2,6),$$
$$(2,3,3),\ (2,3,4),\ (2,3,5),\ (2,3,7),\ (2,3,8),\ (2,3,9),\ (2,3,10).$$

(ii) $f_{a,b,c}(x,y,z)$ is universal over $\mathbb{Z}$ if (a,b,c) is among

$$(1,2,3),\ (1,2,4),\ (1,2,5),\ (2,2,4),\ (2,2,5),\ (2,3,3),\ (2,3,4).$$

Conjecture (Sun). $f_{a,b,c}(x,y,z)$ is universal over $\mathbb{Z}$ if (a,b,c) is among (2,2,6), (2,3,5), (2,3,7), (2,3,8), (2,3,9), (2,3,10).

In 2017, Ju and Oh [arXiv:1701.02974] proved that

$$f_{2,2,6}(x,y,z)\ \text{ and }\ f_{2,3,c}(x,y,z)\ (c=5,7)$$

are universal over $\mathbb{Z}$. The universality of $f_{2,3,c}(x,y,z)$ over $\mathbb{Z}$ for $c=8,9,10$ remains open.

Part II. On universal sums $x(ax+b)/2+y(cy+d)/2+z(ez+f)/2$ over $\mathbb{N}$

Universal tuples (a, b, c, d, e, f) over $\mathbb{N}$

For $c \in \mathbb{Z}^+$ and $m \in \{3, 4, \ldots\}$, clearly

$$cp_m(x) = \frac{x(a_0x + b_0)}{2} \text{ with } a_0 = c(m-2) \text{ and } b_0 = -c(m-4) \in (-a, a].$$

Instead of $cp_m(x)$, we may consider more general polynomials

$$\psi_{a,b}(x) := \frac{x(ax + b)}{2} \text{ with } a \in \mathbb{Z}^+,\ b \in \mathbb{Z},\ b > -a \text{ and } a \equiv b \pmod 2.$$

Clearly, $\psi_{a,b}(\mathbb{N}) \subseteq \mathbb{N}$, $\psi_{a,a}(x) = aT_x$ and $\psi_{2a,0}(x) = ax^2$.

For positive integers a, c, e and integers $b > -a,\ d > -c,\ f > -e$ with $a + b, c + d, e + f$ all even, if

$$\psi_{a,b}(x) + \psi_{c,d}(y) + \psi_{e,f}(z) = \frac{x(ax + b)}{2} + \frac{y(cy + d)}{2} + \frac{z(ez + f)}{2}$$

is universal over $\mathbb{N}$ then we simply call the ordered tuple (a, b, c, d, e, f) *universal over* $\mathbb{N}$.

Note that those tuples (a, b, c, d, e, f) with $b(a - b) = d(c - d) = f(e - f) = 0$ have been determined.

Universal tuples (a, b, c, d, e, f) with $a \mid b$, $c \mid d$ and $e \mid f$

Sun (arXiv:1502.03056): The following tuples are universal over $\mathbb{N}$.

$(1,3,1,1,1,1)$, $(1,3,1,3,1,1)$, $(1,5,1,1,1,1)$, $(1,5,1,3,1,1)$,
$(1,7,1,1,1,1)$, $(1,7,1,3,1,1)$, $(1,9,1,1,1,1)$, $(2,0,1,3,1,1)$,
$(2,0,1,3,1,3)$, $(2,0,1,5,1,1)$, $(2,0,1,5,1,3)$, $(2,0,1,7,1,1)$,
$(2,0,1,7,1,3)$, $(2,0,1,9,1,1)$, $(2,0,1,9,1,3)$, $(2,0,1,11,1,1)$,
$(2,0,1,11,1,3)$, $(2,0,1,13,1,1)$, $(2,0,1,13,1,3)$, $(2,0,1,15,1,1)$,
$(2,0,2,0,1,3)$, $(2,2,1,3,1,1)$, $(2,2,1,5,1,1)$, $(2,2,1,7,1,1)$,
$(2,2,2,0,1,3)$, $(2,2,2,0,1,5)$, $(2,2,2,0,1,7)$, $(2,2,2,0,1,9)$,
$(2,4,1,1,1,1)$, $(2,4,2,0,1,1)$, $(2,4,2,0,1,3)$, $(2,4,2,2,1,1)$,
$(2,4,2,2,2,0)$, $(2,6,1,1,1,1)$, $(2,6,1,3,1,1)$, $(2,6,2,0,1,1)$,
$(2,6,2,0,1,3)$, $(2,6,2,2,1,1)$, $(2,6,2,2,2,0)$, $(2,8,1,1,1,1)$,
$(2,8,2,0,1,1)$, $(2,8,2,0,1,3)$, $(2,8,2,2,2,0)$, $(2,10,2,0,1,1)$,
$(2,10,2,0,1,3)$, $(2,12,2,0,1,1)$, $(2,12,2,0,1,3)$, $(2,14,2,0,1,1)$,
$(3,3,2,0,1,3)$, $(3,9,2,0,1,1)$, $(3,9,2,0,1,3)$, $(4,0,1,3,1,1)$,
$(4,0,1,5,1,1)$, $(4,0,1,7,1,1)$, $(4,4,1,3,1,1)$, $(8,0,1,3,1,1)$.

Universal tuples (a, b, c, d, e, f) with $a \mid b$, $c \mid d$ and $e \mid f$

We have the following conjecture concerning other possible universal tuples (a, b, c, d, e, f) over $\mathbb{N}$ with $a \mid b$, $c \mid d$ and $e \mid f$.

Conjecture (Sun, arXiv:1502.03056) The following 10 tuples

$$(4,0,2,0,1,3),\ (4,0,2,0,1,5),\ (4,0,2,6,1,1),\ (4,0,2,6,2,0),$$
$$(4,4,2,0,1,3),\ (4,8,2,0,1,1),\ (4,8,2,0,1,3),\ (4,12,2,0,1,1),$$
$$(6,0,2,0,1,3),\ (6,6,2,0,1,3)$$

are universal over $\mathbb{N}$.

Theorem (Sun, arXiv:1502.03056) Let a, c, e be positive integers and let $b > -a$, $d > -c$ and $f > -e$ be integers with $a+b, c+d, e+f$ all even. Suppose that $a \geqslant c \geqslant e$, and $b \geqslant d$ if $a = c$, and $d \geqslant f$ if $c = e$, and that the ordered tuple (a, b, c, d, e, f) is universal over $\mathbb{N}$. If $a \mid b$, $c \mid d$ and $e \mid f$, but $b(a-b), d(c-d), f(e-f)$ are not all zero, then (a, b, c, d, e, f) must be among the 56 tuples in the above result and the 10 tuples in the above conjecture.

A key lemma

Lemma. Let $a, b, c, d \in \mathbb{Z}$ with $a \geqslant c \geqslant 1$, $b > -a$, $d > -c$, $a \equiv b \pmod 2$ and $c \equiv d \pmod 2$. Then one of $1, \ldots, 18$ cannot be written as $\psi_{a,b}(x) + \psi_{c,d}(y)$ with $x, y \in \mathbb{N}$.

Proof. If $x(ax+b)/2 = 1$ and $y(ay+b)/2 = 2$ with $x, y \in \mathbb{N}$, then

$$(y-x)\frac{a(x+y)+b}{2} = \frac{y(ay+b)}{2} - \frac{x(ax+b)}{2} = 2 - 1 = 1,$$

hence $y = x+1$ and $ax + (a+b)/2 = 1$, thus $x = 0$ and $x(ax+b)/2 \neq 1$. So $\{1,2\} \not\subseteq \{\psi_{a,b}(x) + \psi_{c,d}(y) : \ x, y \in \mathbb{N}\}$ if $\psi_{c,d}(1) = (c+d)/2 > 2$. Similarly, if $\psi_{a,b}(1) = (a+b)/2 > 2$ then $\{1,2\} \not\subseteq \{\psi_{a,b}(x) + \psi_{c,d}(y) : \ x, y \in \mathbb{N}\}$.

Below we suppose that $a + b \leqslant 4$ and $c + d \leqslant 4$.

In the case $ac < 212$, via a computer we find that one of $1, \ldots, 9$ cannot be written as $x(ax+b)/2 + y(cy+d)/2$ with $x, y \in \mathbb{N}$. Note that 9 is the least positive integer not in the form $p_3(x) + p_5(y)$ with $x, y \in \mathbb{N}$.

A key Lemma

Now we assume that $ac \geqslant 212$. If $c = 1$, then $a \geqslant 212$ and hence

$$\frac{1}{a} + \frac{1}{c} \leqslant \frac{1}{250} + 1 = \frac{213}{212}.$$

If $c > 1$, then $a \geqslant c \geqslant 2$ and hence

$$\frac{1}{a} + \frac{1}{c} \leqslant \frac{1}{2} + \frac{1}{2} = 1 < \frac{251}{250}.$$

Fix a positive integer N. For any $x \in \mathbb{N}$, clearly

$$\frac{x(ax+b)}{2} \leqslant N \iff (2ax+b)^2 \leqslant 8aN+b^2 \iff x \leqslant \frac{\sqrt{8aN+b^2}-b}{2a}.$$

If $-a < b \leqslant a$, then

$$|\{x \in \mathbb{N} : \psi_{a,b}(x) \leqslant N\}| \leqslant 1 + \frac{\sqrt{8aN+b^2}-b}{2a} < \frac{3}{2} + \sqrt{\frac{2N}{a} + \frac{1}{4}};$$

when $b > 0$ we have

$$|\{x \in \mathbb{N} : \psi_{a,b}(x) \leqslant N\}| \leqslant 1 + \frac{\sqrt{8aN}}{2a} < \frac{3}{2} + \sqrt{\frac{2N}{a} + \frac{1}{4}}.$$

A key Lemma

Similarly,

$$|\{x \in \mathbb{N} : \ \psi_{c,d}(x) \leqslant N\}| < \frac{3}{2} + \sqrt{\frac{2N}{c} + \frac{1}{4}}.$$

Note that

$$\sqrt{u} + \sqrt{v} \leqslant \sqrt{2u + 2v} \quad \text{for all } u, v \geqslant 0.$$

Therefore

$$\begin{aligned}
&|\{\psi_{a,b}(x) + \psi_{c,d}(y) : \ x, y \in \mathbb{N}\} \cap [0, N]| \\
\leqslant &|\{x \in \mathbb{N} : \ \psi_{a,b}(x) \leqslant N\}| \times |\{y \in \mathbb{N} : \ \psi_{c,d}(y) \leqslant N\}| \\
< &\left(\frac{3}{2} + \sqrt{\frac{2N}{a} + \frac{1}{4}}\right)\left(\frac{3}{2} + \sqrt{\frac{2N}{c} + \frac{1}{4}}\right) \\
\leqslant &\frac{9}{4} + \sqrt{\frac{4N^2}{ac} + \frac{N}{2}\left(\frac{1}{a} + \frac{1}{c}\right) + \frac{1}{16}} + \frac{3}{2}\sqrt{4N\left(\frac{1}{a} + \frac{1}{c}\right) + 1} \\
\leqslant &\frac{9}{4} + \sqrt{\frac{4N^2}{212} + \frac{N}{2} \cdot \frac{213}{212} + \frac{1}{16}} + \frac{3}{2}\sqrt{4N \times \frac{213}{212} + 1}.
\end{aligned}$$

A key Lemma

Now, take $N = 18$. Then

$$\sqrt{\frac{4N^2}{212} + \frac{N}{2} \cdot \frac{213}{212} + \frac{1}{16}} + \frac{3}{2}\sqrt{4N \times \frac{213}{212} + 1} < N - \frac{5}{4}$$

and thus

$$|\{\psi_{a,b}(x) + \psi_{c,d}(y) : \ x, y \in \mathbb{N}\} \cap [0, N]| < 1 + N.$$

Therefore, one of $1, \ldots, N$ cannot be written as $\psi_{a,b}(x) + \psi_{c,d}(y)$ with $x, y \in \mathbb{N}$.

A Lemma

Lemma 1. Let $n > 1$ be an odd integer. Then there are $x, y, z \in \mathbb{N}$ for which $\max\{x, z\} > 0$, $\max\{y, z\} > 0$ and $x^2 + y^2 + 2z^2 = n^2$.

Proof. By a result of Cooper and Lam [JNT 133(2013)], we have

$$\begin{aligned}
&|\{(x,y,z) \in \mathbb{Z}^3 : \ x^2 + y^2 + 2z^2 = n^2\}| \\
=&4 \prod_{p \mid n} \frac{p^{\mathrm{ord}_p(n)+1} - 1 - (\frac{-2}{p})(p^{\mathrm{ord}_p(n)} - 1)}{p-1} \\
\geqslant&4 \prod_{p \mid n} \frac{p^{\mathrm{ord}_p(n)+1} - p^{\mathrm{ord}_p(n)}}{p-1} = 4n > 4,
\end{aligned}$$

where $\mathrm{ord}_p(n)$ is the p-adic order of n and $(\frac{\cdot}{p})$ is the Legendre symbol. So the equation $x^2 + y^2 + 2z^2 = n^2$ has integral solutions other than the 4 trivial solutions $(\pm n, 0, 0)$ and $(0, \pm n, 0)$. This proves the desired result.

Another Lemma

Lemma 2. Let $m > 3$ be an integer. Then we can write $T_m = x^2 + y^2 + T_z$ with $x, y, z \in \mathbb{N}$, $y \geqslant 3$ and $z \geqslant 1$.

The result for $m = 4, 5, 6, 7, 8$ can be easily checked. When $m = a^2$ with $a \in \{3, 4, \ldots\}$, the desired result also holds since

$$T_m = m + T_{m-1} = 0^2 + a^2 + T_{a^2-1}.$$

Now we assume that m is greater than 8 and not a square. Since

$$\mathbb{N} \setminus \{x^2 + y^2 + 2z^2 : \ x, y, z \in \mathbb{Z}\} = \{4^k(16l + 14) : \ k, l \in \mathbb{N}\}.$$

If $2m + 1$ is not a square then $2m + 1 = x^2 + y^2 + 2z^2$ for some $x, y, z \in \mathbb{N}$ with $(x^2 + z^2)(y^2 + z^2) > 0$. If $2m + 1$ is a square, then by Lemma 1 we can write $n = x^2 + y^2 + 2z^2$ with $x, y, z \in \mathbb{N}$ and $(x^2 + z^2)(y^2 + z^2) > 0$. Anyway, there are $x, y, z \in \mathbb{N}$ with $(x^2 + z^2)(y^2 + z^2) > 0$ such that $2m + 1 = x^2 + y^2 + 2z^2$. Note that $x^2 + y^2 > 1$. If $(x^2 + z^2)(y^2 + z^2) \leqslant 16$, then $xy \leqslant 4$ and $z \leqslant 2$, hence $2m + 1 = x^2 + y^2 + 2z^2 \leqslant 17$ which contradicts $m > 8$. Thus $(x^2 + z^2)(y^2 + z^2) > 16$.

Another Lemma

Observe that

$$\begin{aligned} 8T_m + 1 =& (2m+1)^2 = (x^2 + z^2 + (y^2 + z^2))^2 \\ =& (x^2 + z^2 - (y^2 + z^2))^2 + 4(x^2 + z^2)(y^2 + z^2) \\ =& (x^2 - y^2)^2 + 4((xy + z^2)^2 + (xz - yz)^2). \end{aligned}$$

Since $x \not\equiv y \pmod 2$ and $|x^2 - y^2| \geqslant x + y > 1$, we have $|x^2 - y^2| = 2w + 1$ for some $w \in \mathbb{Z}^+$. Note that $(xy + z^2)^2 + (|x - y|z)^2 = (x^2 + z^2)(y^2 + z^2) \equiv 0 \pmod 2$. Thus

$$u = \frac{xy + z^2 + |x - y|z}{2} \in \mathbb{N} \quad \text{and} \quad v = \left| \frac{xy + z^2 - |x - y|z}{2} \right| \in \mathbb{N}.$$

Since

$$u^2 + v^2 = \frac{(xy + z^2)^2 + (xz - yz)^2}{2} = \frac{(x^2 + z^2)(y^2 + z^2)}{2} > 8,$$

we have $\max\{u, v\} \geqslant 3$. Finally, $8T_m + 1 = (2w + 1)^2 + 8(u^2 + v^2)$ and hence $T_m = u^2 + v^2 + T_w$. This concludes the proof.

$(2,4,2,0,1,3)$ is universal over $\mathbb{N}$

Note that

$$\frac{x(2x+4)}{2}+\frac{y(2y+0)}{2}+\frac{z(z+3)}{2}=(x+1)^2-1+y^2+T_{z+1}-1.$$

So $(2,4,2,0,1,3)$ is universal over $\mathbb{N}$ if and only if any integer $n\geqslant 2$ can be written as $x^2+y^2+T_z$ with $x\in\mathbb{N}$ and $y,z\in\mathbb{Z}^+$.

Clearly, $T_2=3=1^2+1^2+T_1$ and $T_3=6=2^2+1^2+T_1$. If $n=T_m$ with $m>3$, then by Lemma 2 we have $n=x^2+y^2+T_z$ with $x\in\mathbb{N}$ and $y,z\in\mathbb{Z}^+$.

Now assume that $n\geqslant 2$ is not a triangular number. For $\delta=0,1$ let $r_\delta(n)$ be the number of $(x,y,z)\in\mathbb{Z}\times\mathbb{Z}\times\mathbb{N}$ for which $x^2+y^2+T_z=n$ and $x-y\equiv\delta \pmod 2$. By Sun [Acta Arith. 127(2017)], we have $r_0(n)=r_1(n)>0$. So, there are $a,b,c,u,v,w\in\mathbb{N}$ such that $n=a^2+b^2+T_c=u^2+v^2+T_w$ with $a\not\equiv b \pmod 2$ and $u\equiv v \pmod 2$, hence we cannot have $c=w=0$ since a^2+b^2 is odd and u^2+v^2 is even. Thus $n=x^2+y^2+T_z$ for some $x,y\in\mathbb{N}$ and $z\in\mathbb{Z}^+$. Since $n\neq T_z$, we have $\max\{x,y\}>0$.

$(2, 8, 2, 0, 1, 3)$ and $(2, 12, 2, 0, 1, 1)$ are universal over $\mathbb{N}$

To prove that $(2, 8, 2, 0, 1, 3)$ and $(2, 12, 2, 0, 1, 1)$ are universal over $\mathbb{N}$, we need the known list of those squarefree numbers $n \in \mathbb{Z}^+$ such that the class number $h(-n)$ of the field $\mathbb{Q}(\sqrt{-n})$. does not exceed 8, as well as the following lemma.

Lemma 3 (Gauss). Let $n \in \mathbb{N}$ and define

$$R_3(n) := |\{(x, y, z) \in \mathbb{Z}^3 : \ x^2+y^2+z^2 = n \text{ and } \gcd(x, y, z) = 1\}|.$$

Then $R_3(1) = 6$, $R_3(2) = 12$, $R_3(3) = 8$, and

$$R_3(n) = \begin{cases} 12h(-n) & \text{if } n > 3 \text{ and } n \equiv 1, 2 \pmod 4, \\ 24h(-n) & \text{if } n > 3 \text{ and } n \equiv 3 \pmod 8, \\ 0 & \text{if } 4 \mid n \text{ or } n \equiv 7 \pmod 8. \end{cases}$$

Universal tuples (a,b,c,d,e,f) with $a\nmid b$ or $c\nmid d$ or $e\mid f$

Theorem (Sun, arXiv:1502.03056) Let a,c,e be positive integers and let $b>-a$, $d>-c$ and $f>-e$ be integers with $a+b, c+d, e+f$ all even, and $a\nmid b$ or $c\nmid d$ or $e\nmid f$. Suppose that $a\geqslant c\geqslant e$, and $b\geqslant d$ if $a=c$, and $d\geqslant f$ if $c=e$, and that the ordered tuple (a,b,c,d,e,f) is universal over $\mathbb{N}$. Then (a,b,c,d,e,f) must be among our list of 407 tuples.

Let $N(m)$ be the number of tuples (a,b,c,d,e,f) on our list with $a=m$. Then the values of $N(m)$ $(m=3,\ldots,15)$ are
65, 48, 50, 65, 29, 22, 21, 29, 14, 8, 8, 8, 24 respectively.

In our list, those tuples (a,b,c,d,e,f) with $a\geqslant 16$ are as follows.

$$(16,-14,2,0,1,1),\ (16,-10,2,0,1,1),\ (16,-10,2,0,1,3),$$
$$(16,-8,3,-1,1,1),\ (16,-4,2,0,1,1),\ (17,-15,3,1,1,1),$$
$$(17,-15,3,1,1,3),\ (18,-10,2,0,1,1),\ (20,-16,2,2,1,1),$$
$$(20,-16,2,6,1,1),\ (20,-12,3,-1,1,1),\ (20,-4,2,0,1,1),$$
$$(21,-19,2,2,1,1),\ (21,-9,2,0,1,1),\ (21,-5,2,0,1,1),$$
$$(25,-23,2,0,1,1).$$

Conjecture

Conjecture (Sun, arXiv:1502.03056). All the 407 tuples (a, b, c, d, e, f) are universal over $\mathbb{N}$.

I'm unable to prove none of the 407 tuples is universal over $\mathbb{N}$ but many of the tuples can be proved to be universal over $\mathbb{Z}$.

Part III. On universal sums $x(ax+b)/2+y(cy+d)/2+z(ez+f)/2$ over $\mathbb{Z}$

Universal tuples (a, b, c, d, e, f) over $\mathbb{Z}$

For $a \in \mathbb{Z}^+$, clearly $\psi_{a,-b}(\mathbb{Z}) = \psi_{a,b}(\mathbb{Z})$ for all $b = 0, \ldots, a$ with $b \equiv a \pmod 2$, and $\psi_{a,a}(\mathbb{Z}) = \psi_{4a,2a}(\mathbb{Z})$ since $\{T_x : x \in \mathbb{Z}\} = \{x(2x+1) : x \in \mathbb{Z}\}$. Thus we are led to find all the sums

$$\psi_{a,b}(x) + \psi_{c,d}(y) + \psi_{e,f}(z) = \frac{x(ax+b)}{2} + \frac{y(cy+d)}{2} + \frac{z(ez+f)}{2} \qquad (*)$$

which are universal over $\mathbb{Z}$, where $a, c, e \in \mathbb{Z}^+$, $b, d, f \in \mathbb{N}$, $b < a$ and $a \equiv b \pmod 2$, $d < c$ and $c \equiv d \pmod 2$, and $f < e$ and $e \equiv f \pmod 2$. If the sum in $(*)$ is universal over $\mathbb{Z}$, then we also say that the ordered tuple (a, b, c, d, e, f) is universal over $\mathbb{Z}$.

Theorem (Sun, arXiv:1502.03056). Let $a, b, c, d, e, f \in \mathbb{N}$ with $a > b$, $c > d$, $e > f$, $a \equiv b \pmod 2$, $c \equiv d \pmod 2$, $e \equiv f \pmod 2$, $a \geqslant c \geqslant e \geqslant 2$, and $b \geqslant d$ if $a = c$, and $d \geqslant f$ if $c = e$. Suppose that the ordered tuple (a, b, c, d, e, f) is universal over $\mathbb{Z}$. Then (a, b, c, d, e, f) must be among the 12082 tuples listed at `http://oeis.org/A286944`.

Universal tuples (a, b, c, d, e, f) over $\mathbb{Z}$

Conjecture (Sun, arXiv:1502.03056). All the 12082 tuples listed at `http://oeis.org/A286944` are universal over $\mathbb{Z}$.

On the list of 12082 tuples, those tuples (a, b, c, d, e, f) with $a > 100$ are as follows:

$(a, a-22, 6, 2, 5, 3)$ $(a = 102, 105, 109, 110, 112, 116, 117, 121, 128)$,

$(a, a-4, 7, 5, 3, 1)$ $(a = 101, 103, 104, 105, 107, 111, 112, 114, 116, 117, 118, 119, 121, 123, 124, 127, 129, 130, 131)$,

$(a, a-4, 3, 1, 2, 0)$ $(a = 101, 102, 104, 105, 107, 111, 112, 114, 116, 120, 122, 123, 126, 128, 129, 130, 132, 133)$,

$(a, a-38, 7, 1, 3, 1)$ $(a = 102, 103, 104, 105, 106, 108, 111, 115, 117, 118, 119)$,

$(a, a-28, 7, 1, 3, 1)$ $(a = 101, 103, 104, 105, 107, 108, 109, 110, 112, 114, 116, 117, 118, 119, 120, 122, 125, 126, 127, 130, 133, 134, 137, 139, 140, 142, 143, 145, 146, 151, 153, 155, 158, 160, 161, 163, 164, 165, 170, 171)$.

Conjectural universal sums $ax^2 + by^2 + z(cz + d)/2$ over $\mathbb{Z}$

Conjecture (Sun, arXiv:1502.03056). The following polynomials are universal over $\mathbb{Z}$.

$$x^2+y^2+\frac{z(5z+1)}{2},\ x^2+y^2+\frac{z(5z+3)}{2},\ x^2+y^2+\frac{z(9z+5)}{2},$$

$$x^2+y^2+z(5z+3),\ x^2+2y^2+\frac{z(5z+1)}{2},\ x^2+2y^2+\frac{z(5z+3)}{2},$$

$$x^2+2y^2+\frac{z(7z+1)}{2},\ x^2+2y^2+\frac{z(7z+3)}{2},\ x^2+2y^2+z(4z+3),$$

$$x^2+2y^2+\frac{z(9z+1)}{2},\ x^2+2y^2+z(5z+1),\ x^2+2y^2+z(5z+2),$$

$$x^2+2y^2+z(5z+4),\ x^2+2y^2+\frac{z(13z+11)}{2},\ x^2+2y^2+z(7z+3),$$

$$x^2+2y^2+\frac{z(15z+7)}{2},\ 2x^2+2y^2+\frac{z(5z+3)}{2},\ x^2+3y^2+z(3z+1),$$

$$2x^2+3y^2+\frac{z(3z+1)}{2},\ x^2+4y^2+\frac{z(3z+1)}{2},\ x^2+4y^2+z(5z+3),$$

$$2x^2+4y^2+\frac{z(3z+1)}{2},\ 3x^2+4y^2+\frac{z(3z+1)}{2},\ x^2+5y^2+\frac{z(3z+1)}{2},$$

Conjectural universal sums $ax^2 + by^2 + z(cz + d)/2$ over $\mathbb{Z}$

$$x^2 + 5y^2 + \frac{z(5z+1)}{2},\ x^2 + 6y^2 + z(3z+1),\ x^2 + 6y^2 + \frac{z(3z+1)}{2},$$
$$2x^2 + 6y^2 + \frac{z(3z+1)}{2},\ x^2 + 7y^2 + \frac{z(3z+1)}{2},\ x^2 + 7y^2 + z(3z+1),$$
$$x^2 + 7y^2 + \frac{z(7z+3)}{2},\ x^2 + 8y^2 + \frac{z(3z+1)}{2},\ x^2 + 10y^2 + \frac{z(3z+1)}{2},$$
$$x^2 + 11y^2 + \frac{z(3z+1)}{2},\ x^2 + 15y^2 + \frac{z(3z+1)}{2}.$$

Remark. I have shown that the following sums are universal over $\mathbb{Z}$.

$$x^2 + y^2 + \frac{z(3z+1)}{2},\ x^2 + y^2 + z(3z+r)\ (r = 1, 2),$$
$$x^2 + y^2 + z(4z+r)\ (r = 1, 3),\ x^2 + y^2 + 2z(3z+2),$$
$$x^2 + 2y^2 + c\frac{z(3z+1)}{2}\ (c = 1, 2, 4),\ x^2 + 3y^2 + \frac{z(3z+1)}{2},$$
$$x^2 + 3y^2 + z(3z+2),\ 2x^2 + 3y^2 + z(3z+2),\ 2x^2 + 6y^2 + \frac{z(3z+1)}{2}.$$

Progress on the 12082 candidates

Sun (arXiv:1502.03056): Among those tuples (a, b, c, d, e, f) with $a \leqslant 5$ of the 12082 candidates, only the 10 tuples

$$(5,1,2,0,2,0),\ (5,3,2,0,2,0),\ (5,1,4,0,2,0),\ (5,3,4,0,2,0),$$
$$(5,1,4,0,3,1),\ (5,3,4,0,3,1),\ (5,1,5,1,2,0),\ (5,3,5,3,2,0),$$
$$(5,3,4,0,4,0),\ (5,3,5,3,4,0)$$

have *not yet* proved to be universal over $\mathbb{Z}$. Also, the tuple $(5,3,5,3,4,0)$ is universal over $\mathbb{Z}$ under the GRH.

After the work of **Hai-Liang Wu and Sun (arXiv:1707.06223)**, among those tuples $(6, b, c, d, e, f)$ of the 12082 candidates, only the 13 tuples

$$(6,0,5,1,4,2),\ (6,0,5,3,4,2),\ (6,2,5,3,4,0),\ (6,2,5,3,5,3),$$
$$(6,2,6,0,5,3),\ (6,2,6,2,5,3),\ (6,4,5,1,4,0),\ (6,4,5,1,5,1),$$
$$((6,4,5,3,2,0),\ (6,4,5,3,4,0),\ (6,4,5,3,5,3),\ (6,4,6,0,5,1),$$
$$(6,4,6,0,5,3)$$

have *not yet* proved to be universal over $\mathbb{Z}$.

Progress on the little 1-3-5 conjecture

Little 1-3-5 Conjecture (Sun, arXiv:1502.03056; Prize \$135). The tuple $(5,1,3,1,1,1)$ is universal over $\mathbb{N}$, i.e., any $n \in \mathbb{N}$ can be written as

$$\frac{x(x+1)}{2} + \frac{y(3y+1)}{2} + \frac{z(5z+1)}{2} \quad \text{with } x, y, z \in \mathbb{N}.$$

Remark. This is different from the following 1-3-5 conjecture whose integral version was proved Hai-Liang Wu and Sun (arXiv:1710.08763) for sufficiently large integers $n \not\equiv 0 \pmod{16}$.

1-3-5 Conjecture (Sun, JNT 175(2017); Prize \$1350). Each $n \in \mathbb{N}$ can be written as $x^2 + y^2 + z^2 + w^2$ with $x, y, z, w \in \mathbb{N}$ such that $x + 3y + 5z$ is a square.

Though we cannot solve the little 1-3-5 conjecture, we are able to show its integral version.

Theorem (Sun, arXiv:1502.03056). Any $n \in \mathbb{N}$ can be written as $x(x+1)/2 + y(3y+1)/2 + z(5z+1)/2$ with $x, y, z \in \mathbb{Z}$.

Some lemmas

Lemma 1. Let f be an integral quadratic form with nonzero determinant. If an integer m is represented by f over the field of real numbers as well as the ring $\mathbb{Z}_p$ of p-adic integers for each prime p, then m is represented over $\mathbb{Z}$ by some form f^* in the same genus as f.

This is a well known result in the theory of quadratic forms.

Lemma 2. Let $w = 3u^2 + 5v^2 \in \mathbb{Z}^+$ with $u, v \in \mathbb{Z}$ and $8 \mid w$. Then $w = 3x^2 + 5y^2$ for some odd integers x and y.

Proof of Lemma 2

Proof. Let $k = \mathrm{ord}_2 \gcd(u, v)$ and write $u = 2^k u_0$ and $v = 2^k v_0$ with $u_0, v_0 \in \mathbb{Z}$ not all even. If $k \in \{0, 1\}$, then u_0 and v_0 are both odd since $8 \mid w$. If $u_0 \not\equiv v_0 \pmod 2$, then $k \geqslant 2$ and $4^2(3u_0^2 + 5v_0^2) = 3u_2^2 + 5v_2^2$ with $u_2 = u_0 - 5v_0$ and $v_2 = 3u_0 + v_0$ both odd.

Let $j \in \mathbb{N}$. If $4^j(3u_0^2 + 5v_0^2)$ can be written as $3u_j^2 + 5v_j^2$ with u_j and v_j odd, then we may assume $u_j \not\equiv v_j \pmod 4$ without loss of generality, hence

$$4^{j+1}(3u_0^2 + 5v_0^2) = 4(3u_j^2 + 5v_j^2) = 3u_{j+1}^2 + 5v_{j+1}^2$$

with $u_{j+1} = (v_j - u_j)/2 + 2v_j$ and $v_{j+1} = (v_j - u_j)/2 + 2u_j$ both odd.

By the above, $w = 4^k(3u_0^2 + 5v_0^2) = 3u_k^2 + 5v_k^2$ for some odd integers u_k and v_k. This ends our proof.

Proof of the Theorem

Fix a nonnegative integer n. It is easy to see that

$$n = \frac{x(x+1)}{2} + \frac{y(3y+1)}{2} + \frac{z(5z+1)}{2}$$
$$\iff 120n + 23 = 15(2z+1)^2 + 5(6y+1)^2 + 3(10z+1)^2.$$

There are two classes in the genus of $3x^2 + 5y^2 + 15z^2$, and the one not containing $3x^2 + 5y^2 + 15z^2$ has the representative

$$\begin{aligned} 2x^2 + 8y^2 + 15z^2 - 2xy =& 3\left(\frac{x}{2} + y\right)^2 + 5\left(\frac{x}{2} - y\right)^2 + 15z^2 \\ =& 3\left(\frac{x-3y}{2}\right)^2 + 5\left(\frac{x+y}{2}\right)^2 + 15z^2. \end{aligned} \quad (*)$$

If $120n + 23 = 2x^2 + 8y^2 + 15z^2 - 2xy$ for some $x, y \in \mathbb{Z}$ with $2 \nmid x$ and $y \not\equiv x \pmod 2$, then $23 \equiv 2x^2 + 15z^2 \equiv 17 \pmod 4$ which is impossible. Thus, in view of $(*)$ and Lemma 1, there are $x, y, z \in \mathbb{Z}$ such that $120n + 23 = 3x^2 + 5y^2 + 15z^2$.

Proof of the Theorem

If $2 \nmid x$, then $5(y^2+3z^2) \equiv 23-3x^2 \equiv 20 \pmod 8$ and hence $y^2+3z^2 = s^2+3t^2$ for some odd integers s and t. If $2 \nmid z$, then $3x^2+5y^2 \equiv 23-15z^2 \equiv 0 \pmod 8$ and hence $3x^2+5y^2 = 3u^2+5v^2$ for some odd integers u and v (by Lemma 2). If x and z are both even, then $y^2 \equiv 5y^2 \equiv 23 \equiv 3 \pmod 4$ which is impossible. Therefore x, y, z are all odd.

Since $3x^2 \equiv 23 \equiv 3 \pmod 5$, x or $-x$ is congruent to 1 modulo 10. As $y \not\equiv 0 \pmod 3$, y or $-y$ is congruent to 1 modulo 6. Thus, for some $u, v, w \in \mathbb{Z}$ we have

$$120n+23 = 3(10w+1)^2+5(6v+1)^2+15(2u+1)^2$$

and hence

$$n = \frac{u(u+1)}{2}+\frac{v(3v+1)}{2}+\frac{w(5w+1)}{2}.$$

This concludes the proof.

Joint work with Hai-Liang Wu

In the preprint by Hai-Liang Wu and Sun (arXiv:1707.06223), we prove that 47 tuples on Sun's 12082 listed tuples are indeed universal over $\mathbb{Z}$.

For example, we show that the sum $x^2 + T_y + 2z(4z+1)$ (corresponding to the tuple $(16,4,4,2,2,0)$) is universal over $\mathbb{Z}$, this is related to the sophisticated form $x^2 + y^2 + 32z^2$.

References

For main sources of the work introduced here, you may look at two preprints:

1. Zhi-Wei Sun, *On universal sums* $x(ax+b)/2+y(cy+d)/2+z(ez+f)/2$, http://arxiv.org/abs/1502.03056.

2. Hai-Liang Wu and Zhi-Wei Sun, *Some universal quadratic sums over the integers*, http://arxiv.org/abs/1707.06223.

Thank you!