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Zero-sum Problems for Abelian Groups and Covers of the Integers by Residue Classes

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Abstract

Zero-sum problems for abelian groups and covers of the integers by residue classes, are two different active topics initiated by P. Erdős more than 60 years ago and investigated by many researchers separately since then. In 2003 the speaker announced some surprising connections among these seemingly unrelated fascinating areas. In a paper published in 2009, the speaker established further connections between zero-sum problems for abelian p-groups and covers of the integers.

In this talk we introduce the above work and mention some related open problems. For example, we extend the famous Erdős-Ginzburg-Ziv theorem in the following way: If $\{a_s(\text{mod } n_s)\}_{s=1}^k$ covers each integer either exactly $2q-1$ times or exactly $2q$ times where q is a prime power, then for any $c_1, \ldots, c_k \in \mathbb{Z}/q\mathbb{Z}$ there exists an $I \subseteq \{1, \ldots, k\}$ such that $\sum_{s \in I} 1/n_s = q$ and $\sum_{s \in I} c_s = 0$. We conjecture that the prime power q here can be replaced by any positive integer.

Part I. Properties of Covers involving Unit Fractions

Covering systems of residue classes

For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$, let $a(n) = a + n\mathbb{Z}$.

For a finite system $A = \{a_s(n_s)\}_{s=1}^k$ of residue classes, if $\bigcup_{s=1}^k a_s(n_s) = \mathbb{Z}$ then we call A a *covering system* or a *cover of* $\mathbb{Z}$; if A covers each integer exactly once then A is called an *exact cover* (or *disjoint cover*) of $\mathbb{Z}$.

The concept of covering system was introduced by Paul Erdős who gave the following example:

$$\{0(2),\ 0(3),\ 1(4),\ 5(6),\ 7(12)\}.$$

Another Example.

$$A = \{1(2),\, 2(2^2),\, \ldots,\, 2^{k-1}(2^k),\, 0(2^k)\}$$

is an exact cover of $\mathbb{Z}$.

Covering function

For $A = \{a_s(n_s)\}_{s=1}^k$, its *covering function* is defined by

$$w_A(x) = |\{1 \leqslant s \leqslant k: \ x \equiv a_s \pmod{n_s}\}|.$$

Clearly w_A is periodic modulo $N_A = [n_1, \ldots, n_k]$. We call $m(A) = \min_{x\in\mathbb{Z}} w_A(x)$ the *covering multiplicity* of A.

$$\begin{aligned}\frac{1}{N_A}\sum_{x=0}^{N_A-1} w_A(x) =& \frac{1}{N_A}\sum_{x=0}^{N_A-1}\sum_{\substack{s=1\\ n_s\mid x-a_s}}^{k} 1\\ =& \sum_{s=1}^{k}\frac{1}{N_A}|\{0 \leqslant x < N_A: \ x \equiv a_s \pmod{n_s}\}|\\ =& \sum_{s=1}^{k}\frac{1}{N_A}\cdot\frac{N_A}{n_s} = \sum_{s=1}^{k}\frac{1}{n_s}.\end{aligned}$$

If $m(A) \geqslant m$, then we call A an *m-cover* (of $\mathbb{Z}$) and note that $\sum_{s=1}^k \frac{1}{n_s} \geqslant m$. If A covers each integer exactly m times, then we call A an *exact m-cover* and note that $\sum_{s=1}^k \frac{1}{n_s} = m$ in this case.

Ming-Zhi Zhang's Result

In 1989, by using the Riemann zeta function, M. Z. Zhang [J. Sichuan Univ. (Nat. Sci. Ed.)] showed the following surprising result.

Zhang's Result (1989): If $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ then $\sum_{s \in I} 1/n_s \in \mathbb{Z}^+$ for some $I \subseteq [1,k] = \{1, \ldots, k\}$.

The starting point of Zhang is that A forms a cover of $\mathbb{Z}$ if and only if

$$\prod_{s=1}^{k} \left(1 - e^{2\pi i (n+a_s)/n_s}\right) = 0 \quad \text{for all } n = 1, 2, 3, \ldots.$$

The crucial trick in Zhang's proof is that for a real number c the series $\sum_{n=1}^{+\infty} \frac{e^{2\pi i c n}}{n}$ diverges if and only if c is an integer.

On exact m-covers

If $A = \{a_s(n_s)\}_{s=1}^k$ is an exact m-cover, then $\sum_{s=1}^k \frac{1}{n_s} = m$.

In 1976 Š. Porubský asked whether every exact m-cover is a union of m disjoint covers. Choi supplied the following exact 2-cover

$$\{1(2); 0(3); 2(6); 0, 4, 6, 8(10); 1, 2, 4, 7, 10, 13(15); 5, 11, 12, 22, 23, 29(30)\},$$

which is not a union of two exact covers.

Ming-Zhi Zhang [J. Sichuan Univ. (Nat. Sci. Ed.), 1991]: For each $m = 2, 3, \ldots$ there are infinitely many exact m-covers of $\mathbb{Z}$ which cannot be a union of an n-cover and an $(m-n)$-cover with $0 < n < m$. (By a *graph-theoretic approach*.)

Hao Pan and Li-Lu Zhao [Adv. in Appl. Math. 43(2009)]: For each $m = 2, 3, \ldots$ there is an exact m-cover of $\mathbb{Z}$ which is not a union of two covers of $\mathbb{Z}$.

On exact m-covers

For any exact m-cover $\{a_s(n_s)\}_{s=1}^k$, $\sum_{s=1}^k 1/n_s = m \in \mathbb{Z}^+$. So Zhang's result for exact m-covers is trivial.

Z.-W. Sun [Israel J. Math. 77(1992)]. Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact m-cover of $\mathbb{Z}$. Then, for any $n = 0\ldots,m$ we have

$$\left|\left\{I \subseteq \{1,\ldots,k\} :\ \sum_{s\in I}\frac{1}{n_s} = n\right\}\right| \geqslant \binom{m}{n}.$$

Remark. Considering the exact m-cover of $\mathbb{Z}$ consisting m copies of $0(1)$, we see that the lower bound $\binom{m}{n}$ is best possible.

Lemma. $A = \{a_s(n_s)\}_{s=1}^k$ forms an exact m-cover of $\mathbb{Z}$ if and only if we have the identity

$$\prod_{s=1}^{k}\left(1 - x^{N/n_s}e^{2\pi i a_s/n_s}\right) = (1 - x^N)^m,$$

where N is the least common multiple of $n_1,\ldots,n_k$.

Proofs

Proof of the Lemma. It suffices to note that

$$1 - x^{N/n_s} e^{2\pi i a_s/n_s} = 0 \Longrightarrow x^N = 1$$

and

$$1 - (e^{-2\pi i r/N})^{N/n_s} e^{2\pi i a_s/n_s} = 0 \iff r \equiv a_s \pmod{n_s}.$$

Proof of the Theorem. Comparing the coefficients of x^{nN} in both sides of the identity

$$\prod_{s=1}^{k} \left(1 - x^{N/n_s} e^{2\pi i a_s/n_s}\right) = (1 - x^N)^m,$$

we get

$$\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \sum_{s\in I} \frac{1}{n_s} = n}} (-1)^{|I|} e^{2\pi i \sum_{s\in I} a_s/n_s} = (-1)^n \binom{m}{n}.$$

So there are at least $\binom{m}{n}$ subsets I of $\{1, \ldots, k\}$ with $\sum_{s\in I} \frac{1}{n_s} = n$.

Further extensions

Z.-W. Sun [Acta Arith. 81(1997)]: Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact m-cover. Then, for any $t = 1, \ldots, k$ and $a = 0, 1, 2, \ldots$ we have

$$\left|\left\{I \subseteq [1,k] \setminus \{t\} :\ \sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_t}\right\}\right| \geqslant \binom{m-1}{\lfloor a/n_t \rfloor}.$$

Z.-W. Sun [Bull. Austral. Math. Soc. 81(2010)]: If $\{a_s(n_s)\}_{s=0}^k$ covers each integer more than $m = \lfloor \sum_{s=1}^k \frac{1}{n_s} \rfloor$ times, then

$$\left|\left\{I \subseteq [1,k] :\ \sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_0}\right\}\right| \geqslant \binom{m}{\lfloor a/n_0 \rfloor}$$

for all $a \in \mathbb{N}$. In particular, if $A = \{a_s(n_s)\}_{s=1}^k$ has covering multiplicity $m(A) = \lfloor \sum_{s=1}^k \frac{1}{n_s} \rfloor$, then for any $n \in \mathbb{N}$ we have

$$\left|\left\{I \subseteq [1,k] :\ \sum_{s\in I} \frac{1}{n_s} = n\right\}\right| \geqslant \binom{m(A)}{n}.$$

Characterizing m-covers

As usual, the fractional part of a real number x is denoted by $\{x\}$.

For real numbers α and $\beta > 0$, we define

$$\alpha + \beta\mathbb{Z} := \{\alpha + \beta x : \ x \in \mathbb{Z}\}.$$

Z.-W. Sun [Acta Arith. 72(1995)]: Let $\alpha_1, \ldots, \alpha_k$ be real numbers and $\beta_1, \ldots, \beta_k$ be positive real numbers. Then $A = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k$ covers all the integers at least m times if and only if

$$\sum_{\substack{I \subseteq \{1,\ldots,k\} \\ \{\sum_{s\in I} \frac{1}{\beta_s}\} = \theta}} (-1)^{|I|} \binom{\lfloor \sum_{s\in I} 1/\beta_s \rfloor}{n} e^{2\pi i \sum_{s\in I} \alpha_s/\beta_s} = 0$$

for all $n = 0, \ldots, m-1$ and $0 \leqslant \theta < 1$.

Remark. The starting point is that $A = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k$ covers x at least m times if and only if

$$\prod_{s=1}^{k} \left(1 - r^{1/\beta_s} e^{2\pi i(\alpha_s - x)/\beta_s}\right) = o((1-r)^{m-1}) \ \ (r \to 1).$$

m-covers and unit fractions

Using the above characterization of m-covers, we deduced the following properties of m-covers related to unit fractions.

Let $\{a_s(n_s)\}_{s=1}^k$ be an m-cover of $\mathbb{Z}$ and let $m_1, \ldots, m_k$ be positive integers.

Z.-W. Sun [Trans. Amer. Math. Soc. 348(1996)]: There are at least m positive integers in the form $\sum_{s\in I} m_s/n_s$ with $I \subseteq [1, k]$.

Z.-W. Sun [Proc. Amer. Math. Soc. 127(1999)]: For any $J \subseteq \{1, \ldots, k\}$, there are at least m subsets I of $\{1, \ldots, k\}$ with $I \neq J$ such that $\{\sum_{s\in I} \frac{m_s}{n_s}\} = \{\sum_{s\in J} \frac{m_s}{n_s}\}$.

H. Pan and Z.-W. Sun [Proc. Amer. Math. Soc. 135(2007)]: For any $0 \leqslant \theta < 1$, if the set

$$\left\{I \subseteq \{1, \ldots, k\} : \left\{\sum_{s\in I} \frac{m_s}{n_s}\right\} = \theta\right\}$$

is nonempty, then its cardinality is at least 2^m.

Part II. Connections between Zero-sum Problems and Covers of $\mathbb{Z}$

EGZ Theorem

In 1961 P. Erdős, A. Ginzburg and A. Ziv [Bull. Research Council. Israel] established the following celebrated theorem which initiated the study of zero-sums.

EGZ Theorem (1961): For any $c_1, \ldots, c_{2n-1} \in \mathbb{Z}$, there is an $I \subseteq [1, 2n-1]$ with $|I| = n$ such that $\sum_{s \in I} c_s \equiv 0 \pmod{n}$. In other words, given $2n-1$ (not necessarily distinct) elements of $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, we can select n of them with the sum vanishing.

The EGZ theorem can be easily reduced to the case where n is a prime (and hence $\mathbb{Z}_n$ is a field), and then deduced from the well-known Cauchy-Davenport theorem or the Chevalley-Warning theorem.

Davenport's constant

For a finite abelian group G (written additively), the **Davenport constant** $D(G)$ is defined as the smallest positive integer k such that any sequence $(c_1, \ldots, c_k)$ (repetition allowed) of elements of G has a subsequence $c_{i_1}, \ldots, c_{i_l}$ $(i_1 < \ldots < i_l)$ with zero-sum (i.e. $c_{i_1} + \ldots + c_{i_l} = 0$). It is easy to see that $D(G) \leqslant |G|$.

In 1966 Davenport showed that if K is an algebraic number field with ideal class group G, then $D(G)$ is the maximal number of prime ideals (counting multiplicity) in the decomposition of an irreducible integer in K.

We may explain $D(\mathbb{Z}_n) = n$ via covers of $\mathbb{Z}$. As $A = \{r(n)\}_{r=1}^{n}$ is a cover of $\mathbb{Z}$, for any $m_1, \ldots, m_n \in \mathbb{Z}^+$ we have $\sum_{s\in I} \frac{m_s}{n} \in \mathbb{Z}^+$ (i.e., $\sum_{s\in I} m_s \equiv 0 \pmod{n}$) for some $\emptyset \neq I \subseteq \{1, \ldots, k\}$.

Olson's Theorem [J. Number Theory 1(1969)]. Let p be any prime. For an abelian p-group $G \cong \mathbb{Z}_{p^{h_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{h_l}}$ we have

$$D(G) = 1 + \sum_{t=1}^{l} (p^{h_t} - 1).$$

Kemnitz's Conjecture

For a finite abelian group G, define $s(G)$ to be the least positive integer k such that any sequence $(a_1, \ldots, a_k)$ of elements of G has a zero-sum subsequence of length $\exp(G)$, where the exponent $\exp(G)$ of G is the least $n \in \mathbb{Z}^+$ with $nx = 0$ for all $x \in G$.

By the EGZ theorem, $s(\mathbb{Z}_n) = 2n - 1$ for any positive integer n.

What is the smallest integer $l = s(\mathbb{Z}_n^2)$ such that every sequence of l elements in $\mathbb{Z}_n^2 = \mathbb{Z}_n \oplus \mathbb{Z}_n$ contains a zero-sum subsequence of length n?

In 1983 Kemnitz [Ars Combin.] conjectured that $s(\mathbb{Z}_n^2) = 4n - 3$, and the conjecture can be reduced to the case with n prime.

In 1993 Alon and Dubiner showed that $s(\mathbb{Z}_n^2) \leqslant 6n - 5$. In 2000 Rónyai [Combinatorica] was able to prove that $s(\mathbb{Z}_p^2) \leqslant 4p - 2$ for every prime p; in 2001 W. D. Gao [J. Combin. Theory Ser. A] deduced that $s(\mathbb{Z}_q^2) \leqslant 4q - 2$ for any prime power q.

These results were obtained by various algebraic methods. In 2003 C. Reiher finally proved the Kemnitz conjecture.

Alon-Dubiner Lemma

The following lemma plays an indispensable role in the study of the Kemnitz conjecture.

Alon-Dubiner Lemma. Let q be a prime power, and let $c_1, \ldots, c_{3q}$ be elements of $\mathbb{Z}_q^2$ with $c_1 + \ldots + c_{3q} = 0$. Then there is an $I \subseteq [1, 3q]$ with $|I| = q$ such that $\sum_{i \in I} c_i = 0$.

Proof. As $3q - 1 \geqslant 1 + (q-1) + (q-1) + (q-1)$, by Olson's theorem there is a nonempty $I \subseteq \{1, \ldots, 3q-1\}$ such that $\sum_{s \in I} c_s = 0$ in $\mathbb{Z}_q^2$ and also $\sum_{s \in I} 1 = 0$ in $\mathbb{Z}_q$. So $q \mid |I|$ and hence $|I| \in \{q, 2q\}$. If $|I| = 2q$, then $\bar{I} = \{1, \ldots, 3q\} \setminus I$ has cardinality q and

$$\sum_{t \in \bar{I}} c_t = \sum_{i=1}^{3q} c_i - \sum_{s \in I} c_s = 0.$$

Connections between covers of $\mathbb{Z}$ and zero-sum problems

In 2003 Z.-W. Sun established connections between covers of $\mathbb{Z}$ and some classical theorems on zero-sums such as $D(\mathbb{Z}_m) = m$, the EGZ theorem, the Alon-Dubiner lemma and Olson's theorem.

The results were first announced in Electron. Res. Announc. Amer. Math. Soc. 9(2003). Full proofs of them were published in Israel J. Math. 170(2009).

Note that the speaker's discovery is **quite different from** Gao and Geroldinger's work in the paper

W. Gao and A. Geroldinger, *Zero-sum problems and coverings by proper cosets*, European J. Combin. **24** (2003), 531–549.

Connections between covers of $\mathbb{Z}$ and zero-sum problems

The following theorem in the case $n_1 = \cdots = n_k = 1$ reduces to known results on zero-sums.

Theorem (Z.-W. Sun [Israel J. Math. 170(2009)]). Let $A = \{a_s(n_s)\}_{s=1}^k$ and let G be an abelian group with $|G| = q$ a prime power.

(i) If A forms a q-cover of $\mathbb{Z}$, then for any $m_1, \ldots, m_k \in \mathbb{Z}$ there exists a nonempty $I \subseteq [1, k]$ such that $\sum_{s \in I} m_s/n_s \in q\mathbb{Z}$.

(ii) If $\{w_A(x) : x \in \mathbb{Z}\} \subseteq \{D(G) + q - 1, \ldots, 2q\}$, then for any $c_1, \ldots, c_k \in G$ there exists an $I \subseteq [1, k]$ such that $\sum_{s \in I} 1/n_s = q$ and $\sum_{s \in I} c_s = 0$.

(iii) If A is an exact $3q$-cover of $\mathbb{Z}$, then for any $c_1, \ldots, c_k \in G \oplus G$ with $c_1 + \ldots + c_k = 0$, there exists an $I \subseteq [1, k]$ such that $\sum_{s \in I} 1/n_s = q$ and $\sum_{s \in I} c_s = 0$.

(iv) Suppose that A is a $D(G)$-cover of $\mathbb{Z}$. Then, for any $m_1, \ldots, m_k \in \mathbb{Z}$ and $c_1, \ldots, c_k \in G$, there is a nonempty $I \subseteq [1, k]$ such that $\sum_{s \in I} c_s = 0$ and $\sum_{s \in I} m_s/n_s \in \mathbb{Z}$.

Conjecture

Conjecture (Z.-W. Sun [Israel J. Math. 170(2009)]) The above theorem remains valid if the prime power q is replaced by any positive integer. In particular, if $\{a_s(n_s)\}_{s=1}^k$ is an m-cover of $\mathbb{Z}$, then $\sum_{s\in I}\frac{1}{n_s}\in m\mathbb{Z}$ for some $\emptyset\neq I\subseteq\{1,\ldots,k\}$.

Why I can prove the theorem with q a prime power? The following lemma is of technical importance.

Lemma (Z.-W. Sun, 2003). Let p be a prime, and let $a\in\mathbb{N}$ and $m\in\mathbb{Z}$. Then we have the following congruence

$$\binom{m-1}{p^a-1}\equiv\begin{cases}1\pmod p & \text{if } p^a\mid m,\\ 0\pmod p & \text{otherwise.}\end{cases}$$

Remark. Let m be an integer and let p be a prime. Fermat's little theorem tells that we can characterize whether p divides m as follows:

$$1-m^{p-1}\equiv\begin{cases}1\pmod p & \text{if } p\mid m,\\ 0\pmod p & \text{if } p\nmid m.\end{cases}$$

Main Theorem

Actually the theorem follows from the following Main Theorem.

Main Theorem (Z.-W. Sun [Israel J. Math. 170(2009)]). Let G be an additive abelian p-group where p is a prime. Suppose that $A = \{a_s(n_s)\}_{s=1}^k$ is a $(D(G) + p^h - 1)$-cover of $\mathbb{Z}$ with $h \in \mathbb{N} = \{0, 1, \ldots\}$. Let $c_1, \ldots, c_k \in G$ and $m_1, \ldots, m_k \in \mathbb{Z}$. Then

$$\left|\left\{ I \subseteq [1,k] : \sum_{s \in I} c_s = c \text{ and } \sum_{s \in I} \frac{m_s}{n_s} \in \alpha + p^h\mathbb{Z} \right\}\right| \neq 1$$

for any $c \in G$ and rational number α. In particular, $(c_1, \ldots, c_k)$ has a zero-sum subsequence $(c_s)_{s \in I}$ with $\emptyset \neq I \subseteq [1,k]$ satisfying the restriction $\sum_{s \in I} m_s/n_s \in p^h\mathbb{Z}$.

A Key Lemma

For a polynomial $f(x_1,\ldots,x_k)$ over the field $\mathbb{C}$ of complex numbers, we use $[x_1^{j_1}\cdots x_k^{j_k}]f(x_1,\ldots,x_k)$ to represent the coefficient of the monomial $x_1^{j_1}\cdots x_k^{j_k}$ in $f(x_1,\ldots,x_k)$. For an assertion A, we set

$$[\![A]\!]=\begin{cases}1 & \text{if } A \text{ holds,}\\ 0 & \text{otherwise.}\end{cases}$$

Key Lemma (Z.-W. Sun [Israel J.math. 170(2009)]). Let $A=\{a_s(n_s)\}_{s=1}^k$ and let $f(x_1,\ldots,x_k)\in\mathbb{C}[x_1,\ldots,x_k]$ with $\deg f\leqslant m(A)$. Let $m_1,\ldots,m_k\in\mathbb{Z}$. If $[\prod_{s\in I_z}x_s]f(x_1,\ldots,x_k)=0$ for all $z\in\mathbb{Z}$ (where $I_z=\{1\leqslant s\leqslant k:\ z\in a_s(n_s)\}$), then we have $\psi(\theta)=0$ for any $0\leqslant\theta<1$, where

$$\psi(\theta)=\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\sum_{s\in I}m_s/n_s\}=\theta}}(-1)^{|I|}f([\![1\in I]\!],\ldots,[\![k\in I]\!])e^{2\pi i\sum_{s\in I}a_sm_s/n_s}.$$

The converse holds when $m_1,\ldots,m_k$ are relatively prime to $n_1,\ldots,n_k$ respectively.

Proof of an extension of the Alon-Dubiner Lemma

Let G be an abelian group with $|G| = q$ a prime power, and let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact $3q$-cover of $\mathbb{Z}$. Let $c_1, \ldots, c_k \in G \oplus G$ with $c_1 + \ldots + c_k = 0$. We now show that $(c_1, \ldots, c_k)$ has a zero-sum subsequence $(c_s)_{s\in I}$ with $\emptyset \neq I \subseteq [1, k]$ and $\sum_{s\in I} 1/n_s = q$.

As $A_* = \{a_s(n_s)\}_{s=1}^{k-1}$ is a $(D(G \oplus G) + q - 1)$-cover of $\mathbb{Z}$, by the Main Theorem $(c_1, \ldots, c_k)$ has a zero-sum subsequence $(c_s)_{s\in I}$ with $\emptyset \neq I \subseteq [1, k-1]$ and $n = \sum_{s\in I} 1/n_s \in q\mathbb{Z}$.

As $n < \sum_{s=1}^k 1/n_s = 3q$, n is q or $2q$. If $n = 2q$, then for $\bar{I} = [1, k] \setminus I$ we have

$$\sum_{s\in\bar{I}} c_s = \sum_{s=1}^k c_s - \sum_{s\in I} c_s = 0$$

and

$$\sum_{s\in\bar{I}} \frac{1}{n_s} = \sum_{s=1}^k \frac{1}{n_s} - n = 3q - 2q = q.$$

References

Main References

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Thank you!