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Write $n = k + m$ with $f(k, m)$ prime

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Abstract

Goldbach's conjecture asserts that any integer $n > 1$ can be written as $k + m$ (with k and m nonnegative integers) such that $n - k = m$ and $n + k = 2k + m$ are both prime. So it is natural to consider representation problems of the following form: Write $n = k + m$ with k and m positive integers such that $f(k, m)$ is prime, where f is a suitable function. In this talk we introduce the speaker's various conjectures in this area, one of which states that any integer $n > 1$ can be written as $k + m$ with $2^k + m$ prime, where k and m are positive integers. This talk can be easily understood, but the problems might be very challenging.

Part I. Some Classical Problems and Their Variants

Goldbach's conjecture and the twin prime conjecture

Goldbach's Conjecture (1742): Every even number $n > 2$ can be written in the form $p + q$ with p and q both prime.

Goldbach's Weak Conjecture [proved by I. M. Vinorgradov (1937) and H. Helfgott (2013)]. Each odd number $n > 6$ can be written as a sum of three primes.

Remark. Goldbach's conjecture implies that for any $n > 2$ there is a prime $p \in [n, 2n]$ since $2n \neq p + q$ if p and q are smaller than n.

Twin Prime Conjecture: There are infinitely many primes p with $p + 2$ also prime.

Remark. In 2013 Yitang Zhang proved that for some positive even number $d < 7 \times 10^7$ there are infinitely many primes p with $p + d$ also prime. Schinzel's Hypothesis is a very general extension of the twin prime conjecture.

Conjectures for twin primes, cousin primes and sexy primes

Conjecture (Sun, 2012-12-22) Any integer $n \geqslant 12$ can be written as $p+q$ with $p, p+6, 6q \pm 1$ all prime.

Remark. I have verified this for n up to 10^9.

Conjecture (2013-01-03) Let

$$\begin{aligned} A =& \{x \in \mathbb{Z}^+ :\ 6x-1 \text{ and } 6x+1 \text{ are both prime}\},\\ B =& \{x \in \mathbb{Z}^+ :\ 6x+1 \text{ and } 6x+5 \text{ are both prime}\},\\ C =& \{x \in \mathbb{Z}^+ :\ 2x-3 \text{ and } 2x+3 \text{ are both prime}\}. \end{aligned}$$

Then

$$A+B = \{2,3,\ldots\},\ B+C = \{5,6,\ldots\},\ \ A+C = \{5,6,\ldots\}\setminus\{161\}.$$

Also, if we set $2X := X + X$ then

$$2A \supseteq \{702,703,\ldots\},\ 2B \supseteq \{492,493,\ldots\},\ 2C \supseteq \{4006,4007,\ldots\}.$$

Refining Goldbach's conjecture for even numbers

Though Goldbach's conjecture is very difficult, it is actually *too weak*!

Conjecture (Sun, 2013-10-12) For any integer $n > 4$ not equal to 76, we can write $2n$ as $p + q$ with $p, 3p - 10, q, 3q - 10$ all prime.

Conjecture (Sun, 2013-10-12) Any even number greater than 6 can be written as $p + q$ with $p, q, 3p - 10, 3q + 10$ all prime.

Remark. If $2n = p + q$, then $6n = (3p - 10) + (3q + 10)$. We have verified the conjecture for $n \leqslant 10^8$.

A new conjecture with prize

Conjecture (Sun, 2013-10-16). Any integer $n > 3$ can be written as $p + q$ ($q \in \mathbb{Z}^+$) with p, $2p^2 - 1$ and $2q^2 - 1$ all prime.

Examples. Each of 7, 12, 68 and 330 has a unique required representation:

$$7 = 3 + 4,\ 2 \cdot 3^2 - 1 = 17,\ 2 \cdot 4^2 - 1 = 31;$$
$$12 = 2 + 10,\ 2 \cdot 2^2 - 1 = 7,\ 2 \cdot 10^2 - 1 = 199;$$
$$68 = 43 + 25,\ 2 \cdot 43^2 - 1 = 3697,\ 2 \cdot 25^2 - 1 = 1249;$$
$$330 = 7 + 323,\ 2 \cdot 7^2 - 1 = 97,\ 2 \cdot 323^2 - 1 = 208657.$$

Prize. I'd like to offer 1000 US dollars for a proof.

A conjecture stronger than Goldbach's weak conjecture

Recall that Goldbach's weak conjecture is now a theorem! But the following stronger conjecture seems quite challenging!

Conjecture (Sun, 2013-10-21) Let

$$S = \{n \in \mathbb{Z}^+ :\ 2n+1 \text{ and } 2n^3+1 \text{ are both prime}\}.$$

Then any integer $n > 2$ is a sum of three elements of S.

Another Conjecture (Sun, 2013-10-22) Any integer $n > 5$ can be written $a + b + c$ $(a, b, c \in \mathbb{Z}^+)$ such that

$$\{a^2+a\pm 1\},\ \{b^2+b\pm 1\},\ \{c^2+c\pm 1\}$$

are twin prime pairs!

Sums of three sparse primes

Conjecture (Sun, 2013-10-11) Let

$$P = \{p : \ p, \ p+6, \ 3p+8 \text{ are all prime}\}.$$

Then, for any integer $n > 6$, we can write $2n+1 = p+q+r$ with $p, q, r \in P$. Moreover, we may require additionally that $p+q+9$ is prime.

Remark. This implies not only Goldbach's weak conjecture but also Goldbach's conjecture for even numbers.

Example. 37 has a unique required representation: $7+13+17$.

$$\begin{aligned} &7, \ 7+6=13, \ 3\times 7+8=29 \text{ are prime;}\\ &13, \ 13+6=19, \ 3\times 13+8=47 \text{ are prime;}\\ &17, \ 17+6=23, \ 3\times 17+8=59 \text{ are prime;}\\ &7+13+9=29 \ \text{ is prime.}\end{aligned}$$

Sums of four sparse primes

Conjecture (Sun, 2013-10-12) Let

$$S = \{p : \ p, \, 3p-4, \, 3p-10, \, 3p-14 \text{ are all prime}\}.$$

Then, for any integer $n > 17$, we can write $2n = p + q + r + s$ with $p, q, r, s \in S$.

Remark. Each such a representation involves 16 primes!

Example. 54 has a unique required representation:
$7 + 11 + 17 + 19$.

$7, \ 3 \cdot 7 - 4 = 17, \ 3 \cdot 7 - 10 = 11, \ 3 \cdot 7 - 14 = 7$ are prime;

$11, \ 3 \cdot 11 - 4 = 29, \ 3 \cdot 11 - 10 = 23, \ 3 \cdot 11 - 14 = 19$ are prime;

$17, \ 3 \cdot 17 - 4 = 47, \ 3 \cdot 17 - 10 = 41, \ 3 \cdot 17 - 14 = 37$ are prime;

$19, \ 3 \cdot 19 - 4 = 53, \ 3 \cdot 19 - 10 = 47, \ 3 \cdot 19 - 14 = 43$ are prime.

Part II. Write $n = x + y$ with $P(x, y)$ Prime

Ming-Zhi Zhang's problem and A. Murthy's conjecture

Ming-Zhi Zhang's Problem (1990s) Whether any odd integer $n > 1$ can be written as $a + b$ with $a, b \in \mathbb{Z}^+$ and $a^2 + b^2$ prime ? (Cf. http://oeis.org/A036468)

Murthy's Conjecture (i) (2001) For any integer $n > 1$ there is an integer $0 < k < n$ such that $kn + 1$ is prime.

(ii) (2005) Any integer $n > 3$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $xy - 1$ prime.

Conjecture (Sun, 2012-12-20) (i) For any integer $n > 3$, there is an integer $k \in \{1, \ldots, n\}$ such that $kn + 1$ and $k(n - k) - 1$ are both prime.

(ii) For any odd integer $n > 1$, there is an integer $k \in \{1, \ldots, n\}$ such that $kn + 1$ and $k^2 + (n - k)^2$ are both prime.

Conjectures stronger than Zhang's conjecture

Conjecture (i) (Sun, 2012-12) Any even number greater than one can be written as $p + q$, where p is a Sophie Germain prime (i.e., $2p + 1$ is also prime), q is a positive integer, and $(p-1)^2 + q^2$ is prime.

(ii) (2013-10-12) For any positive integer $n \neq 1, 16, 292$, we can write $2n = p + q$ with p, q and $(p-1)^2 + q^2$ all prime.

Remark. Part (ii) unifies Goldbach's conjecture and Zhang's conjecture.

Conjecture (Sun, 2013-11-21). Any odd number $n > 1$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) such that $x^2 + y^2$ and $x^3 + y^2$ are both prime.

Example. $31 = 25 + 6$. Both $25^2 + 6^2 = 661$ and $25^3 + 6^2$ are prime.

Conjectures stronger than Murthy's conjecture

Conjecture (Sun, 2013-10-13) For any integer $n > 4$, there is a prime $p < n$ such that $3p + 8$ and $(p-1)n + 1$ are both prime.

Remark. This implies Murthy's conjecture that for any integer $n > 1$ there is a positive integer $k < n$ with $kn + 1$ prime.

Conjecture (Sun, 2013-10-13) Any integer $n > 5$ can be written as $p + q$ with p, $3p - 10$ and $(p-1)q - 1$ all prime, where q is a positive integer.

Remark. This implies Murthy's conjecture that any integer $n > 3$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $xy - 1$ prime.

Conjecture (Olivier Gerard and Z.-W. Sun). Any even number greater than two can be written as $p + q$ with p, q and $(p-1)(q+1) - 1$ all prime.

Remark. Note that $(p-1) + (q+1) = p + q$.

New-type conjectures

Conjecture (Sun, 2012-11). Any positive integer $n > 7$ can be written as $p + q$ ($q \in \mathbb{Z}^+$) with p and $2pq + 1$ both prime.

Conjecture (Sun, 2013-10-14). Any integer $n > 3$ can be written as $p + q$ with p and $(p + 1)q/2 + 1$ both prime, where q is a positive integer.

Example. 30 has a unique representation $2 + 28$. Note that $(2 + 1)28/2 + 1 = 43$ is a prime.

Remark. We have verified both conjectures for n up to 10^8.

Conjecture (Sun, 2013-11-12). Any integer $n > 4$ can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $x^2y + 1$ (or $x^2y - 1$ or $\mathrm{lcm}[x, y] + 1$ or $\mathrm{lcm}[x, y] - 1$) prime.

Refining Bertrand's Postulate

Bertrand's Postulate (proved by Chebyshev in 1850). For any positive integer n, the interval $[n, 2n]$ contains at least a prime.

Goldbach's conjecture asserts that for any integer $n > 1$, there is an integer $k \in \{0, \ldots, n\}$ such that $n - k$ and $n + k$ are both prime.

Conjecture (Sun, 2012-12-18). For each positive integer n, there is an integer $k \in \{0, \ldots, n\}$ such that $n + k$ and $n + k^2$ are both prime.

Conjecture (Sun, 2013-04-15). For any positive integer n there is a positive integer $k \leqslant 4\sqrt{n+1}$ such that $n^2 + k^2$ is prime.

Conjecture (Sun, 2013-10-15). (i) For any integer $n > 5$, there is a prime p with $p + 6$ and $n + (n - p)^2$ both prime.

(ii) For any integer $n > 3$, there is a prime $p < n$ with $3p - 4$ and $n^2 + (n - p)^2$ both prime.

My conjecture on Heath-Brown primes

Heath-Brown's Theorem (2001). There are infinitely many primes of the form $x^3 + 2y^3$ where x and y are positive integers.

Conjecture (Sun, 2012-12-14). Any positive integer n can be written as $x + y$ ($x, y \in \mathbb{N} = \{0, 1, \ldots\}$) with $x^3 + 2y^3$ prime. In general, for each positive *odd* integer m, any sufficiently large integer can be written as $x + y$ ($x, y \in \mathbb{N}$) with $x^m + 2y^m$ prime.

Conjecture (Sun, 2013-04-15) For any integer $n > 4$, there is a positive integer $k < n$ such that $p = 2n + k$ and $2n^3 + k^3 = 2n^3 + (p - 2n)^3$ are both prime.

My conjecture on primes of the form $x^m + 3y^m$

Conjecture (Sun, 2012-12-16). Let m be a positive integer. Then, any sufficiently large odd integer n can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $x^m + 3y^m$ prime (and any sufficiently large even integer n can be written as $x + y$ ($x, y \in \mathbb{Z}^+$) with $x^m + 3y^m + 1$ prime). In particular, if $m \leqslant 6$ or $m = 18$, then each positive odd integer can be written as $x + y$ ($x, y \in \mathbb{N}$) with $x^m + 3y^m$ prime.

Example. 5 can be written as $1 + 4$ with

$$1^{18} + 3 \times 4^{18} = 206158430209$$

prime.

My conjecture on primes of the form $x^2 + ny^2$

Conjecture (Sun, 2013-11-20). Any integer $n > 1$ can be written as $x + y$ with $x, y \in \mathbb{Z}^+$ such that $x + ny$ and $x^2 + ny^2$ are both prime.

Example. Clearly $20 = 11 + 9$. Note that

$$11 + 20 \times 9 = 191$$

and

$$11^2 + 20 \times 9^2 = 121 + 20 \times 81 = 1741$$

are both prime.

Conjecture (Sun, 2013-11-20). Any integer $n > 2$ can be written as $p + q$ ($q \in \mathbb{Z}^+$) with p and $p^3 + nq^2$ both prime.

Example. $10 = 7 + 3$ with $7^3 + 10 \times 3^2 = 433$ prime.

A general hypothesis on representations

Let's recall

Schinzel's Hypothesis H. If $f_1(x), \ldots, f_k(x)$ are irreducible polynomials with integer coefficients and positive leading coefficients such that there is no prime dividing the product $f_1(q)f_2(q)...f_k(q)$ for all $q \in \mathbb{Z}$, then there are infinitely many $n \in \mathbb{Z}^+$ such that $f_1(n), f_2(n), \ldots, f_k(n)$ are all primes.

The following general hypothesis is somewhat similar to Schinzel's hypothesis.

General Conjecture on Representations (Sun, 2012-12-28) Let $f_1(x, y), \ldots, f_m(x, y)$ be non-constant polynomials with integer coefficients. Suppose that for large $n \in \mathbb{Z}^+$, those $f_1(x, n-x), \ldots, f_m(x, n-x)$ are irreducible, and there is no prime dividing all the products $\prod_{k=1}^m f_k(x, n-x)$ with $x \in \mathbb{Z}$. If $n \in \mathbb{Z}^+$ is large enough, then we can write $n = x + y$ $(x, y \in \mathbb{Z}^+)$ such that $|f_1(x, y)|, \ldots, |f_m(x, y)|$ are all prime.

Examples illustrating the general hypothesis

(i) Goldbach's conjecture says that any integer $n > 2$ can be written as $x + y (x, y > 0)$ with $2x + 1$ and $2y - 1$ both prime. (Note that $2(x + y) = (2x + 1) + (2y - 1)$.)

(ii) Each $n > 1$ can be written as $x + y$ $(x, y > 0)$ with $x^3 + 2y^3$ prime. But, for any integer $d > 2$, not every sufficiently large n can be written as $x + y$ $(x, y > 0)$ with $x^3 + dy^3$ prime. For, if n is a multiple of a prime divisor p of $d - 1$, then

$$x^3 + d(n - x)^3 \equiv (1 - d)x^3 \equiv 0 \pmod{p}.$$

(iii) Any even integer $2n > 2$ can be written as $p + q$ $(q > 0)$, where p a Sophie Germain prime and $(p - 1)^2 + q^2$ is also prime. This can be restated as follows: Any integer $n > 1$ can be written as $x + y$ $(x, y > 0)$ with

$$2x + 1,\ 2(2x + 1) = 4x + 3,\ (2x)^2 + (2y - 1)^2$$

all prime.

Two curious conjectures on primes and squares

Conjecture (Sun, 2012-12-09). Any integer $n > 2$ can be written as $x^2 + y$ ($x, y \in \mathbb{Z}^+$) with $2xy - 1$ prime. In other words, for each $n = 3, 4, \ldots$ there is a prime in the form $2k(n - k^2) - 1$ with $k \in \mathbb{Z}^+$.

Remark. I have verified the conjecture for n up to 3×10^9.

Conjecture (Sun, 2013-10-15). Any integer $n > 1$ can be written as $x^2 + y$ with $2y^2 - 1$ prime, where $x, y \in \mathbb{N} = \{0, 1, 2, \ldots\}$. In other words, for each $n = 2, 3, 4, \ldots$ there is an integer $0 \leqslant k \leqslant \sqrt{n}$ such that $2(n - k^2)^2 - 1$ is prime.

Remark. I have verified the conjecture for n up to 10^8. For example, $9 = 1^2 + 8$ with $2 \times 8^2 - 1 = 127$ prime.

Part III. Write $n = k + m$ with $f(k, m)$ Prime

Conjectures involving triangular numbers

A *triangular number* is an integer of the form $T_x = x(x+1)/2$, where x is a nonnegative integer.

Conjecture (Sun, 2008). (i) Any positive integer $n \neq 216$ can be written as $p + T_x$, where p is a prime or zero.
(ii) Any odd number $n > 3$ can be written in the form $p + 2T_x = p + x(x+1)$, where p is a prime and x is a positive integer.

Conjecture (Sun, 2013-11-11). (i) Any integer $n > 1$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $T_x + y^2$ prime.
(ii) Each $n = 2, 3, \ldots$ can be written as $x + y$ $(x, y \in \mathbb{Z}^+)$ with $x(x+1) + F_y$ prime, where $\{F_k\}_{k \geqslant 0}$ is the Fibonacci sequence defined by

$$F_0 = 0,\ F_1 = 1,\ F_{k+1} = F_k + F_{k-1}\ (k = 1, 2, 3, \ldots).$$

Example. $6 = 2 + 4$ with $T_2 + 4^2 = 19$ prime. $30 = 8 + 22$ with $8 \times 9 + F_{22} = 17783$ prime.

Write $n = k + m$ with $2^k + m$ prime

Conjecture (Sun, 2013-11-11). Any integer $n > 1$ can be written as $k + m$ ($k, m \in \mathbb{Z}^+$) with $2^k + m$ prime. In other words, for each $n = 2, 3, \ldots$ there is a positive integer $k < n$ with $n - k + 2^k$ prime.

Remark. I have verified this for $n \leqslant 2000000$ except for $n = 1657977$ for which the least $k \in \mathbb{Z}^+$ with $n - k + 2^k$ prime is greater than 200000.

Examples.

$$8 = 3 + 5 \text{ with } 2^3 + 5 = 13 \text{ prime.}$$

$$64 = 13 + 51 \text{ with } 2^{13} + 51 = 8243 \text{ prime.}$$

$$421801 = 149536 + 272265 \text{ with } 2^{149536} + 272265 \text{ prime.}$$

Conjecture (Sun, 2013-11-13). For any integer $n > 3$, there is a positive integer $k < n$ with $n + k + 2^k$ prime.

Remark. I have verified this for $n \leqslant 2000000$.

Write $n = p + (2^k - k) + (2^m - m)$

In 1971, R. Crocker [Pacific J. Math.] proved that there are infinitely many positive odd numbers not of the form $p + 2^k + 2^m$, where p is a prime, and k and m are positive integers.

In contrast, I made the following conjecture:

Conjecture (Sun, 2013-11-23) Any integer $n > 3$ can be expressed in the form $p + (2^k - k) + (2^m - m)$, where p is a prime, and k and m are positive integers.

Remark. I verified this for n up to 2×10^8, and later Qing-Hu Hou verified this for n up to 10^{10}.

Theorem. (Sun, arXiv:1312.1166) For any integers a and $m > 0$, the set $\{a^n - n : \ n = 1, \ldots, m^2\}$ contains a complete system of residues modulo m. We may also replace $a^n - n$ by $a^n + n$.

New Diophantine equations

Conjecture (Sun, 2013-11-28). The diophantine equation

$$x^n + n = y^m \text{ with } m, n, x, y > 1$$

has only two integral solutions:

$$5^2 + 2 = 3^3 \quad \text{and} \quad 5^3 + 3 = 2^7.$$

Also, the diophantine equation

$$x^n - n = y^m \text{ with } m, n, x, y > 1$$

has only two integral solutions:

$$2^5 - 5 = 3^3 \quad \text{and} \quad 2^7 - 7 = 11^2.$$

Conjectures involving 2^k or $k!$

Conjecture (i) (Sun, 2013-11-11) Any integer $n > 1$ can be written as $k + m$ ($k, m \in \mathbb{Z}^+$) with $2^k m + 1$ prime.

(ii) (Sun, 2013-11-12) Any integer $n > 1$ can be written as $k + m$ ($1 \leqslant k \leqslant m$) with $k!m + 1$ prime.

Conjecture (i) (Sun, 2013-12-05). Any integer $n > 2$ with $n \neq 7$ can be written as $k + m$ ($k, m \in \mathbb{Z}^+$) with $2^k + p_m$ prime, where p_m denotes the m-th prime.

(ii) (Sun, 2013-12-06). Any integer $n > 3$ can be written as $k + m$ ($1 \leqslant k \leqslant m$) with $k! + p_m$ prime.

(iii) (Sun, 2013-12-05). Any integer $n > 2$ can be written as $k + m$ ($0 < k < m$) with $\binom{2k}{k} + p_m$ prime.

Remark. I have verified parts (i)-(iii) for n up to 3×10^7, 10^7 and 10^8 respectively. For $n = 28117716$, the least $k \in \mathbb{Z}^+$ with $2^k + p_{n-k}$ prime is 81539.

A conjecture involving $\sigma(n)$ and $\varphi(n)$

For a positive integer n, define

$$\varphi(n) = |\{1 \leqslant a \leqslant n : \ (a, n) = 1\} \ \text{and} \ \sigma(n) = \sum_{d|n} d.$$

For example,

$$\varphi(6) = 2 \ \text{ and } \sigma(6) = 1 + 2 + 3 + 6 = 12.$$

If p is a prime, then $\varphi(p) = p - 1$ and $\sigma(p) = p + 1$.

Conjecture (Sun, 2013-12-12). Any integer $n > 1$ can be written as $k^2 + m$ with $1 \leqslant k^2 \leqslant m$ such that $\sigma(k^2) + \varphi(m)$ is prime.

Remark. I verified this for n up to 10^8, later C.R. Greathouse IV verified this for n up to 3×10^9.

Example. $16 = 2^2 + 12$ with $\sigma(2^2) + \varphi(12) = 7 + 4 = 11$ prime.

New representation problems motivated by Goldbach's conjecture

For any even number $2n > 2$, by Goldbach's conjecture there should exist two primes p and q with $p + q = 2n$, hence

$$2n-1 = p+(q-1) = p+\varphi(q) \text{ and } 2n+1 = p+(q+1) = p+\sigma(q).$$

Conjecture. (i) (Dec. 17, 2013) Any odd number $n > 1$ can be written as $p + \varphi(m^2)$, where p is a prime, $m \in \mathbb{Z}^+$ and $m^2 < n$.

(ii) (Dec. 14, 2013) Any even number $n > 2$ can be written as $p + \sigma(k)$, where p is an odd prime and k is a positive integer.

Remark. It is easy to show that $\sigma(k)$ is odd if and only if k has the form m^2 or $2m^2$ with $m \in \mathbb{Z}^+$. I have verified both parts of the conjecture for $n \leqslant 10^9$.

Twin primes and Euler's totient function

Conjecture (Sun, 2013-12-12). Let $n > 5$ be an integer.
(i) We can write $n = k^2 + m$ ($k, m \in \mathbb{Z}^+$) with $\varphi(k^2)\varphi(m) - 1$ prime.

(ii) We can write $n = k + m$ ($k, m \in \mathbb{Z}^+$) with $\varphi(k)\varphi(m) \pm 1$ both prime.

Remark. We have verified both parts for n up to 10^8. Clearly part (ii) is much stronger than the twin prime conjecture.

Examples. $23 = 4^2 + 7$ with $\varphi(4^2)\varphi(7) - 1 = 8 \times 6 - 1 = 47$ prime. $25 = 11 + 14$ with

$$\{\varphi(11)\varphi(14) \pm 1\} = \{10 \times 6 \pm 1\} = \{59, 61\}$$

a twin prime pair.

Representations involving $\pi(x)$ or p_k

By the Prime Theorem,

$$\pi(x) = |\{p \leqslant x : \ p \text{ is prime}\}| \sim \frac{x}{\log x} \quad \text{and} \quad p_n \sim n \log n.$$

Conjecture (Sun) (2003-11-24) Any integer $n > 3$ can be written as $p + q - \pi(q)$, where p and q are odd primes not exceeding n.

Conjecture (Sun). (i) (2013-11-25) The set

$$\{i + j : \ 0 < i < j < k : \ p_i, p_j, p_k \text{ form an arithmetic progression}\}$$

coincides with $\{5, 6, 7, \ldots\}$.
(ii) (2013-12-07) For any integer $n > 1$, $kp_{n-k} + 1$ is prime for some $0 < k < n$.
(iii) (2013-12-11) For any integer $n > 5$, $p_k p_{n-k} - 6$ is prime for some $0 < k < n$.
(iv) (2013-12-10) For any integer $n > 3$, $p_k^2 + 4p_{n-k}^2$ is prime for some $0 < k < n$.

A conjecture which might be open for 10000 years

Conjecture (Sun, 2013-12-01). There are infinitely many positive integers n such that all the six numbers

$$n \pm 1, \quad p_n \pm n, \quad np_n \pm 1$$

are prime.

Remark. The first such a number is 22110, and the 2000-th such n is 9761161692.

Conjectures involving partition functions

For a positive integer n, let $p(n)$ denote the number of ways to write n as a sum of positive integers with the order of addends ignored, and let $q(n)$ denote the number of ways to write n as a sum of *distinct* positive integers with the order of addends ignored. For example, $p(3) = 3$ since $3 = 1 + 2 = 1 + 1 + 1$, and $q(3) = 2$ since $3 = 1 + 2$.

It is known that

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4\sqrt{3}n} \quad \text{and} \quad q(n) \sim \frac{e^{\pi\sqrt{n/3}}}{4(3n^3)^{1/4}} \quad \text{as } n \to +\infty.$$

Conjecture (Sun) (i) Any integer $n > 1$ can be written as $k + m$ $(k, m \in \mathbb{Z}^+)$ with $p(k)^2 + q(m)^2$ (or $p(k) + q(m)$) prime.

(ii) Each integer $n > 1$ can be written as $k + m$ $(k, m \in \mathbb{Z}^+)$ with $2^k - 1 + q(m)$ prime.

Remark. I have verified parts (i) and (ii) for n up to 10^5 and 2×10^5 respectively.

For sources of my conjectures, you may visit

http://math.nju.edu.cn/∼zwsun
http://arxiv.org/abs/1211.1588
http://arxiv.org/abs/1309.1678.
http://oeis.org

You are welcome to solve my conjectures!

Thank you!