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Introduction to Zhi-Wei Sun's Papers on Covers

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1. Zhi-Wei Sun and Zhi-Hong Sun, *Some results on covering systems of congruences*, J. Southwest-China Teachers Univ., 1987, no.1, 10–15. Zbl. M. 749.11018.

It was showed that, if $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ but $\{a_s(n_s)\}_{\substack{s=1 \\ d\nmid n_s}}^k$ is not where d is an integer greater than $n_0 = 1$, then

$$|\{a_s \text{mod } d : 1 \leqslant s \leqslant k \ \&\ d \mid n_s\}| \geqslant \frac{d}{(d, [n_s]_{\substack{0\leqslant s\leqslant k \\ d\nmid n_s}})} = \gcd_{\substack{0\leqslant s\leqslant k \\ d\nmid n_s}} \frac{d}{(d, n_s)}. \qquad (1)$$

Thus when A forms a minimal (i.e. irredundant) cover of $\mathbb{Z}$ for any prime power p^α dividing some of $n_1, \cdots, n_k$ we have

$$|\{1 \leqslant s \leqslant k : \ p^\alpha \mid n_s\}| \geqslant p^\delta$$

where δ is the smallest positive integer such that $p^{\alpha-\delta}$ divides one of those $n_0 = 1, \ n_1, \cdots, n_k$ not divisible by p^α.

2. Zhi-Wei Sun, *Systems of congruences with multipliers*, Nanjing Univ. J. Math. Biquarterly, **6**(1989), no. 1, 124–133. MR 90m:11006; Zbl. M. 703.11002.

Let M be an additive commutative monoid (e.g. $M = \mathbb{N} = \{0, 1, 2, \cdots\}$). Let $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^k$ $(0 \leqslant a_s < n_s)$ where those $\lambda_s \in \Lambda \subseteq M$ are *multipliers* or *weights* of the residue classes $a_s(n_s) = a_s + n_s\mathbb{Z}$ respectively. Define the *covering map* $w_A : \mathbb{Z} \to M$ by

$$w_A(x) = \sum_{s=1}^k \lambda_s \chi_s(x) \qquad (2)$$

where $\chi_s(x)$ is 1 if $x \in a_s(n_s)$, and 0 otherwise. If B is also such a system, putting all the triples in A and B together we get the sum-system $A \sqcup B$ (triples in it may be repeated). Theorem 1 of the paper is

The Generating Theorem. *Let $S \subseteq M$ and S_* denote the class of those triple systems A with $w_A(\mathbb{Z}) \subseteq S$. Then we can generate all those $A \in S_*$ as follows:*

(i) *If $\lambda_1, \cdots, \lambda_k \in \Lambda$ and $\lambda_1 + \cdots + \lambda_k \in S$, then $\{(\lambda_s, 0, 1)\}_{s=1}^k \in S_*$;*

(ii) *Let p be a prime, and*

$$A_r = \{(\lambda_s, a_{sr}, n_s)\}_{s=1}^k \sqcup \{(\lambda_j^{(r)}, a_j^{(r)}, n_j^{(r)})\}_{j=1}^{h(r)} \qquad (3)$$

lie in S_* *for all* $r \in R(p) = \{0, 1, \cdots, p-1\}$ *where* $\max_{r\in R(p)} h(r) > 0$, *and for each* $s = 1, \cdots, k$ *the modulus* n_s *is prime to* p *and there is (a unique)* $a_s \in R(n_s)$ *such that* $a_s \equiv r + pa_{sr} \pmod{n_s}$ *for all* $r \in R(p)$. *Then*

$$A = \{(\lambda_s, a_s, n_s)\}_{s=1}^{k} \sqcup \bigsqcup_{r=0}^{p-1} \{(\lambda_j^{(r)}, r + pa_j^{(r)}, pn_j^{(r)})\}_{j=1}^{h(r)} \tag{4}$$

belongs to S_*.

By Corollary 2 of this theorem, we can generate all those exact m-covers $A = \{a_s(n_s)\}_{s=1}^k$ (identified with $\{(1, a_s, n_s)\}_{s=1}^k$) as follows:

(a) *The system of* m *copies of* $0(1)$ *is an exact* m*-cover;*

(b) *Let* p *be a prime and* A_r $(r \in R(p))$ *(all the* λ*'s are 1) be as in the above theorem and* $h(r) \geqslant 1$ *for all* $r \in R(p)$. *Then the system* A *given by (4) is an exact* m*-cover.*

Also, we can generate all those m-covers $A = \{a_s(n_s)\}_{s=1}^k$ ($w_A(\mathbb{Z}) \subseteq \{m, m+1, \cdots\}$) by means of (a), (b) and the following (c).

(c) *If* A *is an* m*-cover, then so is* $A \sqcup \{a(n)\}$ *where* $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots\}$ *and* $a \in R(n)$.

Let $B = \{(\mu_t, b_t, m_t)\}_{t=1}^l$ with $\mu_t \in \Lambda \subseteq M$. If $w_A = w_B$ then we said that A and B are *equivalent* which is written as $A \sim B$. Clearly $\{r(n)\}_{r=0}^{n-1} \sim \{0(1)\}$, and $A = \{a_s(n_s)\}_{s=1}^k$ forms an exact m-cover if and only if $A \sim \{(m, 0, 1)\} = \{\underbrace{0(1), \cdots, 0(1)}_{m \text{ times}}\}$.

We deduced from the Generating Theorem the following (Theorem 4 in the paper) by induction.

Main Theorem on the Equivalence. *Let* P *be a set of primes and* f *a mapping into a left* R*-module* M *such that* $(\frac{x+r}{p}, py) \in Dom(F)$ *for all* $r \in R(p)$ *if* $p \in P$ *and* $(x, y) \in Dom(F)$. *Then the following* $(\star)$ *and* $(*)$ *are equivalent:*

$(\star)$ *Whenever* $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^k \sim B = \{(\mu_t, b_t, m_t)\}_{t=1}^l$ *with weights in* R *and prime divisors of the moduli in* P, *we have*

$$\sum_{s=1}^{k} \lambda_s F\left(\frac{x+a_s}{n_s}, n_s y\right) = \sum_{t=1}^{l} \mu_t F\left(\frac{x+b_t}{m_t}, m_t y\right) \quad \text{for all } (x, y) \in Dom(F); \tag{5}$$

$$\sum_{r=0}^{p-1} F\left(\frac{x+r}{p}, py\right) = F(x, y) \quad \text{for all } p \in P \text{ and } (x, y) \in Dom(F). \tag{$*$}$$

This theorem is very powerful, it unifies almost all known identities for systems with a given covering map (such as exact m-covers). A slight generalization and a direct proof were given in my paper *Products of binomial coefficients modulo* p^2, Acta Arith. 97(2001), 87–98. We call those F satisfying the above (P, M)-*equivalent maps.* For M being the complex field $\mathbb{C}$, we listed the following examples of equivalent maps in the paper:

$$F(x,y) = [x],\ B_m(x)y^{m-1},\ \frac{\cot(\pi x)}{y},\ \sum_{a=-\infty}^{+\infty} g\left(\frac{a}{y}\right) e^{2\pi i a x}. \tag{6}$$

($B_m(x)$ is the mth Bernoulli polynomial and the corresponding example is another form of Raabe's formula $\sum_{r=0}^{n-1} B_m(z+\frac{r}{n}) = n^{1-m}B_m(nz)$.) For M being the $\mathbb{Z}$-module $\mathbb{C}^* = \mathbb{C}\setminus\{0\}$, we gave examples

$$F(x,y) = 2\sin(\pi x),\ \Gamma(x)y^{x-1/2}/\sqrt{2\pi}$$

where the last example is an equivalent form of Gauss' multiplication formula $\prod_{r=0}^{n-1}\Gamma(z+\frac{r}{n}) = (2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz)$.

The empty system $\emptyset$ has covering map $w_\emptyset = 0$. When multipliers are in an abelian group, $A \sim B \iff A \sqcup (-B) \sim \emptyset$. For example, $\{a_s(n_s)\}_{s=1}^k$ forms an exact m-cover if and only if $\{(1,a_1,n_1),\cdots,(1,a_k,n_k),(-m,0,1)\} \sim \emptyset$. Thus, to study the equivalence, we need only to consider those $A = \{(\lambda_s,a_s,n_s)\}_{s=1}^k \sim \emptyset$.

By use of the equivalent map $F(x,y) = \Gamma(x)y^{x-1/2}/\sqrt{2\pi}$, we concluded the paper with

Corollary 3. *Let* $A = \{(\lambda_s,a_s,n_s)\}_{s=1}^k \sim \emptyset$ *where* $\lambda_s,a_s,n_s \in \mathbb{Z}$ *and* $0 \leqslant a_s < n_s$. *Then*

$$\prod_{s=1}^{n}\left(\Gamma\left(\frac{z+a_s}{n_s}\right) n_s^{\frac{z+a_s}{n_s}-\frac{1}{2}}/\sqrt{2\pi}\right)^{\lambda_s} = 1 \tag{7}$$

for all $n \in \mathbb{Z}^+$ *and* $z \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}$.

Consequently, if $A = \{a_s(n_s)\}_{s=1}^k$ is an exact cover of $\mathbb{Z}$ with $a_1 = 0$, then we have the relative formula

$$\frac{n_1^{\frac{z}{n_1}-\frac{1}{2}}\prod_{s=2}^k\left(\Gamma\left(\frac{z+a_s}{n_s}\right)n_s^{\frac{z+a_s}{n_s}-\frac{1}{2}}\right)}{n_1^{-\frac{1}{2}}\prod_{s=2}^k\left(\Gamma\left(\frac{a_s}{n_s}\right)n_s^{\frac{a_s}{n_s}-\frac{1}{2}}\right)} = \frac{(2\pi)^{\frac{k-1}{2}}\Gamma(z)/\Gamma\left(\frac{z}{n_1}\right)}{(2\pi)^{\frac{k-1}{2}}\lim_{z'\to 0}\frac{\Gamma(z')z'}{\Gamma\left(\frac{z'}{n_1}\right)\frac{z'}{n_1}\cdot n_1}}$$

for $z \neq 0,-1,-2,\cdots$, i. e.,

$$\Gamma(z) = \frac{\Gamma\left(\frac{z}{n_1}\right)}{n_1^{1-\frac{z}{n_1}}}\prod_{s=2}^k\frac{\Gamma\left(\frac{z+a_s}{n_s}\right)}{n_s^{-\frac{z}{n_s}}\Gamma\left(\frac{a_s}{n_s}\right)} \quad \text{for } z\in\mathbb{C} \text{ with } z\neq 0,-1,-2,\cdots.$$

In view of the above, as you can find, most of the identities in the following papers essentially follow from my earlier work.

B1. J. Beebee, *Some trigonometric identities related to exact covers*, Proc. Amer. Math. Soc. **112**(1991), 329–338. MR 91i:11013.

B2. J. Beebee, *Bernoulli numbers and exact covering systems*, Amer. Math. Monthly, **99**(1992), 946–948. MR 93i:11025.

B3. J. Beebee, *Exact covering systems and the Gauss—Legendre multiplication formula for the gamma function*, Proc. Amer. Math. Soc., **120**(1994), 1061–1065. MR 94f:33001.

P1. Š. Porubský, *Identities involving covering systems I*, Math. Slovaca, **44**(1994), 153–162. MR 95f:11002.

P2. Š. Porubský, *Identities involving covering systems II*, Math. Slovaca, **44**(1994), 555–568.

3. Zhi-Wei Sun, *Several results on systems of residue classes*, Adv. Math. (China), **18**(1989), no.2, 251–252.

In this note we announced several results. Here is Theorem 3 in it.

Theorem 3. *Let f be a complex valued function defined on $D = \{(r,n) : r,n \in \mathbb{Z},\ 0 \leqslant r < n\}$. Then $\sum_{s=1}^k \lambda_s f(a_s,n_s) = 0$ for all those $A = \{(\lambda_s,a_s,n_s)\}_{s=1}^k \sim \emptyset$ ($\lambda_s \in \mathbb{C},\ a_s \in R(n_s)$), if and only if f can be written in the form*

$$f(a,n) = \frac{1}{n}\sum_{m=0}^{n-1} g\left(\frac{m}{n}\right)e^{2\pi i\frac{m}{n}a}. \tag{8}$$

In the paper we said that the key idea of the proof is that both are equivalent to

$$\sum_{j=0}^{n-1} f(a+jd,nd) = f(a,d) \quad \text{for all } n \in \mathbb{Z}^+ \text{ and } (a,d) \in D. \tag{9}$$

A detailed proof is contained in my recent paper *Algebraic approaches to periodic arithmetical maps*, J. Algebra, 240(2001), 723–743.

4. Zhi-Wei Sun, *Finite coverings of groups*, Fund. Math. **134**(1990), no. 1, 37–53. MR 91g:20031; Zbl. M. 717.20020.

For a subnormal subgroup H of finite index in a group G we let

$$d(G,H) = \sum_{i=1}^{n}([H_i : H_{i-1}] - 1) \tag{10}$$

where $H_0 = H \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ is a maximal chain of subgroups of G such that H_{i-1} is normal in H_i for every $i = 1, \cdots, n$. It is easy to see that $d(G, H) \geqslant f([G : H])$ where the Mycielski function $f : \mathbb{Z}^+ \to \mathbb{N}$ is given by $f(\prod_{i=1}^r p_i^{\alpha_i}) = \sum_{i=1}^r \alpha_i(p_i - 1)$ ($p_1, \cdots, p_r$ are distinct primes).

The main result of the paper is the following:

Let H be a subnormal subgroup of finite index in a group G. Then $1 + d(G, H)$ is the least $k \in \mathbb{Z}^+$ such that there exists a partition of G into cosets $a_1G_1, \cdots, a_kG_k$ in which all the G_i are subnormal in G and $\bigcap_{i=1}^k G_i = H$. Moreover, if $\{a_iG_i\}_{i=1}^k$ is a partition of G with all the G_i subnormal, then for any subgroup $K \not\subseteq \bigcap_{i=1}^k G_i$ we have

$$|\{1 \leqslant i \leqslant k : \ K \not\subseteq G_i\}| \geqslant 1 + d\Big(K, K \cap \bigcap_{i=1}^k G_i\Big). \tag{11}$$

As an application Corollary 3 of the paper states as follows:

Let G be a group and H its subnormal subgroup of finite index. Let H_G denote the core of H in G (i.e. the largest normal subgroup of G contained in H). Then

$$2^{[G:H]-1} \geqslant [G : H_G] \geqslant [G : H] \geqslant 1 + d(G, H_G) \geqslant 1 + f([G : H_G]). \tag{12}$$

5. Zhi-Wei Sun, *A theorem concerning systems of residue classes*, Acta Math. Univ. Comenian. (N. S.) **60**(1991), no. 1, 123–131. MR 92f:11007; Zbl. M. 734.11022.

Below is the main theorem of the paper.

Theorem. *Let n_0 be the smallest covering period of $X = \bigcup_{s=1}^k a_s(n_s)$ (i.e. the least positive integer such that $x \pm n_0 \in X$ whenever $x \in X$). Then*

$$\frac{(n_1, \cdots, n_k)}{(n_0, n_1, \cdots, n_k)} \leqslant \max_{n \in \mathbb{Z}^+} |\{1 \leqslant i \leqslant k : \ n_i = n\}| \sum_{d \mid \frac{[n_1, \cdots, n_k]}{(n_1, \cdots, n_k)}} \frac{1}{d}. \tag{13}$$

A consequence of this theorem is the following strengthenment of the Burshtein conjecture, which is better than corresponding results of R.J. Simpson (1986) and Berger-Felzenbaum-Fraenkel (1987).

Corollary. *Let n_0 be the smallest positive covering period of $\{a_s(n_s)\}_{s=1}^k$, and $N = [n_1, \cdots, n_k]$ have the standard form $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ where $p_1, \cdots, p_r$ are distinct primes. Suppose that $0 \notin I = \{0 \leqslant s \leqslant k : p_t^{\alpha} \mid n_s\} \neq \emptyset$, and that $a_i(n_i) \cap a_j(n_j) = \emptyset$ whenever $i \in I$ and $j \notin I$. Then*

$$p_t^{\delta_t(\alpha)} \leqslant \varepsilon_t(\alpha) \max_{s \in I} |\{i \in I : n_i = n_s\}| \prod_{i=1}^r \frac{p_i}{p_i - 1}, \tag{15}$$

where $\delta_t(\alpha) = \min\{\delta \geqslant 1 : p_t^{\alpha-\delta} \| n_s$ *for some* $0 \leqslant s \leqslant k\}$ *and*

$$\varepsilon_t(\alpha) = \left(1 - \frac{1}{p_t^{\alpha_t - \alpha + 1}}\right) \prod_{\substack{i=1 \\ i \neq t}}^{r} \left(1 - \frac{1}{p_t^{\alpha_t + 1}}\right) < 1.$$

(*Observe that* $\varepsilon_t(\alpha_t) \leqslant (1 - 1/p_t)$.

6. Zhi-Wei Sun, *An improvement to the Znám–Newman result*, Chinese Quart. J. Math., **6**(1991), no.3, 90–96.

Lemma 3 of this paper says that if $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^k \sim \emptyset$ (where $\lambda_s \in \mathbb{C},\ 0 \leqslant a_s < n_s$) then

$$\sum_{\substack{s=1 \\ \alpha n_s \in \mathbb{Z}}}^{k} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} = 0 \quad \text{for any } \alpha \in \mathbb{R}. \tag{16}$$

This identity brings out many number-theoretical properties. From it we deduced the following improvement to the Znám–Newman result which is better and general than all other improvements before.

Theorem 1. *Let* $A = \{(\lambda_s, a_s.n_s)\}_{s=1}^k$ *where* $\lambda_s \in \mathbb{C}$, *and* $n_0 \in \mathbb{Z}^+$ *be a period of* $w_A(x)$. *If* $d \in \mathbb{Z}^+$ *does not divide* n_0 *and*

$$\sum_{\substack{1 \leqslant s \leqslant k \\ d | (n_s, a_s - a)}} \frac{\lambda_s}{n_s} \neq 0 \quad \textit{for some } a \in \mathbb{Z},$$

then

$$|\{a_s \text{mod } d : 1 \leqslant s \leqslant k \ \& \ d \mid n_s\}| \geqslant \min_{\substack{0 \leqslant s \leqslant k \\ d \nmid n_s}} \frac{d}{(d, n_s)} \geqslant p(d) \tag{17}$$

where $p(d)$ *is the least prime divisor of* d.

This result is somewhat similar to the one stated in item 1. (Compare (17) with (1).) We noted that, if $A = \{a_s(n_s)\}_{s=1}^k$ is an exact m-cover with $n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k$, then by taking $\lambda_1 = \cdots = \lambda_k = n_0 = 1$ and $d = n_k$ we get

$$l \geqslant \min_{1 \leqslant s \leqslant k-l} \frac{n_k}{(n_s, n_k)} \geqslant \max\left\{p(n_k), \frac{n_k}{n_{k-l}}\right\}. \tag{18}$$

This lower bound for l is the best one up to now. [Y.G. Chen and Š. Porubský (1995) proved further that l can be written as $\sum_{s=1}^{k-l} x_s \frac{n_s}{(n_s, n_k)}$ with $x_s \in \mathbb{N}$, starting from the vital lemma above (the key equality (3) in their paper is our (16)).

7. Zhi-Wei Sun, *On a generalization of a conjecture of Erdös*, Nanjing Univ. J. Natur. Sci., **27**(1991), no.1, 8–15. MR 92f:11008; Zbl. M. 805.11007.

In this paper we gave the following generalization of the Davenport-Mirsky-Newman-Rado result.

Theorem 2. *For* $s = 1, \cdots, k$ *let* ψ_s *be an arithmetical function regularly periodic mod* n_s. *(By regularity we mean* $\sum_{r=0}^{n_s-1} \psi_s(r)\xi^r \neq 0$ *for some primitive* n_s*-th root* ξ *of unity.) If* $[n_1, \cdots, n_k]$ *is not the smallest positive period of the function* $\psi = \psi_1 + \cdots + \psi_k$ *then there must exist some* s, t *for which* $n_s = n_t$ *and* $\psi_s \neq \psi_t$.

Note that $A = \{a_s(n_s)\}_{s=1}^k$ is an exact m-cover for some $m \in \mathbb{Z}^+$ if and only if $w_A(x) = \sum_{s=1}^k \chi_s(x)$ has period $n_0 = 1$.

8. Zhi-Wei Sun, *On covering systems with distinct moduli*, J. Yangzhou Teachers College (Nat. Sci. Ed.), **11**(1991), no.3, 21–27.

In this paper we noted that the question of V.Billik and H. M. Edgar in 1973 (whether for any $d \in \mathbb{Z}^+$ there exists a minimal (irredundant) cover with all the moduli distinct and having greatest common divisor d) is equivalent to the well known problem of P. Erdös (whether for any $c > 0$ there exists a cover with all the moduli distinct and greater than c). If $\{a_s(n_s)\}_{s=1}^k$ is a cover with $n_1 < \cdots < n_k$ and there are no covers with all the moduli distinct and greater than n_1, then we showed that some n_s is divisible by $3n_1$ or $4n_1$.

9. Zhi-Wei Sun, *Solutions to two problems of Huhn and Megyesi*, Chinese Ann. Math. Ser. A, **13**(1992), no.6, 722–727. MR 94c:11001; Zbl. M. 770.11003.

In this paper we solved two problems on disjoint systems raised by A.P. Huhn and L. Megyesi [Discrete Math. **41**(1982), 327–330]. They called a finite sequence $\{n_s\}_{s=1}^k$ of positive integers *harmonic* if there are $a_1, \cdots, a_k \in \mathbb{Z}$ such that $A = \{a_s(n_s)\}_{s=1}^k$ is disjoint (i.e. those $a_s(n_s)$ are pairwise disjoint). Let statements (†) and (‡) be as follows:

(†) For any $I \subseteq \{1, \cdots, k\}$ with $|I| \geqslant 2$ we have

$$\sum_{s \in I} \frac{1}{\tilde{n}_s(I)} \leqslant 1 \quad \text{where } \tilde{n}_s(I) = (n_s, [n_t]_{t \in I \setminus \{s\}}) = [(n_s, n_t)]_{t \in I \setminus \{s\}}.$$

(‡) For any $I \subseteq \{1, \cdots, k\}$ with $|I| \geqslant 2$, there exist $s, t \in I$ with $s \neq t$ such that $(n_s, n_t) \geqslant |I|$.

Their first question is whether (†) is sufficient for $\{n_s\}_{s=1}^k$ to be harmonic. (Huhn and Megyesi noted the necessity.) The second one asks whether (‡) is necessary and sufficient for $\{n_s\}_{s=1}^k$ to be harmonic.

We first determined all those non-harmonic sequences $\{n_s\}_{s=1}^k$ with $k \leqslant 4$, by means of covers consisting of $k \leqslant 4$ residue classes.

Theorem 1. *Let n_1, n_2, n_3, n_4 be positive integers. Then*

i) $\{n_1\}$ *is harmonic;*

ii) $\{n_i\}_{i=1}^2$ *is not harmonic if and only if* $(n_1, n_2) = 1$*;*

iii) $\{n_i\}_{i=1}^3$ *is not harmonic but* $\{n_i\}_{i\in I}$ *is harmonic for all* $\emptyset \neq I \subset \{1,2,3\}$*, if and only if* $(n_1, n_2) = (n_1, n_3) = (n_2, n_3) = 2$*;*

iv) $\{n_i\}_{i=1}^4$ *is not harmonic but* $\{n_i\}_{i\in I}$ *is harmonic for all* $\emptyset \neq I \subset \{1,2,3,4\}$*, if and only if all the* (n_i, n_j) *with* $1 \leqslant i < j \leqslant 4$ *are* 3*, or we can rearrange* n_1, n_2, n_3, n_4 *so that*

$$(n_1, n_2) = (n_1, n_3) = (n_1, n_4) = 2, \ (n_2, n_3) = (n_2, n_4) = (n_3, n_4) = 4.$$

As a consequence we obtained

Corollary 1. *Let* $_1, \cdots, n_k \in \mathbb{Z}^+$*. For* $k \leqslant 4$*,* $\{n_s\}_{s=1}^k$ *is harmonic if and only if* (†) *holds. For* $k \leqslant 3$*,* $\{n_s\}_{s=1}^k$ *is harmonic if and only if we have* (‡)*; when* $k = 4$*,* $\{n_s\}_{s=1}^k$ *may not be harmonic even if we have* (‡) *together with the inequality* $\sum_{s=1}^k 1/n_s \leqslant 1$*.*

The second Theorem in the paper is

Theorem 2. *Let* $k, a, b, c, d \in \mathbb{Z}^+$*,* $k, a \geqslant 5$*, and* $6, a, b, c, d$ *be relatively prime. Put*

$$n_1 = 2a, n_2 = 3a, n_3 = 6b, n_4 = 6c, n_5 = 6d, n_6 = \cdots = n_k = 6kabcd. \tag{19}$$

Then $\sum_{s=1}^k 1/n_s \leqslant 1$*, both* (†) *and* (‡) *hold, but* $\{n_s\}_{s=1}^k$ *is not harmonic.*

In view of the above we answered the two questions negatively (but we don't know whether (‡) is necessary for $\{n_s\}_{s=1}^k$ to be harmonic).

10. Zhi-Wei Sun, *On disjoint residue classes*, Discrete Math., **104**(1992), no.3, 321–326. SCI 1992:5D; MR 93d:11005; Zbl. M. 755.11002.

In 1982 A.P. Huhn and L. Megyesi [Discrete Math.] proved (by a graph-theoretic method) that the sequence $\{n_s\}_{s=1}^k$ of positive integers is harmonic if those greatest common divisors (n_i, n_j) $(1 \leqslant i < j \leqslant k)$ are distinct and greater than one. By establishing connections with covers, we gave the following extension of the result.

Theorem. *Let* $n_1, \cdots, n_k \in \mathbb{Z}^+$*. If for any* $d \in \mathbb{Z}^+$ *with* $f(d) \leqslant k-2$ *(and hence* $d \leqslant 2^{k-2}$*) we have*

$$|\{\{i,j\} : \ 1 \leqslant i < j \leqslant k \ \& \ (n_i, n_j) = d\}| < \sqrt{\frac{d+7}{8}}, \tag{20}$$

then $\{n_s\}_{s=1}^k$ *is harmonic.*

We conjectured that the above $\sqrt{(d+7)/8}$ can be replaced by $2d-1$.

11. Zhi-Wei Sun, *On exactly m times covers*, Israel J. Math., **77**(1992), no.3, 345–348. SCI 1993:2D; MR 93k:11007; Zbl. M. 768.11001.

Let $A = \{a_s(n_s)\}_{s=1}^k$ forms an exact m-cover of $\mathbb{Z}$. It is well known that $\sum_{s=1}^k 1/n_s = m$. Also, A may not have a proper subcover which is an exact n-cover for some $n < m$. In the paper we proved the following result analytically.

Theorem. *Let* $A = \{a_s(n_s)\}_{s=1}^k$ *be an exact* m*-cover. Then for each* $n = 0, 1, \cdots, m$ *there exist at least* $\binom{m}{n}$ *subsets* I *of* $\{1, \cdots, k\}$ *such that* $\sum_{s=1}^k 1/n_s = n$. *The bounds* $\binom{m}{n}$ $(0 \leqslant n \leqslant m)$ *are best possible.*

12. Zhi-Wei Sun, *Covering the integers by arithmetic sequences*, Acta Arith. **72**(1995), no.2, 109–129. MR 96k:11013; Zbl. M. 841.11011.

For $\alpha \in \mathbb{R}$ and $\beta > 0$ we let $\alpha + \beta\mathbb{Z} = \{\alpha + \beta x : x \in \mathbb{Z}\}$. Instead of systems of residue classes we may consider a general system in the form

$$\mathcal{A} = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k. \tag{21}$$

By inventing a combined method involving linear algebraic, analysis and Stirling numbers, we were able to characterize general covers (not having a fixed covering function) for the first time.

Theorem 1. *For system* (21) *the following statements are equivalent:*

a) (21) *forms an* m*-cover of* $\mathbb{Z}$.

b) (21) *covers* $|S(\mathcal{A})|$ *consecutive integers at least* m *times where*

$$S(\mathcal{A}) = \left\{\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\} :\ I \subseteq \{1, \cdots, k\}\right\}. \tag{22}$$

c) *For any* $\theta \in [0, 1)$ *and* $n \in R(m)$ *we have*

$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \{\sum_{s=1}^k 1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s=1}^k 1/\beta_s]}{n} e^{2\pi i\sum_{s=1}^k \alpha_s/\beta_s} = 0. \tag{23}$$

A conjecture of P. Erdös in the early 1960's asserts that $A = \{a_s(n_s)\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ if it covers integers from 1 to 2^k. R.B. Crittenden and C.L. Vanden Eynden [Proc. Amer. Math. Soc. **24**(1970), 475–481] provided a long awkward proof for $k \geqslant 20$. Note that $|S(\mathcal{A})| \leqslant 2^k$ depends on those β's! So that b) implies a) gave more detailed information than the original conjecture of Erdös. In other words, the covering function $w_{\mathcal{A}}(x)$ $(x \in \mathbb{Z})$ takes the least value when x ranges over an interval $[a, a+|S(\mathcal{A})|)$ of length $|S(\mathcal{A})|$.

Crittenden and Vanden Eynden [Amer. Math. Monthly, **79**(1972), 630] made a further conjecture which says that $A=\{a_s(n_s)\}_{s=1}^k$ is a cover if it covers integers in $[1, 2^{k-l}(l+1)]$ where $0 \leqslant l < \min\{k, n_1, \cdots, n_k\}$. In contrast with this conjecture, we noted that 'b)$\Rightarrow$ a)' yields the following

Theorem 7. (21) *is an m-cover of $\mathbb{Z}$ if it covers $2^{k-M}(M+1)$ consecutive integers at least m times, where*

$$M = \max_{1\leqslant t\leqslant k} |\{1 \leqslant s \leqslant k : \beta_s = \beta_t\}|. \tag{24}$$

That $a) \Leftrightarrow c)$ is useful, we derived from it many new properties of the moduli in an m-cover.

By part (ii) of Theorem 4 in the paper, *if $A=\{a_s(n_s)\}_{s=1}^k$ forms an exact m-cover then for any $\emptyset \neq J \subset \{1,\cdots,k\}$ there exists an $I \subseteq \{1,\cdots,k\}$ with $I \neq J$ such that $\sum_{s\in I} 1/n_s = \sum_{s\in J} 1/n_s$.* Actually this also follows from our work in the paper in Israel J. Math. (1992). By the first part of Theorem 3, *if* (21) *is an m-cover of $\mathbb{Z}$ and J is a subset of $\{1,\cdots,k\}$ with $\sum_{s\in I} 1/\beta_s = \sum_{s\in J} 1/\beta_s$ for no other $I \subseteq \{1,\cdots,k\}$, then there are at least m nonzero integers in the form $\sum_{s\in I} 1/\beta_s - \sum_{s\in J} 1/\beta_s$ with $I \subseteq \{1,\cdots,k\}$.*

13. Zhi-Wei Sun, *Covering the integers by arithmetic sequences.II*, Trans. Amer. Math. Soc., **348**(1996), no.11, 4279–4320. MR 97c:11011; Zbl. M. 884.11013

Let $A=\{a_s(n_s)\}_{s=1}^k$ and $n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k$ where $0 < l \leqslant k$. By applying results in Part I [Acta Arith. 72(1995)] to the system $\mathcal{A} = \{a_s + \frac{n_s}{m_s}\mathbb{Z}\}_{s=1}^k$ where $m_1,\cdots,m_k \in \mathbb{Z}^+$, we obtained lots of results on the moduli in an m-cover A.

By Theorem 1(i), $A=\{a_s(n_s)\}_{s=1}^k$ *forms an m-cover if it covers W consecutive integers where*

$$W = \min_{\substack{m_1,\cdots,m_k\in\mathbb{Z} \\ (m_s,n_s)=1\ (1\leqslant s\leqslant k)}} \left|\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\} : I \subseteq \{1,\cdots,k\}\right\}\right|. \tag{25}$$

In Example 2 we noted that if $n_1 = n_2 = 4$ and $n_3 = n_4 = n_5 = 6$ then $|S(\{a_s(n_s)\}_{s=1}^5)| = 10 > W = 9$.

Here we listed several properties of m-cover A proved in the paper:

1) *For any* $m_1, \cdots, m_k \in \mathbb{Z}^+$ *there are at least* m *positive integers in the form* $\sum_{s\in I} m_s/n_s$ *where* $I \subseteq \{1, \cdots, k\}$. [This is an extension of Zhang's result in 1989.]

2) *If* $l \neq k$, *then either* $l \geqslant n_k/n_{k-l}$ *or* $\sum_{s=1}^{k-l} 1/n_s \geqslant m$. [This improves the Davenport-Mirsky-Newman-Rado result which says that if $l = 1$ then $\sum_{s=1}^{k} 1/n_s > 1$ (i.e. A is not a disjoint cover).]

3) *If* $a_t(n_t)$ *is essential (i.e.* $A_t = \{a_s(n_s)\}_{s\neq t}$ *fails to be an* m*-cover), then for any* $a \in \mathbb{Z}$ *there exist* $I, J \subseteq \{1, \cdots, k\}$ *such that*

$$\frac{a}{n_t} \equiv \sum_{s\in I} \frac{1}{n_s} - \sum_{s\in J} \frac{1}{n_t} \pmod 1. \tag{26}$$

14. Zhi-Wei Sun, *Exact* m*-covers and the linear form* $\sum_{s=1}^{k} x_s/n_s$, Acta Arith., **81**(1997), no. 2, 175–198. MR 98h:11019; Zbl. M. 871.11011

In this paper we characterized exact m-covers in several ways. One of the characterizations is

Theorem 4.2. $A = \{a_s(n_s)\}_{s=1}^{k}$ *forms an exact* m*-cover if and only if we have*

$$\begin{aligned}&\sum_{\substack{J\subseteq\{1,\cdots,k\}\setminus I\\ \{\sum_{s\in J} 1/n_s\}=\theta}} (-1)^{|J|} e^{2\pi i \sum_{s\in J} a_s/n_s}\\ =&\sum_{\substack{x_s\in R(n_s)\ (s\in I)\\ \{\sum_{s\in I} x_s/n_s\}=\theta}} e^{2\pi i \sum_{s\in I} a_s x_s/n_s}\end{aligned} \tag{27}$$

for all $\theta \in [0, 1)$ *and* $I \subseteq \{1, \cdots, k\}$ *with* $|I| = m$.

From the characterizations we deduced some properties of exact m-covers. Here we summarize the central results.

Theorem. *Let* $A = \{a_s(n_s)\}_{s=1}^{k}$ *be an exact* m*-cover. Then*

(I) *For* $a = 0, 1, 2, \cdots$ *and* $t = 1, \cdots, k$ *we have*

$$\left|\left\{I \subseteq \{1, \cdots, k\} : t \notin I \ \&\ \sum_{s\in I} \frac{1}{n_s} = \frac{a}{n_t}\right\}\right| \geqslant \binom{m-1}{[a/n_t]} \tag{28}$$

where the lower bounds are best possible.

(II) If $\emptyset \neq I \subseteq \{1, \cdots, k\}$ and $(n_s, n_t) \mid a_s - a_t$ for all $s, t \in I$, then we have

$$\left\{\left\{\sum_{s\in J} \frac{1}{n_s}\right\} : J \subseteq \bar{I}\right\} \supseteq \left\{\frac{r}{[n_s]_{s\in I}} :\ r \in R([n_s]_{s\in I})\right\} \tag{29}$$

where $\bar{I}=\{1,\cdots,k\}\setminus I$, *moreover for any* $r\in R([n_s]_{s\in I})$ *we have*

$$\left|\left\{J\subseteq\bar{I}:\left\{\sum_{s\in J}\frac{1}{n_s}\right\}=\frac{r}{[n_s]_{s\in I}}\right\}\right|\geqslant\frac{\prod_{s\in I}n_s}{[n_s]_{s\in I}}. \tag{30}$$

(III) *The number of solutions of the equation* $\sum_{s=1}^{k}x_s/n_s=c$ *with* $x_s\in R(n_s)$ *for* $s=1,\cdots,k$, *is the sum of finitely many (not necessarily distinct) prime factors of* $n_1,\cdots,n_k$ *if* $c\neq 0,1,2,\cdots$, *and at least* $\binom{k-m}{n}$ *if* c *equals a nonnegative integer* n.

15. Si-Man Yang and Zhi-Wei Sun, *Covers with less than* 10 *moduli and their applications*, J. Southeast Univ. (English Edition), **14**(1998), no.2, 106–114.

In the paper we essentially determined all covers $A=\{a_s(n_s)\}_{s=1}^{k}$ with $k<10$, in the Appendix we listed out all those irreducible minimal covers $\{a_s(n_s)\}_{s=1}^{k}$ (up to possible changes of the residues) with $k<10$ for which $n_1,\cdots,n_k$ are distinct, and $\{2,3,6\}\not\subseteq\{n_1,\cdots,n_k\}$ if $k=9$. Our algorithm is based on the recent results of Sun and actually valid for any $k\in\mathbb{Z}^+$. As an application of the data yielded by the algorithm, for any minimal cover $A=\{a_s(n_s)\}_{s=1}^{k}$ with $k<10$, $n_1<\cdots<n_k$ and $\{3,6\}\not\subseteq\{n_1,\cdots,n_k\}$, we found (uniformly) an explicit residue class $a(2p_0p_1\cdots p_k\}$ containing no integers of the form 2^h+p (p is a prime) where $2\nmid a$, $p_0\in\{31,43\}$ and $p_1,\cdots,p_k$ are distinct primes different from 2 and p_0.

The second theorem in the paper is as follows:

Theorem 2. *Let* $A=\{a_s(n_s)\}_{s=1}^{k}$ *be a minimal cover with* $n_1<\cdots<n_k$. *Suppose that for each* $s=1,\cdots,k$ *prime* p_s *divides* $2^{n_s}-1$ *but not* 2^n-1 *with* $0<n<n_s$. *Let* $X=\bigcap_{s=1}^{k}2^{a_s}(p_s)=a(p_1\cdots p_k)$, *and* c *be any integer divisible by a unique prime among* $p_1,\cdots,p_k$. *Then there exists an* $n\in\mathbb{Z}^+$ *such that* $2^n+cp\in X$ *for infinitely many primes* p.

16. Zhi-Wei Sun, *On covering multiplicity*, Proc. Amer. Math. Soc., **127**(1999), no. 5, 1293–1390. MR 99h:11012; Zbl. M. 917.11006

Theorem. Let $A=\{a_s(n_s)\}_{s=1}^{k}$ form an m-cover and $J\subseteq\{1,\cdots,k\}$. Put $\bar{J}=\{1,\cdots,k\}\setminus J$.

(i) For any $m_1,\cdots,m_k\in\mathbb{Z}$ we have

$$\left|\left\{I\subseteq\{1,\cdots,k\}:\ I\neq J\ \&\ \left\{\sum_{s\in I}\frac{m_s}{n_s}\right\}=\left\{\sum_{s\in J}\frac{m_s}{n_s}\right\}\right\}\right|\geqslant m. \tag{31}$$

(ii) Suppose that there is an $x\in\bigcap_{s\in J}a_s(n_s)$ ($J\neq\emptyset$) with $w_A(x)=m$. For each $s\in\bar{J}$ let m_s be a positive integer prime to n_s. Then there exists an $\alpha\in[0,1)$ such that

$$\left\{\left\{\sum_{s\in I}\frac{m_s}{n_s}\right\} : I\subseteq \bar{J},\ \left[\sum_{s\in I}\frac{m_s}{n_s}\right]\geqslant m-|J|\right\}$$
$$\supseteq\left\{\frac{a}{[n_s]_{s\in J}} : 0\leqslant a<[n_s]_{s\in J},\ \{a\}=\alpha\right\}. \tag{32}$$

In view of this theorem, an m-cover $A=\{a_s(n_s)\}_{s=1}^k$ possesses the following properties:

(a) *For each $J\subseteq\{1,\cdots,k\}$, there exist at least m subsets I of $\{1,\cdots,k\}$ with $I\neq J$ such that $\sum_{s\in I}1/n_s-\sum_{s\in J}1/n_s\in\mathbb{Z}$.*

(b) *If $a_t(n_t)$ is essential, then the set*

$$S_t(A)=\left\{\left\{\sum_{s\in I}\frac{1}{n_s}\right\} : I\subseteq\{1,\cdots,k\}\setminus\{t\},\ \left[\sum_{s\in I}\frac{1}{n_s}\right]\geqslant m-1\right\} \tag{33}$$

contains an arithmetic progression of length n_t with common difference $1/n_t$.

Part (a) is another generalization of Zhang's result. In the case $J=\emptyset$, this and 1) in item **13** are implied by our following *conjecture: If A forms an m-cover and $m_1,\cdots,m_k\in\mathbb{Z}^+$, then there exist a chain $\emptyset\neq I_1\subset\cdots\subset I_m\subseteq\{1,\cdots,k\}$ such that $\sum_{s\in I_t}m_s/n_s\in\mathbb{Z}$ for all $t=1,\cdots,m$.*

Part b) yields 3) in item **13** and the inequality $|S_t(A)|\geqslant n_t$. Now *we conjecture that for any minimal m-cover $A=\{a_s(n_s)\}_{s=1}^k$, $|S(A)|\leqslant n_1+\cdots+n_k$, and $S(A)\supseteq\{r/d : r\in R(d)\}$ if $d\in\mathbb{Z}^+$ and $1/d\in S(A)$.*

17. Zhi-Wei Sun, *On integers not of the form $\pm p^a\pm q^b$*, Proc. Amer. Math. Soc. **128**(2000), no. 4, 997–1002. MR 2000i:11157; Zbl. M. 991.18749

In 1975 F. Cohen and J.L. Selfridge found a 94-digit positive integer which cannot be written as the sum or difference of two prime powers. Following their basic construction and introducing a new method to avoid a bunch of extra congruences, we proved that if

$$P=\{2,\ 3,\ 5,\ 7,\ 11,\ 13,\ 17,\ 19,\ 31,\ 37,\ 41,\ 61,\ 73,\ 97,\ 109,\ 151,\ 241,\ 257,\ 331\}$$

and

$$x\equiv 47867742232066880047611079\ \left(\mathrm{mod}\ \prod_{p\in P}p\right)$$

then x is not of the form $\pm p^a\pm q^b$ where p,q are primes and a,b are nonnegative integers.

18. Zhi-Wei Sun, *Exact m-covers of groups by cosets*, European J. Combin., **22**(2001), no. 3, 415–429.

Let G be a group and $\mathcal{A} = \{a_iG_i\}_{i=1}^k$ be a finite system of left cosets in G where each G_i is a subgroup of G.

The main results in the paper are as follows:

(I) *Let $\mathcal{A}$ be an exact m-cover of G with all the G_i subnormal in G. Then $k \geqslant m + d(G, \bigcap_{i=1}^k G_i)$, and the lower bound is best possible. Moreover, for any subgroup K of G not contained in all the G_i we have*

$$|\{1 \leqslant i \leqslant k : K \not\subseteq G_i\}| \geqslant 1 + d\bigg(K, K \cap \bigcap_{i=1}^k G_i\bigg).$$

(This generalizes the author's work in 1990.)

(II) *Let $\mathcal{A}$ be an exact m-cover of G. Whenever $G/(G_i)_G$ is solvable, we have $k \geqslant m + f([G : G_i])$ and hence $[G : G_i] \leqslant 2^{k-m}$.*

Concerning result (II) we have a further conjecture.

Conjecture. *Let $\mathcal{A}$ be an exact m-cover of a group G with all the $G/(G_i)_G$ solvable. Then $k \geqslant m + f(N)$ where N is the least common multiple of the indices $[G : G_1], \cdots, [G : G_k]$.*

For a cover $\mathcal{A}$ of G, if it doesn't form an exact m-cover for any $m = 1, 2, 3, \cdots$, then we don't have a similar inequality in general. When G is cyclic, or $|G|$ is square-free and all the G_i are subnormal in G, if $m(\mathcal{A}') < m(\mathcal{A})$ for any proper subsystem $\mathcal{A}'$ of $\mathcal{A}$ then we can show that $k \geqslant m(\mathcal{A}) + f([G : \bigcap_{i=1}^k G_i])$.

For a subgroup H of G we let G/H denote the set of all left cosets of H in G. To prove Results (I) and (II), we need the following

Theorem 2.1. *Assume that $\mathcal{A}$ forms an exact m-cover of G For a subgroup H of G we have*

$$\{C \in G/H : C \supseteq a_iG_i \text{ for some } i = 1, \cdots, k\} = \emptyset \text{ or } G/H.$$

in the following cases:

(a) *H is the group G or a normal subgroup of prime index in G;*

(b) *$G_1, \cdots, G_k$ are normal in G and H is maximal in G;*

(c) *$G_1, \cdots, G_k$ are subnormal and H is maximal normal in G.*

The paper also includes an interesting application in group theory.

Corollary 4.6. *Let H be a subnormal subgroup of a group G with $[G : H] < \infty$. Then H is normal in G if and only if*

$$|N_G(H)/H| + d(H, H_G) \geqslant [G : H].$$

When H is a subnormal subgroup of a group G with $[G:H]<\infty$, we conjectured that $|N_G(H)/H|\geqslant d(G,H)$.

19. Zhi-Wei Sun, *Products of binomial coefficients modulo* p^2, Acta Arith. **97**(2001) 87–98.

Let m be an integer and M an additive abelian group. Let f be a map from a subset of $\mathbb{C}\times\mathbb{C}$ into M. If for any ordered pair $\langle x,y\rangle$ in the domain $\mathrm{Dom}(f)$ of f and each positive integer n prime to m, we have

$$\left\{\left\langle \frac{x+mr}{n},ny\right\rangle : r=0,1,\cdots,n-1\right\}\subseteq \mathrm{Dom}(f) \qquad (*)$$

and

$$\sum_{r=0}^{n-1} f\left(\frac{x+mr}{n},ny\right)=f(x,y),$$

then we call f an *m-uniform map* (into M).

The following result presented in the paper is an extension of the Main Theorem on Equivalence mentioned in item 2.

Theorem 2.1. *Let m be an integer and M a left R-module where R is a ring with identity. Let f be a map into M with* $\mathrm{Dom}(f)\subseteq\mathbb{C}\times\mathbb{C}$ *such that* $(*)$ *holds for any* $\langle x,y\rangle\in\mathrm{Dom}(f)$ *and* $n\in\mathbb{Z}^+$ *with* $(m,n)=1$. *Then the following two statements are equivalent:*

(a) *f is an m-uniform map into M.*

(b) *Whenever*

$$\sum_{\substack{1\leqslant s\leqslant k\\ x\in a_s(n_s)}}\lambda_s=\sum_{\substack{1\leqslant t\leqslant l\\ x\in b_t(m_t)}}\mu_t \qquad \text{for all } x\in\mathbb{Z}$$

(*with* $\lambda_s,\mu_t\in R$, $a_s,n_s,b_t,m_t\in\mathbb{Z}$, $0\leqslant a_s<n_s$, $0\leqslant b_t<m_t$ *and* $(n_sm_t,m)=1$), *we have*

$$\sum_{s=1}^{k}\lambda_s f\left(\frac{x+ma_s}{n_s},n_sy\right)=\sum_{t=1}^{l}\mu_t f\left(\frac{x+mb_t}{m_t},m_ty\right)\quad \text{for } \langle x,y\rangle\in\mathrm{Dom}(f).$$

The proof of this result is very simple. As for examples of uniform maps, we gave

Proposition 2.1. (i) *Let* $m \in \mathbb{Z}$. *Then the function* $[\ ]_m : \mathbb{R} \times \mathbb{R} \to \mathbb{Q}$ *given by*

$$[\ ]_m(x,y) = [x] + \frac{1-m}{2}$$

is an m*-uniform map into the rational field* $\mathbb{Q}$.

(ii) *For each* $m = 0, 1, 2, \cdots$ *the functions* $b_m : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C}$ *and* $e_m : \mathbb{C} \times \mathbb{Z} \to \mathbb{C}$ *given by*

$$b_m(x,y) = y^{m-1} B_m(x)$$

and

$$e_m(x,y) = \begin{cases} e^{\pi i x y} y^m E_m(x) & \text{if } y \text{ is odd,} \\ -\frac{2}{m+1} e^{\pi i x y} y^m B_{m+1}(x) & \text{if } y \text{ is even,} \end{cases}$$

are 1*-uniform maps into the complex field* $\mathbb{C}$, *where* $B_m(x)$ *and* $E_m(x)$ *are the* m*th Bernoulli polynomial and the* m*th Euler polynomial respectively.*

Proposition 2.2. *Let* p *be an odd prime. For* $x \geqslant 0$ *and* $m \in \mathbb{Z} \setminus p\mathbb{Z}$ *let*

$$q(x,m) = \frac{q_p(m)}{m} + \sum_{\substack{0<j\leqslant [x] \\ p\nmid j}} \frac{1}{jm} \quad \text{where } q_p(m) = \frac{m^{p-1}-1}{p}.$$

Then the function $\bar{q}(x,m) = q(x,m) \bmod p$ *is a* p*-uniform map into the finite field* $\mathbb{Z}/p\mathbb{Z}$.

From the above, we deduced the following

Theorem 1.2. *Let* p *be an odd prime. Let* $A = \{a_s(n_s)\}_{s=1}^k$ $(0 \leqslant a_s < n_s)$ *and* $B = \{b_t(m_t)\}_{t=1}^l$ $(0 \leqslant b_t < m_t)$ *be covering equivalent systems with the moduli* n_s *and* m_t *not divisible by* p *but dividing integer* N. *Then for any* $x \in [0,p)$ *we have*

$$\prod_{s=1}^k \binom{p\frac{N}{n_s}-1}{[\frac{x+pa_s}{n_s}]} \Big/ \prod_{t=1}^l \binom{p\frac{N}{m_t}-1}{[\frac{x+pb_t}{m_t}]}$$
$$\equiv (-1)^{(k-l)\frac{p-1}{2}} \left(1 + pN\left(\sum_{s=1}^k \frac{q_p(n_s)}{n_s} - \sum_{t=1}^l \frac{q_p(m_t)}{m_t}\right)\right) \pmod{p^2}.$$

Actually we may not require the integer N in Theorem 1.2 to be a common multiple of those moduli n_s and m_t. For example $N = 1$ is allowed if we don't mind using $x \notin \mathbb{Z}$ in the notation $\binom{x}{n}$.

Corollary 1.2. *Let* $A = \{a_s(n_s)\}_{s=1}^k$ $(0 \leqslant a_s < n_s)$ *be an exact* m*-cover of* $\mathbb{Z}$. *Let* N *be the least common multiple of* $n_1, \cdots, n_k$ *and* p *an odd prime not dividing* N. *Then*

$$\prod_{s=1}^k \binom{p\frac{N}{n_s}-1}{[\frac{pa_s}{n_s}]} \equiv (-1)^{(k-m)\frac{p-1}{2}} \left(1 + pN\sum_{s=1}^k \frac{q_p(n_s)}{n_s}\right) \pmod{p^2}.$$

Applying Corollary 1.2 to the trivial disjoint cover $A = \{r(n)\}_{r=0}^{n-1}$ we then get a congruence due to A. Granville.

20. Zhi-Wei Sun, *Algebraic approaches to periodic arithmetical maps*, J. Algebra **240**(2001), no. 2, 723–743.

A residue class $a + n\mathbb{Z}$ with weight λ is denoted by $\langle \lambda, a, n\rangle$. For a finite system $\mathcal{A} = \{\langle \lambda_s, a_s, n_s\rangle\}_{s=1}^k$ of such triples, the periodic map $w_{\mathcal{A}}(x) = \sum_{n_s \mid x-a_s} \lambda_s$ is called the covering map of $\mathcal{A}$. Some interesting identities for those $\mathcal{A}$ with a fixed covering map have been known, in this paper we mainly determine out all those functions $f : \Omega \to \mathbb{C}$ such that $\sum_{s=1}^k \lambda_s f(a_s + n_s\mathbb{Z})$ depends only on $w_{\mathcal{A}}$ where Ω denotes the family of all residue classes. We also study algebraic structures related to such maps f, and periods of arithmetical functions $\psi(x) = \sum_{s=1}^k \lambda_s e^{2\pi i a_s x/n_s}$ and $\omega(x) = |\{1 \leqslant s \leqslant k : (x + a_s, n_s) = 1\}|$.

Theorems 3 and 4 announced in my earlier paper *Several results on systems of residue classes* (see item 3) were proved in this paper.

Let $\Omega = \bigcup_{n\in\mathbb{Z}^+} \mathbb{Z}/n\mathbb{Z}$. A map $f : \Omega \to M$ is said to be *equivalent* if

$$\sum_{j=0}^{n-1} f(a + jd + nd\mathbb{Z}) = f(a + d\mathbb{Z}) \quad \text{for any } a \in \mathbb{Z} \text{ and } d, n \in \mathbb{Z}^+.$$

If R is a ring with identity and M is a left R-module, then $E(R)$ forms a ring with identity with respect to the functional addition and convolution, and the set $P(M)$ of all periodic maps from $\mathbb{Z}$ to M forms an $E(R)$-module where the scalar multiplication is defined by

$$f \circ \psi(x) = \sum_{r=0}^{n-1} f(r + n\mathbb{Z})\psi(x - r) \text{ where } n \in \mathbb{Z}^+ \text{ is a period of } \psi.$$

Theorem 6. *Let* $n_1, \cdots, n_k \in \mathbb{Z}^+$ *and* $f \in \mathrm{E}(\mathbb{C})$. *Then*

$$\sum_{r=0}^{n_s-1} f(r + n_s\mathbb{Z}) e^{2\pi i \frac{a}{n_s} r} \neq 0 \quad \textit{for all } a \in \mathbb{Z} \textit{ and } s = 1, \cdots, k,$$

if and only if for any $\psi_1 \in \mathrm{P}(\mathbb{C})$ *periodic* mod n_1, $\cdots$, $\psi_k \in \mathrm{P}(\mathbb{C})$ *periodic* mod n_k *we have*

$$\psi_1 + \cdots + \psi_k = 0 \iff f \circ \psi_1 + \cdots + f \circ \psi_k = 0.$$

This was announced by me in 1989 (see item 3).

Now we state two more results in this paper:

I. *Let* $\lambda_1,\cdots,\lambda_k\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, *and* $\xi_1,\cdots,\xi_k$ *be distinct roots of unity. Then the smallest (positive) period of the arithmetical function* $\psi(x)=\sum_{s=1}^k\lambda_s\xi_s^x$, *coincides with* $[n_1,\cdots,n_k]$ *where* n_s *is the least* $n\in\mathbb{Z}^+$ *with* $\xi_s^n=1$ *(i.e.,* ξ_s *is a primitive* n_s*th root of unity).*

II. *Let* $A=\{a_s(n_s)\}_{s=1}^k$ *be a system of residue classes with* $n_1,\cdots,n_k$ *square-free and* $n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k$ $(0<l<k)$. *If* $|\{1\leqslant s\leqslant k:(x+a_s,n_s)=1\}|=m$ *for all* $x\in\mathbb{Z}$, *then* $l\geqslant\min_{1\leqslant s\leqslant k-l}n_k/(n_s,n_k)$, *furthermore*

$$\frac{l}{n_k}=\sum_{s=1}^{k-l}\frac{x_s}{(n_s,n_k)}\quad\textit{for some } x_1,\cdots,x_{k-l}\in\mathbb{N}=\{0,1,2,\cdots\}.$$

21. Zhi-Wei Sun, *On covering equivalence*, in: 'Analytic Number Theory' (Beijing/Kyoto, 1999), 277–302, Dev. Math., 6, Kluwer Acad. Publ., Dordrecht, 2002.

In this paper we characterize covering equivalence in various ways, our characterizations involve the Γ-function, the Hurwitz ζ-function, trigonometric functions, the greatest integer function and Egyptian fractions.

Here we collect some results of the paper.

Theorem. *Let* $n_s,m_t\in\mathbb{Z}^+$, $a_s\in R(n_s)$ *and* $b_t\in R(m_t)$ *for* $s=1,\cdots,k$ *and* $t=1,\cdots,l$. *Then the following statements are equivalent:*

$$A=\{a_s(n_s)\}_{s=1}^k\sim B=\{b_t(m_t)\}_{t=1}^l;$$

$$\sum_{\substack{I\subseteq\{1,\cdots,k\}\\ \sum_{s\in I}\frac{1}{n_s}=c}}(-1)^{|I|}e^{2\pi i\sum_{s\in I}\frac{a_s}{n_s}}=\sum_{\substack{J\subseteq\{1,\cdots,l\}\\ \sum_{t\in J}\frac{1}{m_t}=c}}(-1)^{|J|}e^{2\pi i\sum_{t\in J}\frac{b_t}{m_t}}\quad\textit{for all } c\geqslant 0;$$

$$2^k\prod_{\substack{s=1\\ s\notin S_z}}^{k}\sin\pi\frac{a_s-z}{n_s}\cdot\prod_{s\in S_z}\frac{(-1)^{[\frac{z}{n_s}]}}{n_s}=2^l\prod_{\substack{t=1\\ t\notin T_z}}^{l}\sin\pi\frac{b_t-z}{m_t}\cdot\prod_{t\in T_z}\frac{(-1)^{[\frac{z}{m_t}]}}{m_t}\quad\textit{for } z\in\mathbb{C}$$

where $S_z=\{1\leqslant s\leqslant k:\ z\in a_s(n_s)\}$ *and* $T_z=\{1\leqslant t\leqslant l:\ z\in b_t(m_t)\}$;

$$\frac{\prod_{\substack{s=1\\ s\notin U_z}}^{k}\Gamma\left(\frac{a_s-z}{n_s}\right)n_s^{\frac{a_s-z}{n_s}-\frac12}}{\prod_{\substack{t=1\\ t\notin V_z}}^{l}\Gamma\left(\frac{b_t-z}{m_t}\right)m_t^{\frac{b_t-z}{m_t}-\frac12}}=(2\pi)^{\frac{k-l}{2}}\frac{\prod_{s\in U_z}[\frac{z}{n_s}]!(-1)^{[\frac{z}{n_s}]}n_s^{[\frac{z}{n_s}]-\frac12}}{\prod_{t\in V_z}[\frac{z}{m_t}]!(-1)^{[\frac{z}{m_t}]}m_t^{[\frac{z}{m_t}]-\frac12}}\quad\textit{for } z\in\mathbb{C}$$

where $U_z = \{1 \leqslant s \leqslant k:\ z \in a_s + n_s\mathbb{N}\}$ *and* $V_z = \{1 \leqslant t \leqslant l:\ z \in b_t + m_t\mathbb{N}\}$*;*

$$\prod_{s=1}^{k} F\left(\frac{u}{n_s}, \frac{v}{n_s}, \frac{w+a_s}{n_s}, 1\right) = \prod_{t=1}^{l} F\left(\frac{u}{m_t}, \frac{v}{m_t}, \frac{w+b_t}{m_t}, 1\right)$$
for $u, v, w \in \mathbb{C}$ *with* $\mathrm{Re}(w) > \mathrm{Re}(u+v)$ *and* $w, w-u, w-v \notin -\mathbb{N}$,

where $F(\alpha, \beta, \gamma, z)$ $(Re(\gamma) > Re(\alpha+\beta)$ *and* $\gamma \notin -\mathbb{N})$ *denotes the hypergeometric series given by*

$$F(\alpha, \beta, \gamma, z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{n!\gamma(\gamma+1)\cdots(\gamma+n-1)} z^n$$

which converges absolutely for $|z| \leqslant 1$*;*

$$\sum_{i=1}^{k} n_i^{-s} \zeta\left(s, \frac{z+a_i}{n_i}\right) = \sum_{j=1}^{l} m_j^{-s} \zeta\left(s, \frac{z+b_j}{m_j}\right) \text{ for all } z \in \mathbb{C} \setminus (-\infty, 0]$$

where $\zeta(s, v)$ *is the well-known Hurwitz zeta function and* s *is a complex number different from* $1, 0, -1, -2, \cdots$.

22. Zhi-Wei Sun and Si-Man Yang, *A note on integers of the form* $2^n + cp$, Proc. Edinburgh Math. Soc. 45(2002), no. 1, 155–160.

In 1950 P. Erdös proved that if $x \equiv 2036812 \pmod{5592405}$ and $x \equiv 3 \pmod{62}$ then x is not of the form $2^n + p$ where n is a nonnegative integer and p is a prime.

Our main result in this note is as follows:

Theorem. *Let* $A = \{a_s(n_s)\}_{s=1}^k$ *be a minimal cover with* $0 \leqslant a_s < n_s$ *for* $s = 1, \cdots, k$*. Suppose that distinct primes* $p_1, \cdots, p_k$ *are primitive divisors of* $2^{n_1} - 1, \cdots, 2^{n_k} - 1$ *respectively. Put* $\bigcap_{s=1}^k 2^{a_s}(\mathrm{mod}\ p_s) = a(\mathrm{mod}\ d)$ *where* $a \in \mathbb{Z}$ *and* $d = p_1 \cdots p_k$*, and write*

$$\left(a_t(\mathrm{mod}\ n_t) \setminus \bigcup_{\substack{s=1 \\ s \neq t}}^{k} a_s(\mathrm{mod}\ n_s)\right) \cap \{0, 1, \cdots, N-1\} = \{b_1^{(t)}, \cdots, b_{l_t}^{(t)}\}$$

for $t = 1, \cdots, k$ *where* N *is the least common multiple* $[n_1, \cdots, n_k]$ *of the moduli* $n_1, \cdots, n_k$*. Set*

$$S(A) = \bigcup_{t=1}^{k} \bigcup_{j=1}^{l_t} \frac{a - 2^{b_j^{(t)}}}{p_t} \left(\mathrm{mod}\ \frac{d}{p_t}\right) \tag{3}$$

where all the $(a - 2^{b_j^{(t)}})/p_t$ *are integers. Then an integer* c *divisible by none of* $p_1, \cdots, p_k$ *belongs to* $S(A)$ *if and only if* $a \pmod d$ *contains integers of the form* $2^n + cp$ *where* $n \geqslant 0$ *is an integer and* p *is a prime.*

It follows from the theorem that for any integer $c \in [-3150, 20054)$ divisible by none of 3,5,7,13,17,241, the residue class 20036812(mod 5592405) contains no integers of the form $2^n + cp$ where $n \geqslant 0$ is an integer and p is a prime.

23. Zhong Hu and Zhi-Wei Sun, *On n-dimensional covering systems*, Nanjing Univ. J. Natur. Sci., **37**(2001), no.4, 486–492.

Covers of $\mathbb{Z}$ by finitely many residue classes have been investigated for many years. The Newman-Znám result asserts that if $\{a_s(\text{mod } n_s)\}_{s=1}^k$ forms a disjoint cover of $\mathbb{Z}$ with $1 < n_1 \leqslant \cdots \leqslant n_{k-1} \leqslant n_k$ then $n_{k-p+1} = \cdots = n_{k-1} = n_k$ where p is the least prime dividing n_k. In this paper we study covers of $\mathbb{Z}^n$ in the form $\mathcal{A} = \{\vec{a}_s(\vec{m}_s)\}_{s=1}^k$ where $\vec{a}_s(\vec{m}_s) = \{\vec{x} = \langle x_1, \cdots, x_n\rangle : x_t \equiv a_{st}(\text{mod } m_{st}) \text{ for } t = 1, \cdots, n\}$. Some classical results on covers of $\mathbb{Z}$ are generalized. In particular, we show that if each $\vec{x} \in \mathbb{Z}^n$ is covered by $\mathcal{A}$ exactly m times and the moduli $\vec{m}_s$ are not all equal, then any maximal modulus $\vec{m}_r$ with respect to divisiblity $(\vec{m}_r \mid \vec{m}_s \iff \vec{m}_s = \vec{m}_r)$ occurs at least p times among $\vec{m}_1, \cdots, \vec{m}_k$ where p is the smallest prime divisor of $m_{r1} \cdots m_{rn}$.

24. Zhi-Wei Sun, *On the function* $w(x) = |\{1 \leqslant s \leqslant k : x \equiv a_s \pmod{n_s}\}|$, Combinatorica **23**(2003), no.4, 681–691. MR 2004m:11013; Zbl. M. 1047.11014.

The main results of the paper are included in the following theorem.

Theorem. *Let* $A = \{a_s(n_s)\}_{s=1}^k$ *and* $n_0 \in \mathbb{Z}^+$ *be a period of the covering function* $w_A(x)$*. Then we have the following results*:

(a) *For each* $r \in R(n_k/(n_0, n_k))$ *there exists a* $J \subseteq \{1, \cdots, k-1\}$ *such that* $\sum_{s \in J} 1/n_s = r/n_k$*.*

(b) *If* $n_1 \leqslant \cdots \leqslant n_{k-l} < n_{k-l+1} = \cdots = n_k$ $(0 < l < k)$*, then for any positive integer* $r < n_k/n_{k-l}$ *with* $r \not\equiv 0 \pmod{n_k/(n_0, n_k)}$*, the binomial coefficient* $\binom{l}{r}$ *can be written as the sum of some* (*not necessarily distinct*) *prime divisors of* n_k*.*

(c) $M(A) = \max_{x \in \mathbb{Z}} w_A(x)$ can be written in the form $\sum_{s=1}^k m_s/n_s$ where $m_1, \cdots, m_k \in \mathbb{Z}^+$.

25. Zhi-Wei Sun, *On the Herzog-Schönheim conjecture for uniform covers of groups*, J. Algebra **273**(2004), no.1, 153–175.

In the early 1930s P. Erdös initiated the study of covers of $\mathbb{Z}$ by finitely many residue classes, he posed the famous question whether for any arbitrarily large

$c > 0$ there exists a cover $A = \{a_s(n_s)\}_{s=1}^k$ of $\mathbb{Z}$ with all the moduli $n_1, \cdots, n_k$ distinct and greater than c. In the 1950s B. H. Neumann investigated covers of general groups by cosets, in 1974 M. Herzog and J. Schönheim conjectured that if $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ forms a partition of a group G into $k > 1$ left cosets then at least two of the (finite) indices $[G : G_1], \cdots, [G : G_k]$ are equal.

The main result of this paper can be summarized as follows.

Theorem. *Let $\mathcal{A} = \{a_i G_i\}_{i=1}^k$ ($[G : G_1] \leqslant \cdots \leqslant [G : G_k]$) be a uniform cover of a group G by left cosets (i.e. it covers every element of G with the same multiplicity). Assume that $G_1, \cdots, G_k$ are all subnormal in G and not all equal to G. Then the indices $[G : G_1], \cdots, [G : G_k]$ cannot be distinct. Moreover, if each of the indices occurs at most $M > 1$ times then we have the following* (i)–(iii) *where γ is the Euler constant and the O-constants are absolute.*

(i) *M is greater than or equal to the smallest prime divisor of the indices $[G : G_1], \cdots, [G : G_k]$.*

(ii) *All prime divisors of $[G : G_1], \cdots, [G : G_k]$ are smaller than $e^{\gamma} M \log M + O(M \log\log M)$.*

(iii) *The number of distinct prime divisors of $[G : G_1], \cdots, [G : G_k]$ does not exceed $e^{\gamma} M + O(M/\log M)$.*

(iii) $\log[G : G_1] \leqslant \frac{e^{\gamma}}{\log 2} M \log^2 M + O(M \log M \log\log M)$.

We emphasize that, for uniform covers of groups by cosets of subnormal subgroups, the above theorem confirms the generalized Herzog–Schönheim conjecture and answers the analogous question of Erdös negatively!

26. Zhi-Wei Sun, *Arithmetic properties of periodic maps*, Math. Res. Lett. **11**(2004), no. 2-3, 187–196.

We introduce two main results in the paper.

Theorem 1.1. *Let F be a field of characteristic p where p is zero or a prime. Let $n_1, \ldots, n_k$ be positive integers not divisible by p, and let $\psi_1, \ldots, \psi_k$ be maps from $\mathbb{Z}$ to F with periods $n_1, \ldots, n_k$ respectively. Then $\psi_1 + \cdots + \psi_k = 0$ if $\psi_1(x) + \cdots + \psi_k(x) = 0$ for $\sum_{d \in D} \varphi(d)$ consecutive integers x, where φ is Euler's totient function, $D = \bigcup_{s=1}^k D(n_s)$, and $D(n)$ denotes the set of positive divisors of $n \in \mathbb{Z}^+$. (Actually $\sum_{d \in D} \varphi(d)$ also equals $|\bigcup_{s=1}^k \{r/n_s : r = 0, \ldots, n_s - 1\}|$).*

By this local-global theorem, if $A = \{a_s(n_s)\}_{s=1}^k$ (where $a_s \in \mathbb{Z}$, $n_s \in \mathbb{Z}^+$ and $a_s(n_s) = a_s + n_s\mathbb{Z}$) covers $|\{r/n_s : r = 0, 1, \ldots, n_s - 1;\ s = 1, \ldots, k\}|$ consecutive integers exactly m times then it covers all the integers exactly m times.

The following result characterizes the least positive period of the covering function of a finite system of residue classes with weights.

Theorem 1.3. *Let $\lambda_s \in \mathbb{C}$, $a_s \in \mathbb{Z}$ and $n_s \in \mathbb{Z}^+$ for $s = 1, \ldots, k$. Then the smallest positive period n_0 of the (weighted) covering function*

$$w(x) = \sum_{\substack{1 \leqslant s \leqslant k \\ x \in a_s(n_s)}} \lambda_s$$

is the least $n \in \mathbb{Z}^+$ such that $\alpha n \in \mathbb{Z}$ for all those $\alpha \in [0, 1)$ with

$$\sum_{\substack{1 \leqslant s \leqslant k \\ \alpha n_s \in \mathbb{Z}}} \frac{\lambda_s}{n_s} e^{2\pi i \alpha a_s} \neq 0.$$

27. Zhi-Wei Sun, *On the range of a covering function*, J. Number Theory **111**(2005) 190–196.

Let $\{a_s(\text{mod } n_s)\}_{s=1}^k$ $(k > 1)$ be a finite system of residue classes with the moduli $n_1, \ldots, n_k$ distinct. By means of algebraic integers we show that the range of the covering function $w(x) = |\{1 \leqslant s \leqslant k : x \equiv a_s \pmod{n_s}\}|$ is not contained in any residue class with modulus greater one. In particular, the values of $w(x)$ cannot have the same parity.

28. Zhi-Wei Sun, *A connection between covers of $\mathbb{Z}$ and unit fractions*, submitted, `arXiv:math.NT/0411305`.

Suppose that $A = \{a_s(\text{mod } n_s)\}_{s=1}^k$ covers all the integers at least m times with $a_k(\text{mod } n_k)$ irredundant. We show that if n_k is a period of the covering function $w_A(x) = |\{1 \leqslant s \leqslant k : x \equiv a_s \pmod{n_s}\}|$ then for any $r = 0, 1, \ldots, n_k - 1$ there are at least m integers in the form $\sum_{s \in I} 1/n_s - r/n_k$ with $I \subseteq \{1, \ldots, k - 1\}$.