NOTES ON CONJECTURES OF ZHI-WEI SUN

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Our notations are standard. In addition, we set $\mathbb{N} = \{0,1,2,\cdots\}$, $\mathbb{Z}^+ = \{1,2,3,\cdots\}$, and $a(n) = a(\text{mod } n) = a + n\mathbb{Z} = \{x \in \mathbb{Z}\colon x \equiv a \pmod n\}$ for $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$.

Conjecture 1 (1988-04-23)**.** *Let $a_0,\ldots,a_{n-1},b_0,\ldots,b_{n-1} \in \mathbb{N}$. Suppose that $\sum_{r=0}^{n-1} a_r e^{2\pi ir/n} = \sum_{r=0}^{n-1} b_r e^{2\pi ir/n}$, and that the least prime divisor $p = p(n)$ of n is greater than $|\{0 \leqslant r < n : a_r \neq 0\}|$ and $|\{0 \leqslant r < n : b_r \neq 0\}|$. Then $a_r = b_r$ for all $r \in R(n) = \{0,1,\ldots,n-1\}$.*

Remark 1. M. Newman [Math. Ann. 1971] showed that if $c_0,\ldots,c_{n-1} \in \mathbb{Q}$, $\sum_{r=0}^{n-1} c_r e^{2\pi ir/n} = 0$ and $|\{0 \leqslant r < n : c_r \neq 0\}| < p(n)$, then $c_0 = \cdots = c_{n-1} = 0$.

Conjecture 2 (1988-04-23)**.** *For $s = 1,\ldots,k$ let $\psi_s : \mathbb{Z} \to \mathbb{C}$ be an arithmetical function with period $n_s \in \mathbb{Z}^+$. If $\psi = \psi_1 + \cdots + \psi_k$ is not the zero function, then*

$$|\{0 \leqslant r < N = [n_1,\ldots,n_k] : \psi(r) \neq 0\}| \geqslant \min_{1\leqslant s\leqslant k} \frac{N}{n_s}.$$

Remark 2. This is trivial if $n_1 = \cdots = n_k$. In 2005 S. Guo showed that the conjecture is actually equivalent to a result of the author in 1991 (cf. Remark 1.4 in [J. Number Theory 111(2005), 190–196]).

Conjecture 3 [Made on July 16, 1988, and appeared in Integers 7(2)(2007)]**.** *If $N \in \mathbb{Z}^+$ is a covering number (i.e. there are distinct divisors $1 < n_1 < \cdots < n_k$ of N such that $\{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ for some $a_1,\ldots,a_k \in \mathbb{Z}$) but none of its proper divisors is, then we can write N in the form $p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ with $p_1,\ldots,p_r$ distinct primes and $\alpha_1,\ldots,\alpha_r \in \mathbb{Z}^+$, such that*

$$(*) \qquad \prod_{0<i<t} (\alpha_i + 1) \geqslant p_t - 1 + \delta_{tr} \quad \text{for each } t = 1,\ldots,r$$

where $\delta_{tr} = 1$ if $t = r$, and $\delta_{tr} = 0$ if $t \neq r$.

Remark 3. Example 3 of Z. W. Sun [Trans. Amer. Math. Soc., 348(1996)] indicates that $2^{n-1}n$ is a covering number for any odd $n > 1$. From $(*)$ I can deduce that $N = p_1^{\alpha_1}\cdots p_r^{\alpha_r}$ is a covering number. Conjecture 3 implies the well-known conjecture of P. Erdős and J. L. Selfridge which says that no odd integer greater than one can be a covering number.

Conjecture 4 [1989-05-04; Discrete Math., 1992]**.** *Let $n_1,\ldots,n_k$ be positive integers. If $\#d=|\{\{i,j\}:1\leqslant i<j\leqslant k\ \&\ (n_i,n_j)=d\}|$ is less than $2d-1$ for every positive integer $d\leqslant 2^{k-2}$, then $\{n_i\}_{i=1}^k$ is harmonic, i.e. there are integers $a_1,\ldots,a_k$ such that the residue classes $a_1(\mathrm{mod}\ n_1),\ldots,a_k(\mathrm{mod}\ n_k)$ are pairwise disjoint.*

Remark 4. A. P. Huhn and L. Megyesi [Discrete Math., 1982] showed that $\{n_i\}_{i=1}^k$ is harmonic if $\#1=0$ and $\#d\leqslant 1$ for all $d\in\mathbb{Z}^+$. I [Discrete Math., 1992] proved the weaker version of the conjecture with $2d-1$ replaced by $\sqrt{(d+7)/8}$. Y. G. Chen [Discrete Math., 1996] showed that if $\#1=0,\ \#2\leqslant 1,\ \#3\leqslant 1$ and $\#d\leqslant d/4$ for all $d=4,5,\ldots,2k-2$, then $\{n_s\}_{s=1}^k$ is harmonic.

Conjecture 5 [Made in 2004, and appeared in Internat. J. Math. 17(2006)]**.** *Let $a_1G_1,\ldots,a_kG_k$ $(k>1)$ be finitely many left cosets in a group G with all the indices $[G:G_i]$ finite. If the k cosets are pairwise disjoint, then $\gcd([G:G_i],[G:G_j])\geqslant k$ for some $1\leqslant i<j\leqslant k$.*

Remark 5. In the case $G=\mathbb{Z}$, Z. W. Sun [Chin. Ann. Math. Ser. A 13(1992)] showed that the above conjecture holds for $k\leqslant 4$, and that the converse of the conjecture fails for $k\geqslant 4$. It is easy to show Conjecture 5 for $k=2$ and for p-groups.

Conjecture 6 (1989-05-10)**.** *Let $A_0=\{a_s^{(0)}(\mathrm{mod}\ n_s)\}_{s=1}^k$ be an minimal cover of $\mathbb{Z}$ (i.e. A_0 covers all the integers but none of its proper subsystems does).*

(i) *There exists an $S\subseteq\{\{i,j\}:1\leqslant i<j\leqslant k\}$ such that $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a minimal cover of $\mathbb{Z}$ if and only if $(n_i,n_j)\nmid a_i-a_j$ for all $\{i,j\}\in S$.*

(ii) *If $A=\{a_s(\mathrm{mod}\ n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ and the moduli $n_1,\ldots,n_k$ are distinct, then A must be a minimal cover of $\mathbb{Z}$.*

Remark 6. (a) As for part (i) we give an example, $A_0=\{0(2),0(3),1(4),5(6),7(12)\}$ is a minimal cover, $A=\{a_i(n_i)\}_{i=1}^5$ $(n_1=2<n_2=3<n_3=4<n_4=6<n_5=12)$ forms a minimal cover if and only if $(n_i,n_j)\nmid a_i-a_j$ for all $\{i,j\}\in S$ where $S=\{\{1,3\},\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{3,5\},\{4,5\}\}$.

(b) If we don't require that $n_1,\ldots,n_k$ are distinct, then part (ii) fails as illustrated by the following example pointed out by Y.-G. Chen:

$$A_0=\{0(3),0(4),1(6),4(6),1(8),3(8),5(8),2(8),14(24),23(24)\}$$

and

$$A_1=\{0(3),0(4),1(6),5(6),2(8),14(24),23(24)\}$$

are minimal covers of $\mathbb{Z}$, A_1 together with $1(8),3(8),5(8)$ forms a redundant cover.

Conjecture 7 (1990-11-31)**.** *Let G be a group and $A=\{a_iG_i\}_{i=1}^k$ a system of left cosets of subnormal subgroups of G.*

(i) *If A is a cover of G, then $\sum_{i\in I}[G:G_i]^{-1}\in\mathbb{Z}^+$ for some $I\subseteq\{1,\ldots,k\}$.*

(ii) *If A forms an exact m-cover of G (i.e. $|\{1 \leqslant s \leqslant k : x \in a_iG_i\}| = m$ for all $x \in G$), then for each $n = 0, 1, \dots, m$ there are at least m subsets I of $\{1, \dots, k\}$ such that $\sum_{i\in I}[G : G_i]^{-1} = n$.*

Remark 7. When $G = \mathbb{Z}$, part (i) was first discovered by M.-Z. Zhang in 1989. I [Israel J. Math., 77(1992)] proved part (ii) in the case $G = \mathbb{Z}$.

Conjecture 8 (with S. Guo, 2004)**.** *Let $G_1, \dots, G_k$ be subnormal subgroups of a group G. If $A = \{G_i\}_{i=1}^k$ forms a minimal m-cover of G and $[G : \bigcap_{i=1}^k G_i]$ has the factorization $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ where $p_1, \dots, p_r$ are distinct primes and $\alpha_1, \dots, \alpha_r \in \mathbb{Z}^+$, then $k \geqslant m + 1 + \sum_{t=1}^r (\alpha_t - 1)(p_t - 1)$.*

Remark 8. For a minimal m-cover $\{a_iG_i\}_{i=1}^k$ of a group by left cosets, it is known that $[G : \bigcap_{i=1}^k G_i] \leqslant k! < +\infty$ (see Z. W. Sun [Fund. Math. 134(1990)]). The current version of Conjecture 8 is Guo's slight modification of Sun's original conjecture, it appeared as Conjecture 1.1 in a paper of Z. W. Sun [Internat. J. Math. 17(2006)].

Conjecture 9 (1997)**.** *Let $A = \{a_s(n_s)\}_{s=1}^k$ be a minimal cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$. Assume that $N = [n_1, \dots, n_k]$ has the factorization $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ where $p_1 < \cdots < p_r$ are primes and $\alpha_1, \dots, \alpha_r \in \mathbb{Z}^+$. For $n \in \{n_1, \dots, n_k\}$, if p_s is the largest prime divisor of n and α is the order of n at p_s, then we let $\lambda(n)$ be the ordered pair $\langle s, \alpha\rangle$. Set*

$$\Lambda = \{\langle 1, 0\rangle\} \cup \{\lambda(n_1), \dots, \lambda(n_k)\}.$$

If $\langle s, \alpha\rangle \in \Lambda \setminus \{\langle r, \alpha_r\rangle\}$, and $\langle s', \alpha'\rangle$ is the least element of Λ greater than $\langle s, \alpha\rangle$ in alphabetical order, then

$$|\{1 \leqslant i \leqslant k : \lambda(n_i) = \langle s', \alpha'\rangle\}| \geqslant f\left(p_{s'}^{\alpha'} \prod_{0<t<s'} p_t^{\alpha_t}\right) - f\left(p_s^{\alpha} \prod_{0<t<s} p_t^{\alpha_t}\right) + \delta(\langle s', \alpha'\rangle)$$

where the Mycielski function $f : \mathbb{Z}^+ \to \mathbb{N}$ is given by $f(n) = \sum_{p|n} \operatorname{ord}_p(n)(p - 1)$, and $\delta(\langle s', \alpha'\rangle)$ takes 1 or 0 according as $\langle s', \alpha'\rangle = \langle r, \alpha_r\rangle$ or not.

Remark 9. (a) Benefited from some discussion with me, my twin brother Zhi-Hong Sun ever conjectured that if $A = \{a_i(n_i)\}_{s=1}^k$ forms a minimal cover of $\mathbb{Z}$ with $1 < n_1 < \cdots < n_k$, and $[n_1, \dots, n_k]$ has the primary factorization $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ $(p_1 < \cdots < p_r)$, then, for any $s = 1, \dots, r$ and $\alpha = 1, \dots, \alpha_s$, we have

$$|\{1 \leqslant i \leqslant k : n_i = p_1^{\beta_1} \cdots p_{s-1}^{\beta_{s-1}} p_s^{\alpha} \text{ for some } \beta_1 \leqslant \alpha_1, \dots, \beta_{s-1} \leqslant \alpha_{s-1}\}| \geqslant p_s - 1.$$

This and those variations given by me in [Fund. Math. 134(1990), Problem 6] and [Trans. Amer. Math. Soc. 348(1996), Conjecture IV], are actually refuted by Example 3 of Z. W. Sun [Trans. Amer. Math. Soc. 348(1996)].

(b) Clearly the assertion in Conjecture 8 implies the known inequality $k \geqslant 1 + f(N)$ (see, e.g. Z. W. Sun [Fund. Math. 134(1990)]). If $p_1^{\alpha} \in \{n_1, \dots, n_k\}$, then by the conjecture we should have $1 = |\{1 \leqslant i \leqslant k :\ n_i = p_1^{\alpha}\}| \geqslant p_1 - 1$ and hence $p_1 = 2$. So Conjecture 9 also implies the well-known conjecture of P. Erdös and J. L. Selfridge stated in Remark 3.

Conjecture 10 [1994-02-16; Combinatorica, 24(2004)]**.** *Let $A=\{a_s(n_s)\}_{s=1}^k$ be a minimal cover of $\mathbb{Z}$. Then the set*

$$S(A)=\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\}: I\subseteq\{1,\cdots,k\}\right\}.$$

contains $0,1/d,\ldots,(d-1)/d$ whenever $1/d$ lies in it, where $\{\alpha\}$ denotes the fractional part of $\alpha\in\mathbb{R}$.

Remark 10. Let $A=\{a_s(n_s)\}_{s=1}^k$ be a minimal cover of $\mathbb{Z}$. By Theorem 1 (ii) of Z. W. Sun [Proc. Amer. Math. Soc. 127(1999)], for any $t=1,\ldots,k$ there is an $\alpha_t\in[0,1)$ such that $S(A)\supseteq\{\frac{\alpha_t+r}{n_t}: r=0,1,\ldots,n_t-1\}$. Z. W. Sun [Adv. in Appl. Math. 38(2007)] showed that we can take $\alpha_t=0$ if n_t is a period of the covering function $w_A(x)=|\{1\leqslant s\leqslant k: x\in a_s(n_s)\}|$. Perhaps, if A is a minimal m-cover of $\mathbb{Z}$ but not an exact m-cover of $\mathbb{Z}$, then $S(A)=\{r/N: r=0,1,\ldots,N-1\}$ where N is the least common multiple of the moduli $n_1,\ldots,n_k$.

Conjecture 11 [Proc. Amer. Math. Soc. 127(1999)]**.** *Let $A=\{a_s(n_s)\}_{s=1}^k$ form a m-cover of $\mathbb{Z}$. Then there exists a chain $\emptyset\neq I_1\subset\cdots\subset I_m\subseteq\{1,\ldots,k\}$ such that*

$$\sum_{s\in I_t}\frac{1}{n_s}\in\mathbb{Z}\qquad \text{for all } t=1,\ldots,m.$$

Remark 11. Let $A=\{a_s(n_s)\}_{s=1}^k$ be any m-cover of $\mathbb{Z}$. By Theorem I of Z. W. Sun [Trans. Amer. Math. Soc. 348(1996)], there are at least m positive integers in the form $\sum_{s\in I}\frac{1}{n_s}$ with $I\subseteq\{1,\ldots,k\}$. By Theorem 1 (i) of Z. W. Sun [Proc. Amer. Math. Soc. 127(1999)], for any $J\subseteq\{1,\ldots,k\}$ there are at least m subsets I of $\{1,\ldots,k\}$ such that $I\neq J$ and $\sum_{s\in I}\frac{1}{n_s}-\sum_{s\in J}\frac{1}{n_s}\in\mathbb{Z}$.

Conjecture 12 (1998-03-04)**.** *78557 is the smallest positive integer x such that both $x+2^n$ and $x2^n+1$ always have at least two distinct odd prime divisors.*

Remark 12. J. L. Selfridge showed that $78557\times 2^n+1$ are divisible by one of $3,5,7,13,19,37,73$. In Guy's book *Unsolved Problems in Number Theory*, there is a list of 35 positive integers x less than 78557 for which $x2^n+1$ are composite for all $n\leqslant 50000$. It is believed that 78557 might be the smallest positive integer x such that $x2^n+1$ are always composite. I have proved that neither $78557+2^n$ nor $78557\times 2^n+1$ can be a prime power. By computation via the software Maple (version 4.0), we find that no number in the list different from 7013, 19249, 60443, 67607 possesses the strong property as 78557 has. Namely,

$$\begin{aligned}&4847+2^{33},\ 5297+2^{129},\ 5359+2^{170},\ 10223+2^{19},\ 13787+2,\\&21181+2^{28},\ 22699+2^{26},\ 24737+2^{17},\ 25819+2^{70},\ 27653+2^{39},\\&27923+2^{7},\ 33661+2^{72},\ 34999+2^{14},\ 39781+2^{8},\ 44131+2^{436},\\&46157+2^{49},\ 46187+2^{5},\ 46471+2^{8},\ 47897+2^{9},\ 48833+2^{175},\\&50693+2^{7},\ 54767+2^{5},\ 55459+2^{14},\ 59569+2^{26},\ 60541+2^{20},\\&63017+2^{13},\ 65567+2^{5},\ 69109+2^{26},\ 74191+2^{120},\ 74269+2^{22}\end{aligned}$$

are all primes, also $5297+2^5=73^2$ and $28433+2^7=13^4$ are prime powers. (Z. W. Sun thanks Yun-Zhi Zou and Si-Man Yang for their helps in obtaining the above data.)

Conjecture 13 [Acta Arith. 99(2001)]**.** *There are infinitely many odd integers not in the form $\pm 2^a \pm 2^b \pm p^\alpha$ where $a,b,\alpha \in \mathbb{N}$, p is a prime and any choice of signs can be made.*

Remark 13. R. Crocker [Pacific J. Math. 1971] showed that there are infinitely many positive odd integers not in the form $p+2^a+2^b$ where $a,b\in\mathbb{N}$ and p is a prime. Z. W. Sun and M. H. Le [Acta Arith., 99(2001)] showed that the only solutions of the diophantine equation

$$2^{2^n}-1=2^a+2^b+p^\alpha$$

with $n,a,b,\alpha\in\mathbb{N},\ a>b$ and p being a prime, are as follows:

$$\begin{aligned}&2^{2^2}-1=2^2+2+3^2=2^3+2^2+3=2^3+2+5,\\&2^{2^3}-1=2^3+2^2+3^5=2^7+2+5^3.\end{aligned}$$

On the basis of these work, P. Z. Yuan [Acta Arith. 115(2004), 23-28] confirmed Conjecture 13 completely.

Conjecture 14 [European J. Combin. 22(2001)]**.** *Let $A=\{a_iG_i\}_{i=1}^k$ be an exact m-cover of a group G by left cosets. If all the $G/(G_i)_G$ are solvable, then*

$$k\geqslant m+f([[G:G_1],\dots,[G:G_k]])$$

where f is the Mycielski function as in Conjecture 9.

Remark 14. I [European J. Combin. 22(2001)] proved the weaker inequality $k\geqslant m\geqslant \max_{1\leqslant i\leqslant k} f([G:G_i])$, and that $k\geqslant m+f([G:\bigcap_{i=1}^k G_i])$ if all the G_i are subnormal in G.

Conjecture 15 [European J. Combin. 22(2001), Remark 4.4]**.** *Let G be a group and H a subnormal subgroup of G with $[G:H]<\infty$. Let $H=H_0\subseteq H_1\subseteq\cdots\subseteq H_{n-1}\subseteq H_n=G$ be a finite chain of subgroups of G such that H_i is maximal normal in H_{i+1} for all $i=0,1,\ldots,n-1$, and define*

$$d(G,H)=\sum_{i=0}^{n-1}(|H_{i+1}/H_i|-1)$$

as in Z. W. Sun [Fund. Math. 134(1990)]*. Then $|N_G(H)/H|\geqslant d(G,H)$ where $N_G(H)$ denotes the normalizer of H in G.*

Remark 15. Z. W. Sun [Fund. Math. 134(1990)] showed that $[G:H]-1\geqslant d(G,H_G)=d(G,H)+d(H,H_G)$; Corollary 4.5 of Sun [European J. Combin. 22(2001)] says that $[G;H]-1\geqslant|N_G(H)/H|+d(H,H_G)$ if H is not normal in G.

Conjecture 16 [Acta Arith. 102(2002)]**.** *Let F be any field, and $A_1,\ldots,A_n$ be subsets of F which are finite and nonempty. For $1\le i<j\le n$ let S_{ij} and S_{ji} be finite subsets of F with $|S_{ij}|\equiv|S_{ji}|\pmod 2$. Then, for the set*

$$C=\{a_1+\cdots+a_n:\ a_1\in A_1,\ldots,a_n\in A_n\ \text{and}\ a_i-a_j\notin S_{ij}\ \text{if}\ i\neq j\}|,$$

we have

$$|C|\geq\min\left\{p(F),\sum_{i=1}^{n}|A_i|-\sum_{1\leq i<j\leq n}(|S_{ij}|+|S_{ji}|)-n+1\right\}$$

where $p(F)$ denotes the characteristic of the field F if it is a prime, and $p(F)=+\infty$ if F is of characteristic 0*.*

Remark 16. The conjecture is open even when F is the rational field $\mathbb{Q}$, the reader may consult Z. W. Sun [Acta Arith. 99(2001)] for related results. Qing-Hu Hou and I [Acta Arith. 102(2002)] confirmed Conjecture 16 on the condition that all the S_{ij} have the same cardinality.

Conjecture 17 [Discrete Math. 257(2002)]**.** *For positive integers $h_1,\ldots,h_n$ define*

$$m(h_1,\ldots,h_n)=\sum_{r_1=0}^{h_1-1}\cdots\sum_{r_n=0}^{h_n-1}\min\left\{\frac{r_1}{h_1},\ldots,\frac{r_n}{h_n}\right\}.$$

If $h_1,\ldots,h_n$ are pairwise coprime, then $m(h_1,\ldots,h_n)$ is a rational function in $h_1,\ldots,h_n$.

Remark 17. The conjecture is true for $n\leqslant 3$. Furthermore, Z. W. Sun [Discrete Math. 257(2002)] determined $m(h_1,h_2,h_3)$ explicitly.

Conjecture 18 [Electron. Res. Announc. Amer. Math. Soc. 9(2003)]. *Let $A = \{a_s(n_s)\}_{s=1}^k$ be a (finite) system of residue classes, and let n be a positive integer.*

(i) *If A covers every integer either exactly $2n-1$ times or exactly $2n$ times, then for any $c_1, \ldots, c_k \in \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ there exists an $I \subseteq \{1, \ldots, k\}$ such that $\sum_{s\in I} 1/n_s = n$ and $\sum_{s\in I} c_s = 0$.*

(ii) *If A covers every integer exactly $3n$ times, then for any $c_1, \ldots, c_k \in \mathbb{Z}_n \oplus \mathbb{Z}_n$ with $c_1 + \cdots + c_k = 0$, there exists an $I \subseteq \{1, \ldots, k\}$ such that $\sum_{s\in I} 1/n_s = n$ and $\sum_{s\in I} c_s = 0$.*

Remark 18. Z. W. Sun [Electron. Res. Announc. Amer. Math. Soc. 9(2003); `arXiv:math.NT/0305369`] was able to prove the above conjecture in the case where n is a prime power.

Conjecture 19 [J. Algebra 273(2004)]. *Let $\mathcal{A} = \{a_iG_i\}_{i=1}^k$ be a finite system of left cosets of subnormal subgroups of a group G. If not all the G_i are G and $\mathcal{A}$ covers each element of G with the same multiplicity, then for $n = \max_{1\leqslant i\leqslant k}[G : G_i]$ the number $|\{1 \leqslant i \leqslant k\colon [G : G_i] = n\}|$ is not less than the least prime divisor of n.*

Remark 19. Z. W. Sun [J. Algebra 273(2004)] showed that under the conditions of the conjecture, $\max_{n\in\mathbb{Z}^+} |\{1 \leqslant i \leqslant k\colon [G : G_i] = n\}|$ is not less than the least prime divisor of $\prod_{i=1}^k [G : G_i]$.

Conjecture 20 [`arXiv:math.GR/0411289`]. *Let $\mathcal{A} = \{a_iG_i\}_{i=1}^k$ be a finite system of left cosets of subgroups of an abelian group G. Suppose that $\mathcal{A}$ covers all the elements of G at least m times but none of its proper subsystems does. Then $k \geqslant m + f(N)$, where N is the least common multiple of the finite indices $[G : G_1], \ldots, [G : G_k]$, and f is the Mycieski function given in Conjecture 9.*

Remark 20. G. Lettl and Z. W. Sun [Acta Arith. 131(2008)] showed that under the conditions of the conjecture, we have $k \geqslant m + f([G : G_i])$ for every $i = 1, \ldots, k$.

Conjecture 21 (2005-02-19). *Let $q > 1$ be a power of two. Then the function $f : \mathbb{Z}^+ \to \mathbb{Z}$ given by*

$$f(k) = (4k-1)!! = 1 \times 3 \times \cdots \times (4k-1)$$

is q-normal, i.e., there are $c_r \in \mathbb{Z}$ with $0 < r < q$ and $\gcd(r, q) = 1$ such that

$$f(k) \equiv \sum_{\substack{0<r<q\\ \gcd(r,q)=1}} c_r r^k \pmod{q} \quad \textit{for all } k = 1, 2, 3, \ldots.$$

Remark 21. (i) On Feb. 17, 2005, H. Cohen asked whether the order of $F(n) = \sum_{k=0}^n \binom{n}{k}(-1)^k(4k-1)!!$ at 2 is n. Later K. Buzzard said that $2^n \| F(n)$ follows from the recursion

$$16(k+1)(k+2)f(k) - 32(k+2)^2 f(k+1) + (16k^2 + 80k + 98)f(k+2) + f(k+3) = 0$$

found by the WZ method. If $f(k)=(4k-1)!!$ is 2^n-normal, then

$$F(n)=\sum_{k=0}^n\binom{n}{k}(-1)^kf(k)\equiv\sum_{\substack{0<r<2^n\\ 2\nmid r}}c_r(1-r)^n\equiv 0\ (\mathrm{mod}\ 2^n).$$

(ii) The concept of q-normal function was first introduced by Z. W. Sun in [Discrete Math. 262(2003)]. In the recent preprint `arXiv:math.NT/0502187`, Sun and R. Tauraso proved that the function $g(n)=\sum_{k\equiv r\ (\mathrm{mod}\ p-1)}\binom{n}{k}$ is q-normal for any power q of odd prime p.

Conjecture 22 [Acta Arith. 127(2007)]**.** (i) *Any positive integer n can be written in the form $x^2+(2y+1)^2+T_z$, where x,y,z are integers and $T_z=z(z+1)/2$ is a triangular number.*

(ii) *Each natural number n can be written in the form $x^2+2y^2+3T_z$ (with $x,y,z\in\mathbb{Z}$) except $n=23$, in the form $x^2+5y^2+2T_z$ (or the equivalent form $5x^2+T_y+T_z$) except $n=19$, in the form $x^2+6y^2+T_z$ except $n=47$, in the form $2x^2+4y^2+T_z$ except $n=20$.*

Remark 22. Z. W. Sun [Acta Arith. 127(2007)] has verified this conjecture for $n\leqslant 10^4$, and proved part (i) for $n\notin\{T_{4m}: m\in\mathbb{Z}^+\}$. Part (i) has been confirmed by B.-K. Oh and Z. W. Sun in a preprint available from `http://arxiv.org/abs/0804.3750`

Conjecture 23 (March, 2006)**.** *There are only finitely many intervals (x^m,y^n) with integers $x,y,m,n>1$ that contain no primes; namely, these intervals are as follows:*

$$(2^3,3^2),\ (5^2,3^3),\ (2^5,6^2),\ (11^2,5^3),\ (3^7,13^3),\ (5^5,56^2),$$
$$(181^2,2^{15}),\ (43^3,282^2),\ (46^4,312^2),\ (22434^2,55^5).$$

Remark 23. In a message posted to Number Theory List on March 23, 2006, Stephen Redmond asked whether each interval (x^m,y^n) with $x,y>1$ and $m,n\geqslant 3$ contain a prime and claimed that this had been verified for $y^n\leqslant 5\times 10^7$. Then, on March 25, I posted a message to Number Theory List in which I formulated an extension of Redmond's conjecture: The only intervals (x^m,y^n) with $x,y,m,n>1$ that contain no primes are

$$(2^3,3^2),\ (5^2,3^3),\ (2^5,6^2),\ (11^2,5^3).$$

Later I checked this conjecture up to powers not exceeding 10^6 via computer (with help of S. Guo) and found more exceptions:

$$(3^7,13^3),\ (5^5,56^2),\ (181^2,2^{15}),\ (43^3,282^2),\ (46^3,312^2).$$

J. Mc. Laughlin also noted this 5 exceptions almost simultaneously, and K. Buzzard found the new exception $(22434^2,55^5)$. C. Pomerance informed me that my conjecture for sufficiently large powers is reasonable in view of the famous *abc* conjecture and the conjecture that for each $c > 0$ and all large x there is a prime between x and $x + x^c$. On March 28, I posted a summary of the development to Number Theory List and still believed that Conjecture 23 should be true.

On March 30, Tony D. Noe wrote to me that he had verified Conjecture 23 up to 10^{16} and added a sequence to the On-Line Encyclopedia of Integer Sequences (cf. `http://www.research.att.com/~njas/sequences/A116086`) in view of Conjecture 23. On August 2, 2007, a term *Redmond-Sun conjecture* was created and posted on PlaneMath at the website
`http://planetmath.org/encyclopedia/RedmondSunConjecture.html`

Conjecture 24. (i) (March 23, 2008) *Each natural number $n \neq 216$ can be written in the form $p + T_x$, where p is zero or a prime, and $T_x = x(x+1)/2$ is a triangular number. We can require further that if p is nonzero then $p \equiv 1 \pmod 4$ when $n > 88956$, and $p \equiv 3 \pmod 4$ when $n > 90441$.*

(ii) (April 4, 2008) *In general, for any $a, b = 0, 1, 2, \ldots$ and odd integer r, all sufficiently large integers can be written in the form $2^a p + T_x$ with $x \in \mathbb{Z}$, where p is either zero or a prime congruent to r modulo 2^b.*

Remark 24. The main motivation of Conjecture 24 is the following two known facts: (1) (Fermat) Each prime $p \equiv 1 \pmod 4$ can be written as a sum of an even square and an odd square. (2) [Z. W. Sun, Acta Arith. 127(2007)] If a positive integer is not a triangular number, then we can write it as a sum of an even square, an odd square and a triangular number.

In March and April, 2008, I posted several messages concerning Conjecture 24 and related things to the Number Theory Mailing List. Conjecture 24 and related data and discussion can be found in my preprint *Conjectures on sums of primes and triangular numbers* available from `http://arxiv.org/abs/0803.3737` or `http://math.nju.edu.cn/~zwsun/PrimeTri.pdf`. See also the entry *"Sun's conjecture on sums of primes and triangular numbers"* on PlanetMath with the website `http://planetmath.org/?op=getobj;from=objects;id=10594`
The first assertion in Conjecture 24 has been verified by Douglas McMeil up to 10^{10}, the reader may visit
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0901&L=nmbrthry&T=0&P=840`.

Conjecture 25. (March 30, 2008) *Each natural number $n > 864$ can be written as a sum of two even squares and a triangular number. Any integer $n > 1029$ is either a triangular number, or a sum of a triangular number and two odd squares.*

Remark 25. Since any prime $p \equiv 1 \pmod 4$ can be written as a sum of an even square and an odd square, Conjecture 25 for sufficiently large integers follows from Conjecture 24 concerning the forms $2p + T_x$ and $4p + T_x$ with p zero or a prime congruent to 1 mod 4. Conjecture 25 was first made public via a message sent to the Number Theory Mailing List on March 30, 2008, here is a link to the message:
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0803&L=nmbrthry&T=0&P=3586`.
It also appeared in a preprint of mine available from `http://arxiv.org/abs/0803.3737` or `http://math.nju.edu.cn/~zwsun/PrimeTri.pdf`.

Conjecture 26. (May 10, 2008) (i) *Any odd integer $n > 3$ can be written in the form $p + x(x+1)$ with p a prime and x a positive integer. We can require further that $p \equiv 1 \pmod 4$ if n is not among the following 30 multiples of three:*

$$3, 9, 21, 27, 45, 51, 87, 105, 135, 141, 189, 225, 273, 321, 327, 471, 525,$$
$$627, 741, 861, 975, 1197, 1461, 1557, 1785, 2151, 12285, 13575, 20997, 49755.$$

Also, we can require further that $p \equiv 3 \pmod 4$ if n is not among the following 15 multiples of three:

$$57, 111, 297, 357, 429, 615, 723, 765, 1185, 1407, 2925, 3597, 4857, 5385, 5397.$$

(ii) In general, for any $a = 0, 1, 2, \ldots$ and odd integer r, all sufficiently large integers can be written in the form $p + x(x+1)$ with $x \in \mathbb{Z}$, where p is a prime congruent to r modulo 2^a. For example, if we use N_r to denote the largest odd integer not in the form $p + x(x+1)$ with $p \equiv r \pmod 8$, then

$$N_1 = 358245,\ N_3 = 172995,\ N_5 = 359907,\ N_7 = 444045.$$

Remark 26. I have verified Conjecture 26 for odd integers not exceeding 10^6. The conjecture was first made public via a message sent to the Number Theory Mailing List; here is a link to the message:
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0805&L=nmbrthry&T=0&P=398`.
In Jan. 2009 Dr. Douglas McNeil verified the first assertion in Conjecture 26(i) up to 10^{12}, the reader may consult
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0901&L=nmbrthry&T=0&P=840`.

I'd like to offer 1000 US dollars for a positive solution to the first assertions in Conjectures 24 and 26, and 200$ for the first explicit counterexample to the first assertion in Conjecture 24 or 26.

Conjecture 27. (December 23, 2008) *Any integer $n > 4$ can be written as the sum of an odd prime, an odd Fibonacci number and a positive Fibonacci number.*

Remark 27. On Dec. 23, 2008, I formulated the conjecture and verified it for $n \leqslant 3 \times 10^7$. The verification was continued by my former student Song Guo and Dr. Douglas McNeil (University of London). It is now known that there are no counterexamples below 10^{14}. The conjecture was discussed via messages sent to the Number Theory Mailing List; here are links to the related websites:

(1) *A promising conjecture $n = p + F_s + F_t$* (Zhi-Wei Sun)
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0812&L=nmbrthry&T=0&P=2140`
(2) *A summary concerning my conjecture $n = p + F_s + F_t$* (Zhi-Wei Sun)
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0812&L=nmbrthry&T=0&P=2704`
(3) *A summary concerning my conjecture $n = p + F_s + F_t$ (II)* (Zhi-Wei Sun)
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0812&L=nmbrthry&T=0&P=3124`

(4) *Sun's strong conjecture* (Douglas McNeil)
http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0812&L=nmbrthry&T=0&P=3020
(5) *Sequence A154257 in OEIS* (Zhi-Wei Sun)
http://www.research.att.com/∼njas/sequences/A154257

Conjecture 28. (Dec. 25, 2008–Jan. 8, 2009) *Any integer $n > 4$ can be written in any of the following forms:*

$$\begin{aligned}
&p + F_s + F_t^2 \ (p \textit{ is an odd prime}, \ s,t > 0, \ F_s \textit{ or } F_t \textit{ is odd}),\\
&p + F_s + F_t^3 \ (p \textit{ is an odd prime}, \ s,t > 0, \ F_s \textit{ or } F_t \textit{ is odd}),\\
&p + F_s + 2F_t \ (p \textit{ is an odd prime}, \ s,t > 0),\\
&p + L_s + L_t \ (p \textit{ is an odd prime}, \ L_s \textit{ or } L_t \textit{ is odd}),\\
&p + F_s + F_{3t}/2 \ (p \textit{ is an odd prime}, \ s,t > 0),\\
&p + F_s + L_t \ (p \textit{ is an odd prime}, \ s > 0, \ F_s \textit{ or } L_t \textit{ is odd}),\\
&p + 2F_s + F_t^2 \ (p \textit{ is an odd prime}, \ s,t > 0),\\
&p + L_s + L_t^2, \ p + L_s + L_t^3 \ \ (p \textit{ is an odd prime}, \ L_s \textit{ or } L_t \textit{ is odd}).
\end{aligned}$$

Remark 28. The first to the fifth has been verified up to 10^{12} by Dr. D. McNeil, the sixth has been verified up to 10^{13}, the last three has been verified up to 3×10^7. (Besides Conjecture 28, Conjecture 27 also has other variants.) The reader may visit the following websites to get further information:
http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0901&L=nmbrthry&T=0&P=840

http://www.research.att.com/∼njas/sequences/A154258
http://www.research.att.com/∼njas/sequences/A154263
http://www.research.att.com/∼njas/sequences/A154285
http://www.research.att.com/∼njas/sequences/A154417
http://www.research.att.com/∼njas/sequences/A154290
http://www.research.att.com/∼njas/sequences/A154421

Conjecture 29. (Jan. 10, 2009) *Any integer $n > 5$ can be written as the sum of an odd prime, a positive Pell number and twice a positive Pell number.*

Remark 29. On Jan. 10, 2009, I formulated the conjecture and verified it for $n \leqslant 5 \times 10^7$. The verification was continued by my former student Song Guo and Dr. Douglas McNeil (University of London). It is now known that there are no counterexamples below 10^{13}. See the related sequence A154536 in OEIS via
http://www.research.att.com/∼njas/sequences/A154536.

Conjecture 30. (Jan. 16, 2009) *Any integer $n > 4$ can be written as the sum of an odd prime, a Lucas number and a Catalan number.*

Remark 30. On Jan. 16, 2009, I formulated the conjecture and verified it for $n \leqslant 5 \times 10^6$. The verification was continued by Dr. Douglas McNeil (University of

London). It is now known that there are no counterexamples below 10^{13}. See the related sequence A154940 in OEIS via
`http://www.research.att.com/~njas/sequences/A154940`
The conjecture is similar to Qing-Hu Hou and Jiang Zeng's conjecture that any integer $n > 4$ can be written as the sum of an odd prime, a positive Fibonacci number and a Catalan number (see sequence A154404 in OEIS) which has been verified by McNeil up to 3×10^{13}. Both conjectures were motivated by Conjecture 27 which was posed by me on Dec. 23, 2008.

Conjecture 31. (April 16-17, 2009) *Any natural number n can be written as the sum of two squares and a pentagonal number, as the sum of a triangular number, an even square and a pentagonal number, and as the sum of a square, a pentagonal number and a hexagonal number.*

Remark 31. I have verified the conjecture for $n \leqslant 10^6$. For the number of ways to write $n \in \mathbb{N}$ in a required form, please visit
`http://www.research.att.com/~njas/sequences/A160324`
`http://www.research.att.com/~njas/sequences/A160325`
`http://www.research.att.com/~njas/sequences/A160326`

Conjecture 32. (April 18, 2009) *$p_i + p_j + p_k$ is universal (over $\mathbb{N}$) (i.e., for any $n \in \mathbb{N}$ there are $x, y, z \in \mathbb{N}$ such that $n = p_i(x) + p_j(y) + p_k(z)$) if (i, j, k) is among the following* 31 *triples:*

$$\begin{gathered}(3,3,5),\ (3,3,6),\ (3,3,7),\ (3,3,8),\ (3,3,10),\ (3,3,12),\ (3,3,17),\\ (3,4,5),\ (3,4,6),\ (3,4,7),\ (3,4,8),\ (3,4,9),\ (3,4,10),\ (3,4,11),\\ (3,4,12),\ (3,4,13),\ (3,4,15),\ (3,4,17),\ (3,4,18),\ (3,4,27),\\ (3,5,5),\ (3,5,6),\ (3,5,7),\ (3,5,8),\ (3,5,9),\ (3,5,11),\ (3,5,13),\\ (3,7,8),\ (3,7,10),\ (4,4,5),\ (4,5,6).\end{gathered}$$

Remark 32. For $m = 3, 4, 5, \ldots$ those $p_m(x) = (m-2)\binom{x}{2} + x$ $(x = 0, 1, 2, \ldots)$ are called m-gonal numbers. I have verified the conjecture for $n \leqslant 5 \times 10^5$, and proved that if $p_i + p_j + p_k$ is universal with $3 \leqslant i \leqslant j \leqslant k$ and $k \geqslant 5$ then (i, j, k) must be on the list of 31 triples given in Conjecture 32. See also my message *Various new conjectures involving polygonal numbers and primes* posted to Number Theory List available from
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0905&L=nmbrthry&T=0&P=940`

Conjecture 33. (April 18, 2009) *$ap_i + bp_j + cp_k$ is universal (over $\mathbb{N}$) (i.e., for any $n \in \mathbb{N}$ there are $x, y, z \in \mathbb{N}$ such that $n = ap_i(x) + bp_j(y) + cp_k(z)$) if (ap_i, bp_j, cp_k)*

is on the following list of 64 *triples:*

$$\begin{gathered}(p_3,p_3,2p_5),\ (p_3,p_3,4p_5),\ (p_3,2p_3,p_5),\ (p_3,2p_3,4p_5),\ (p_3,3p_3,p_5),\\ (p_3,4p_3,p_5),\ (p_3,4p_3,2p_5),\ (p_3,6p_3,p_5),\ (p_3,9p_3,p_5),\ (2p_3,3p_3,p_5),\\ (p_3,2p_3,p_6),\ (p_3,2p_3,2p_6),\ (p_3,2p_3,p_7),\ (p_3,2p_3,2p_7),\ (p_3,2p_3,p_8),\\ (p_3,2p_3,2p_8),\ (p_3,2p_3,p_9),\ (p_3,2p_3,2p_9),\ (p_3,2p_3,p_{10},\ (p_3,2p_3,p_{12}),\\ (p_3,2p_3,2p_{12}),\ (p_3,2p_3,p_{15}),\ (p_3,2p_3,p_{16}),\ (p_3,2p_3,p_{17}),\ (p_3,2p_3,p_{23}),\\ (p_3,p_4,2p_5),\ (p_3,2p_4,p_5),\ (p_3,2p_4,2p_5),\ (p_3,2p_4,4p_5),\ (p_3,3p_4,p_5),\\ (p_3,4p_4,p_5),\ (p_3,4p_4,2p_5),\ (2p_3,p_4,p_5),\ (2p_3,p_4,2p_5),\ (2p_3,p_4,4p_5),\\ (2p_3,3p_4,p_5),\ (3p_3,p_4,p_5),\ (p_3,2p_4,p_6),\ (2p_3,p_4,p_6),\ (p_3,p_4,2p_7),\\ (2p_3,p_4,p_7),\ (p_3,p_4,2p_8),\ (p_3,2p_4,p_8),\ (p_3,3p_4,p_8),\ (2p_3,p_4,p_8),\\ (2p_3,3p_4,p_8),\ (p_3,p_4,2p_9),\ (p_3,2p_4,p_9),\ (2p_3,p_4,p_{10}),\ (2p_3,p_4,p_{12}),\\ (p_3,2p_4,p_{17}),\ (2p_3,p_4,p_{17}),\ (p_3,p_5,4p_6),\ (p_3,2p_5,p_6),\ (p_3,p_5,2p_7),\\ (p_3,p_5,4p_7),\ (p_3,2p_5,p_7),\ (3p_3,p_5,p_7),\ (p_3,p_5,2p_9),\ (2p_3,p_5,p_9),\\ (p_3,2p_6,p_8),\ (p_3,p_7,2p_7),\ (p_4,2p_4,p_5),\ (2p_4,p_5,p_6).\end{gathered}$$

Remark 33. Let $a,b,c\in\mathbb{Z}^+$ with $\max\{a,b,c\}>1$, and let $i,j,k\in\{3,4,\dots\}$ with $i\leqslant j\leqslant k$ and $\max\{i,j,k\}\geqslant 5$. Suppose that (ap_i,bp_j,cp_k) is universal (over $\mathbb{N}$) with $a\leqslant b$ if $i=j$, and $b\leqslant c$ if $j=k$. I have proved that (ap_i,bp_j,cp_k) must be among the 64 triples given in Conjecture 33. See my preprint *On universal sums of polygonal numbers* available from `http://arxiv.org/abs/0905.0635`.

Conjecture 34. (May 15, 2009). *$p_5+bp_5+cp_5$ is universal over $\mathbb{Z}$ (i.e., for any $n\in\mathbb{N}$ there are $x,y,z\in\mathbb{Z}$ such that $n=p_5(x)+bp_5(y)+cp_5(z)$) if (b,c) is among the following ordered pairs:*

$$(1,6),(1,8),(1,9),(1,10),(2,8),(3,8).$$

Remark 34. This conjecture together with Theorem 1.1 in my preprint *On universal sums of polygonal numbers* and my another paper with F. Ge available from `http://arxiv.org/abs/0906.2450`, yields the complete determination of all those universal $ap_k+bp_k+cp_k$ over $\mathbb{Z}$.

Conjecture 35. (August 21, 2009). *For any integer $m\geqslant 3$, each natural number n can be written in the form $p_{m+1}(x_1)+p_{m+2}(x_2)+p_{m+3}(x_3)+r$ with $x_1,x_2,x_3\in\mathbb{N}$ and $r\in\{0,\dots,m-3\}$.*

Remark 34. Since each of $0,\dots,m-3$ can be written in the form $p_{m+4}(x_4)+\dots+p_{2m}(x_m)$ with $x_4,\dots,x_m\in\{0,1\}$, the conjecture implies that every natural number is the sum of an $(m+1)$-gonal number, an $(m+2)$-gonal number, $\dots$, and a $2m$-gonal number. It is interesting to compare this with Fermat's assertion

that any natural number is the sum of m m-gonal numbers. I have verified the conjecture in the following cases: (1) $m = 3$ and $n \leqslant 10^6$; (2) $4 \leqslant m \leqslant 10$ and $n \leqslant 5 \times 10^5$; (3) $11 \leqslant m \leqslant 40$ and $n \leqslant 10^5$. I would like to offer 500 US dollars for the first rigorous proof of the whole conjecture. See also my message *A challenging conjecture on sums of polygonal numbers* posted to Number Theory List available from
`http://listserv.nodak.edu/cgi-bin/wa.exe?A2=ind0908&L=nmbrthry&T=0&P=802`