Introduction to Zhi-Wei Sun’s Papers on Covers

Z. W. Sun


It was showed that, if $A = \{a_s(n_s)\}_{s=1}^k$ is a cover of $\mathbb{Z}$ but $\{a_s(n_s)\}_{s=1}^k$ is not where $d$ is an integer greater than $n_0 = 1$, then

$$|\{a_{s \mod d} : 1 \leq s \leq k & d \mid n_s\}| \geq \frac{d}{(d, [n_s]_{0 \leq s \leq k})} \frac{d}{d_{n_s}} \gcd_{0 \leq s \leq k} (d, n_s).$$

(1)

Thus when $A$ forms a minimal (i.e. irredundant) cover of $\mathbb{Z}$ for any prime power $p^\alpha$ dividing some of $n_1, \ldots, n_k$ we have

$$|\{1 \leq s \leq k : p^\alpha \mid n_s\}| \geq p^\delta$$

where $\delta$ is the smallest positive integer such that $p^{\alpha - \delta}$ divides one of those $n_0 = 1$, $n_1, \ldots, n_k$ not divisible by $p^\alpha$.


Let $M$ be an additive commutative monoid (e.g. $M = \mathbb{N} = \{0, 1, 2, \ldots \}$). Let $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^k$ ($0 \leq a_s < n_s$) where those $\lambda_s \in \Lambda \subseteq M$ are multipliers or weights of the residue classes $a_s(n_s) = a_s + n_s\mathbb{Z}$ respectively. Define the covering map $w_A : \mathbb{Z} \rightarrow M$ by

$$w_A(x) = \sum_{s=1}^k \lambda_s \chi_s(x)$$

where $\chi_s(x)$ is 1 if $x \in a_s(n_s)$, and 0 otherwise. If $B$ is also such a system, putting all the triples in $A$ and $B$ together we get the sum-system $A \sqcup B$ (triples in it may be repeated). Theorem 1 of the paper is

The Generating Theorem. Let $S \subseteq M$ and $S_e$ denote the class of those triple systems $A$ with $w_A(\mathbb{Z}) \subseteq S$. Then we can generate all those $A \in S_e$ as follows:

(i) If $\lambda_1, \cdots, \lambda_k \in \Lambda$ and $\lambda_1 + \cdots + \lambda_k \in S$, then $\{(\lambda_s, 0, 1)\}_{s=1}^k \in S_e$;

(ii) Let $p$ be a prime, and

$$A_r = \{(\lambda_s, a_{sr}, n_s)\}_{s=1}^k \sqcup \{(\lambda_j^{(r)}, a_j^{(r)}, n_j^{(r)})\}_{j=1}^{h(r)}$$

(3)
lie in $S_s$ for all $r \in R(p) = \{0, 1, \cdots , p - 1\}$ where $\max_{r \in R(p)} h(r) > 0$, and for each $s = 1, \cdots , k$ the modulus $n_s$ is prime to $p$ and there is (a unique) $a_s \in R(n_s)$ such that $a_s = r + \hat{p}a_{sr} \pmod{n_s}$ for all $r \in R(p)$. Then
\[
A = \{(\lambda_s, a_s, n_s)\}_{s=1}^{k} \bigcup_{r=0}^{p-1} \{(\lambda_j^{(r)}, r + pa_j^{(r)}, pn_j^{(r)})\}_{j=1}^{h(r)}
\] (4)

belongs to $S_s$.

By Corollary 2 of this theorem, we can generate all those exact $m$-covers $A = \{a_s(n_s)\}_{s=1}^{k}$ (identified with $\{(1, a_s, n_s)\}_{s=1}^{k}$) as follows:

(a) The system of $m$ copies of $0(1)$ is an exact $m$-cover;

(b) Let $p$ be a prime and $A_r$ ($r \in R(p)$) (all the $\lambda$’s are 1) be as in the above theorem and $h(r) \geq 1$ for all $r \in R(p)$. Then the system $A_r$ given by (4) is an exact $m$-cover.

Also, we can generate all those $m$-covers $A = \{a_s(n_s)\}_{s=1}^{k}$ ($w_A(Z) \subseteq \{m, m + 1, \cdots \}$) by means of (a), (b) and the following (c).

(c) If $A$ is an $m$-cover, then so is $A \sqcup \{a(n)\}$ where $n \in \mathbb{Z}^+ = \{1, 2, 3, \cdots \}$ and $a \in R(n)$.

Let $B = \{\{(\mu_t, b_t, m_t)\}_{t=1}^{l}\}$ with $\mu_t \in \Lambda \subseteq M$. If $w_A = w_B$ then we said that $A$ and $B$ are equivalent which is written as $A \sim B$. Clearly $\{r(n)\}_{r=0}^{n-1} \sim \{0(1)\}$,
\[A = \{a_s(n_s)\}_{s=1}^{k} \text{ forms an exact } m\text{-cover if and only if } A \sim \{(m, 0, 1)\} = \{(0(1), \cdots , 0(1))\}.
\]

We deduced from the Generating Theorem the following (Theorem 4 in the paper) by induction.

**Main Theorem on the Equivalence.** Let $P$ be a set of primes and $f$ a mapping into a left $R$-module $M$ such that $(\frac{x + r}{p}, py) \in \text{Dom}(F)$ for all $r \in R(p)$ if $p \in P$ and $(x, y) \in \text{Dom}(F)$. Then the following $(\ast)$ and $(\ast)$ are equivalent:

$(\ast)$ Whenever $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^{k} \sim B = \{\{(\mu_t, b_t, m_t)\}_{t=1}^{l}\}$ with weights in $R$ and prime divisors of the moduli in $P$, we have
\[
\sum_{s=1}^{k} \lambda_s F\left(\frac{x + a_s}{n_s}, ny\right) = \sum_{t=1}^{l} \mu_t F\left(\frac{x + b_t}{m_t}, my\right) \text{ for all } (x, y) \in \text{Dom}(F);
\]

\[
\sum_{r=0}^{p-1} F\left(\frac{x + r}{p}, py\right) = F(x, y) \text{ for all } p \in P \text{ and } (x, y) \in \text{Dom}(F). \quad (\ast)
\]

This theorem is very powerful, it unifies almost all known identities for systems with a given covering map (such as exact $m$-covers). A slight generalization and
a direct proof were given in my paper *Products of binomial coefficients modulo* $p^2$, Acta Arith. 97(2001), 87–98. We call those $F$ satisfying the above $(P,M)$-equivalent maps. For $M$ being the complex field $\mathbb{C}$, we listed the following examples of equivalent maps in the paper:

$$F(x,y) = [x], \ B_m(x)y^{m-1}, \ \frac{\cot(\pi x)}{y}, \ \sum_{a=-\infty}^{+\infty} g\left(\frac{a}{y}\right) e^{2\pi i a x}. \quad (6)$$

($B_m(x)$ is the $m$th Bernoulli polynomial and the corresponding example is another form of Raabe’s formula $\sum_{r=0}^{n-1} B_m(z + \frac{r}{n}) = n^{1-m} B_m(nz)$.) For $M$ being the $\mathbb{Z}$-module $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, we gave examples

$$F(x,y) = 2\sin(\pi x), \ \Gamma(x)y^{x-1/2}/\sqrt{2\pi}$$

where the last example is an equivalent form of Gauss’ multiplication formula

$$\prod_{r=0}^{n-1} \Gamma(z + \frac{r}{n}) = (2\pi)^{(n-1)/2}n^{1/2-nz}\Gamma(nz).$$

The empty system $\emptyset$ has covering map $w_\emptyset = 0$. When multipliers are in an abelian group, $A \sim B \iff A \sqcup (-B) \sim \emptyset$. For example, $\{a_s(n_s)\}_{s=1}^k$ forms an exact $m$-cover if and only if $\{(1,a_1,n_1), \cdots, (1,a_k,n_k), (-m,0,1)\} \sim \emptyset$. Thus, to study the equivalence, we need only to consider those $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^k \sim \emptyset$.

By use of the equivalent map $F(x,y) = \Gamma(x)y^{x-1/2}/\sqrt{2\pi}$, we concluded the paper with

**Corollary 3.** Let $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^k \sim \emptyset$ where $\lambda_s, a_s, n_s \in \mathbb{Z}$ and $0 \leq a_s < n_s$.

Then

$$\prod_{s=1}^n \left( \Gamma\left(\frac{z + a_s}{n_s}\right) \frac{z + a_s}{n_s}^{-\frac{1}{2}} / \sqrt{2\pi} \right)^{\lambda_s} = 1 \quad (7)$$

for all $n \in \mathbb{Z}^+$ and $z \in \mathbb{C} \setminus \{0, -1, -2, \cdots\}$.

Consequently, if $A = \{a_s(n_s)\}_{s=1}^k$ is an exact cover of $\mathbb{Z}$ with $a_1 = 0$, then we have the relative formula

$$\prod_{s=2}^k \left( \Gamma\left(\frac{z + a_s}{n_s}\right) \frac{z + a_s}{n_s}^{-\frac{1}{2}} \right) = \frac{(2\pi)^{\frac{k-1}{2}} \Gamma(z)/\Gamma\left(\frac{z}{n_1}\right)}{(2\pi)^{\frac{k-1}{2}} \lim_{z' \to 0} \frac{\Gamma(z')z'}{\Gamma\left(\frac{z'}{n_1}\right) z'} \frac{1}{n_1}}$$

for $z \neq 0, -1, -2, \cdots$, i. e.,

$$\Gamma(z) = \frac{\Gamma\left(\frac{z}{n_1}\right)}{n_1^{-\frac{1}{2}} \prod_{s=2}^k \Gamma\left(\frac{z + a_s}{n_s}\right) n_s^{-\frac{1}{2}} \Gamma\left(\frac{a_s}{n_s}\right)} \quad \text{for } z \in \mathbb{C} \text{ with } z \neq 0, -1, -2, \cdots.$$
In view of the above, as you can find, most of the identities in the following papers essentially follow from my earlier work.


In this note we announced several results. Here is Theorem 3 in it.

**Theorem 3.** Let \( f \) be a complex valued function defined on \( D = \{(r,n) : r,n \in \mathbb{Z}, 0 \leq r < n\} \). Then \( \sum_{s=1}^{k} \lambda_s f(a_s,n_s) = 0 \) for all those \( A = \{(\lambda_s,a_s,n_s)\}_{s=1}^{k} \sim \emptyset \) \((\lambda_s \in \mathbb{C}, a_s \in R(n_s))\), if and only if \( f \) can be written in the form

\[
f(a,n) = \frac{1}{n} \sum_{m=0}^{n-1} g\left(\frac{m}{n}\right) e^{2\pi i \frac{ma}{n}}. \tag{8}\]

In the paper we said that the key idea of the proof is that both are equivalent to

\[
\sum_{j=0}^{n-1} f(a+jd,nd) = f(a,d) \quad \text{for all } n \in \mathbb{Z}^+ \text{ and } (a,d) \in D. \tag{9}\]

A detailed proof is contained in my recent paper Algebraic approaches to periodic arithmetical maps, J. Algebra, 240(2001), 723–743.


For a subnormal subgroup \( H \) of finite index in a group \( G \) we let

\[
d(G,H) = \sum_{i=1}^{n} ([H_i : H_{i-1}] - 1) \tag{10}\]
where $H_0 = H \subseteq H_1 \subseteq \cdots \subseteq H_n = G$ is a maximal chain of subgroups of $G$ such that $H_{i-1}$ is normal in $H_i$ for every $i = 1, \ldots, n$. It is easy to see that $d(G, H) \geq f([G : H])$ where the Mycielski function $f : \mathbb{Z}^+ \to \mathbb{N}$ is given by $f(\prod_{i=1}^r p_i^{\alpha_i}) = \sum_{i=1}^r \alpha_i(p_i - 1)$ ($p_1, \ldots, p_r$ are distinct primes).

The main result of the paper is the following:

Let $H$ be a subnormal subgroup of finite index in a group $G$. Then $1 + d(G, H)$ is the least $k \in \mathbb{Z}^+$ such that there exists a partition of $G$ into cosets $a_1G_1, \ldots, a_kG_k$ in which all the $G_i$ are subnormal in $G$ and $\bigcap_{i=1}^k G_i = H$. Moreover, if $\{a_iG_i\}_{i=1}^k$ is a partition of $G$ with all the $G_i$ subnormal, then for any subgroup $K \not\subseteq \bigcap_{i=1}^k G_i$ we have

$$\left|\{1 \leq i \leq k : K \not\subseteq G_i\}\right| \geq 1 + d\left(K, K \cap \bigcap_{i=1}^k G_i\right).$$

(11)

As an application Corollary 3 of the paper states as follows:

Let $G$ be a group and $H$ its subnormal subgroup of finite index. Let $H_G$ denote the core of $H$ in $G$ (i.e. the largest normal subgroup of $G$ contained in $H$). Then

$$2^{[G : H]} - 1 \geq [G : H_G] \geq [G : H] \geq 1 + d(G, H_G) \geq 1 + f([G : H_G]).$$

(12)


Below is the main theorem of the paper.

**Theorem.** Let $n_0$ be the smallest covering period of $X = \bigcup_{s=1}^k a_s(n_s)$ (i.e. the least positive integer such that $x \pm n_0 \in X$ whenever $x \in X$). Then

$$\frac{(n_1, \ldots, n_k)}{(n_0, n_1, \ldots, n_k)} \leq \max_{n \in \mathbb{Z}^+} \left|\{1 \leq i \leq k : n_i = n\}\right| \sum_{d|\prod_{i=1}^k n_i} \frac{1}{d}.$$  

(13)

A consequence of this theorem is the following strengthenment of the Burshtein conjecture, which is better than corresponding results of R.J. Simpson (1986) and Berger-Felzenbaum-Fraenkel (1987).

**Corollary.** Let $n_0$ be the smallest positive covering period of $\{a_s(n_s)\}_{s=1}^k$, and $N = [n_1, \ldots, n_k]$ have the standard form $p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ where $p_1, \ldots, p_r$ are distinct primes. Suppose that $0 \not\in I = \{0 \leq s \leq k : p_1^{\alpha_1} | n_s\} \neq \emptyset$, and that $a_i(n_i) \cap a_j(n_j) = \emptyset$ whenever $i \in I$ and $j \not\in I$. Then

$$p_i^{\delta_i(\alpha)} \leq \varepsilon_t(\alpha) \max_{s \in I} \left|\{i \in I : n_i = n_s\}\right| \prod_{i=1}^r \frac{p_i}{p_i - 1}.$$  

(15)
where $\delta_t(\alpha) = \min\{\delta \geq 1 : p_t^{\alpha-\delta} \| n_s \text{ for some } 0 \leq s \leq k\}$ and
\[
\varepsilon_t(\alpha) = \left(1 - \frac{1}{p_t^{\alpha_t - \alpha + 1}}\right) \prod_{i=1, i \neq t}^r \left(1 - \frac{1}{p_t^{\alpha_t + 1}}\right) < 1.
\]
(Observe that $\varepsilon_t(\alpha_t) \leq (1 - 1/p_t)$.)


Lemma 3 of this paper says that if $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^k \sim \emptyset$ (where $\lambda_s \in \mathbb{C}$, $0 \leq a_s < n_s$) then
\[
\sum_{s=1, \alpha n_s \in \mathbb{Z}}^k \frac{\lambda_s}{n_s} e^{2\pi i a_s} = 0 \quad \text{for any } \alpha \in \mathbb{R}.
\]
(16)

This identity brings out many number-theoretical properties. From it we deduced the following improvement to the Znám–Newman result which is better and general than all other improvements before.

**Theorem 1.** Let $A = \{(\lambda_s, a_s, n_s)\}_{s=1}^k$ where $\lambda_s \in \mathbb{C}$, and $n_0 \in \mathbb{Z}^+$ be a period of $w_A(x)$. If $d \in \mathbb{Z}^+$ does not divide $n_0$ and
\[
\sum_{1 \leq s \leq k, d \mid (n_s, a_s - a)} \frac{\lambda_s}{n_s} \neq 0 \quad \text{for some } a \in \mathbb{Z},
\]
then
\[
|\{a_s \text{ mod } d : 1 \leq s \leq k \text{ and } d \mid n_s\}| \geq \min_{0 \leq s \leq k} \frac{d}{(d, n_s)} \geq p(d)
\]
(17)

where $p(d)$ is the least prime divisor of $d$.

This result is somewhat similar to the one stated in item 1. (Compare (17) with (1).) We noted that, if $A = \{a_s(n_s)\}_{s=1}^k$ is an exact $m$-cover with $n_1 \leq \cdots \leq n_{k-1} < n_{k-1} + 1 = \cdots = n_k$, then by taking $\lambda_1 = \cdots = \lambda_k = n_0 = 1$ and $d = n_k$ we get
\[
l \geq \min_{1 \leq s \leq k-1} \frac{n_k}{(n_s, n_k)} \geq \max \left\{p(n_k), \frac{n_k}{n_{k-1}}\right\}.
\]
(18)

This lower bound for $l$ is the best one up to now. [Y.G. Chen and Š. Porubský (1995) proved further that $l$ can be written as $\sum_{s=1}^{k-1} x_s \frac{n_s}{(n_s, n_k)}$ with $x_s \in \mathbb{N}$, starting from the vital lemma above (the key equality (3) in their paper is our (16)).}

In this paper we gave the following generalization of the Davenport-Mirsky-Newman-Rado result.

**Theorem 2.** For \( s = 1, \cdots, k \) let \( \psi_s \) be an arithmetical function regularly periodic mod \( n_s \). (By regularity we mean \( \sum_{r=0}^{n_s-1} \psi_s(r) \xi^r \neq 0 \) for some primitive \( n_s \)-th root \( \xi \) of unity.) If \( [n_1, \cdots, n_k] \) is not the smallest positive period of the function \( \psi = \psi_1 + \cdots + \psi_k \) then there must exist some \( s, t \) for which \( n_s = n_t \) and \( \psi_s \neq \psi_t \).

Note that \( A = \{a_s(n_s)\}_{s=1}^k \) is an exact \( m \)-cover for some \( m \in \mathbb{Z}^+ \) if and only if \( w_A(x) = \sum_{s=1}^k \chi_s(x) \) has period \( n_0 = 1 \).


In this paper we noted that the question of V.Billik and H. M. Edgar in 1973 (whether for any \( d \in \mathbb{Z}^+ \) there exists a minimal (irredundant) cover with all the moduli distinct and having greatest common divisor \( d \)) is equivalent to the well known problem of P. Erdős (whether for any \( c > 0 \) there exists a cover with all the moduli distinct and greater than \( c \)). If \( \{a_s(n_s)\}_{s=1}^k \) is a cover with \( n_1 < \cdots < n_k \) and there are no covers with all the moduli distinct and greater than \( n_1 \), then we showed that some \( n_s \) is divisible by \( 3n_1 \) or \( 4n_1 \).


In this paper we solved two problems on disjoint systems raised by A.P. Huhn and L. Megyesi [Discrete Math. 41(1982), 327–330]. They called a finite sequence \( \{n_s\}_{s=1}^k \) of positive integers harmonic if there are \( a_1, \cdots, a_k \in \mathbb{Z} \) such that \( A = \{a_s(n_s)\}_{s=1}^k \) is disjoint (i.e. those \( a_s(n_s) \) are pairwise disjoint). Let statements \( (\dagger) \) and \( (\ddagger) \) be as follows:

\( (\dagger) \) For any \( I \subseteq \{1, \cdots, k\} \) with \( |I| \geq 2 \) we have

\[
\sum_{s \in I} \frac{1}{\tilde{n}_s(I)} \leq 1 \quad \text{where} \quad \tilde{n}_s(I) = (n_s, [n_t]_{t \in I \setminus \{s\}}) = [(n_s, n_t)]_{t \in I \setminus \{s\}}.
\]

\( (\ddagger) \) For any \( I \subseteq \{1, \cdots, k\} \) with \( |I| \geq 2 \), there exist \( s, t \in I \) with \( s \neq t \) such that \( (n_s, n_t) \geq |I| \).

Their first question is whether \( (\dagger) \) is sufficient for \( \{n_s\}_{s=1}^k \) to be harmonic. (Huhn and Megyesi noted the necessity.) The second one asks whether \( (\ddagger) \) is necessary and sufficient for \( \{n_s\}_{s=1}^k \) to be harmonic.
We first determined all those non-harmonic sequences \( \{n_s\}_{s=1}^k \) with \( k \leq 4 \), by means of covers consisting of \( k \leq 4 \) residue classes.

**Theorem 1.** Let \( n_1, n_2, n_3, n_4 \) be positive integers. Then

i) \( \{n_i\}_{i=1}^2 \) is not harmonic if and only if \( (n_1, n_2) = 1 \);

ii) \( \{n_i\}_{i=1}^3 \) is not harmonic but \( \{n_i\}_{i \in I} \) is harmonic for all \( \emptyset \neq I \subset \{1, 2, 3\} \), if and only if \( (n_1, n_2) = (n_1, n_3) = (n_2, n_3) = 2 \);

iv) \( \{n_i\}_{i=1}^4 \) is not harmonic but \( \{n_i\}_{i \in I} \) is harmonic for all \( \emptyset \neq I \subset \{1, 2, 3, 4\} \), if and only if all the \( (n_i, n_j) \) with \( 1 \leq i < j \leq 4 \) are 3, or we can rearrange \( n_1, n_2, n_3, n_4 \) so that

\[
(n_1, n_2) = (n_1, n_3) = (n_1, n_4) = 2, \quad (n_2, n_3) = (n_2, n_4) = (n_3, n_4) = 4.
\]

As a consequence we obtained

**Corollary 1.** Let \( 1, \ldots, n_k \in \mathbb{Z}^+ \). For \( k \leq 4 \), \( \{n_s\}_{s=1}^k \) is harmonic if and only if (†) holds. For \( k \leq 3 \), \( \{n_s\}_{s=1}^k \) is harmonic if and only if we have (‡); when \( k = 4 \), \( \{n_s\}_{s=1}^k \) may not be harmonic even if we have (‡) together with the inequality

\[
\sum_{s=1}^k 1/n_s \leq 1.
\]

The second Theorem in the paper is

**Theorem 2.** Let \( k, a, b, c, d \in \mathbb{Z}^+ \), \( k, a \geq 5 \), and \( 6, a, b, c, d \) be relatively prime. Put

\[
n_1 = 2a, n_2 = 3a, n_3 = 6b, n_4 = 6c, n_5 = 6d, n_6 = \cdots = n_k = 6abcd.
\]

Then \( \sum_{s=1}^k 1/n_s \leq 1 \), both (†) and (‡) hold, but \( \{n_s\}_{s=1}^k \) is not harmonic.

In view of the above we answered the two questions negatively (but we don’t know whether (‡) is necessary for \( \{n_s\}_{s=1}^k \) to be harmonic).


In 1982 A.P. Huhn and L. Megyesi [Discrete Math.] proved (by a graph-theoretic method) that the sequence \( \{n_s\}_{s=1}^k \) of positive integers is harmonic if those greatest common divisors \( (n_i, n_j) \) (\( 1 \leq i < j \leq k \)) are distinct and greater than one. By establishing connections with covers, we gave the following extension of the result.

**Theorem.** Let \( n_1, \ldots, n_k \in \mathbb{Z}^+ \). If for any \( d \in \mathbb{Z}^+ \) with \( f(d) \leq k - 2 \) (and hence \( d \leq 2^{k-2} \)) we have

\[
|\{(i, j) : 1 \leq i < j \leq k \& (n_i, n_j) = d\}| < \sqrt{\frac{d+7}{8}},
\]

(20)
then $\{n_s\}_{s=1}^k$ is harmonic.

We conjectured that the above $\sqrt{(d + 7)/8}$ can be replaced by $2d - 1$.


Let $A = \{a_s(n_s)\}_{s=1}^k$ forms an exact $m$-cover of $\mathbb{Z}$. It is well known that $\sum_{s=1}^k 1/n_s = m$. Also, $A$ may not have a proper subcover which is an exact $n$-cover for some $n < m$. In the paper we proved the following result analytically.

**Theorem.** Let $A = \{a_s(n_s)\}_{s=1}^k$ be an exact $m$-cover. Then for each $n = 0, 1, \ldots, m$ there exist at least $\binom{m}{n}$ subsets $I$ of $\{1, \ldots, k\}$ such that $\sum_{s=1}^k 1/n_s = n$. The bounds $\binom{m}{n}$ ($0 \leq n \leq m$) are best possible.


For $\alpha \in \mathbb{R}$ and $\beta > 0$ we let $\alpha + \beta \mathbb{Z} = \{\alpha + \beta x : x \in \mathbb{Z}\}$. Instead of systems of residue classes we may consider a general system in the form

$$A = \{\alpha_s + \beta_s \mathbb{Z}\}_{s=1}^k.$$  \hfill (21)

By inventing a combined method involving linear algebraic, analysis and Stirling numbers, we were able to characterize general covers (not having a fixed covering function) for the first time.

**Theorem 1.** For system (21) the following statements are equivalent:

a) (21) forms an $m$-cover of $\mathbb{Z}$.

b) (21) covers $|S(A)|$ consecutive integers at least $m$ times where

$$S(A) = \left\{ \left\{ \sum_{s \in I} \frac{1}{\beta_s} \right\} : I \subseteq \{1, \ldots, k\} \right\}.$$  \hfill (22)

c) For any $\theta \in [0, 1)$ and $n \in R(m)$ we have

$$\sum_{\{I \subseteq \{1, \ldots, k\} : \sum_{s=1}^k 1/\beta_s = \theta\}} (-1)^{|I|} \left( \left[ \sum_{s=1}^k 1/\beta_s \right] \right)^n e^{2\pi i} \sum_{s=1}^k \alpha_s/\beta_s = 0.$$  \hfill (23)

A conjecture of P. Erdös in the early 1960’s asserts that $A = \{a_s(n_s)\}_{s=1}^k$ forms a cover of $\mathbb{Z}$ if it covers integers from 1 to $2^k$. R.B. Crittenden and C.L. Vanden Eynden [Proc. Amer. Math. Soc. 24(1970), 475–481] provided a long awkward
proof for $k \geq 20$. Note that $|S(A)| \leq 2^k$ depends on those $\beta$’s! So that b) implies
a) gave more detailed information than the original conjecture of Erdős. In other
words, the covering function $w_A(x)$ ($x \in \mathbb{Z}$) takes the least value when $x$ ranges
over an interval $[a, a + |S(A)|]$ of length $|S(A)|$.

further conjecture which says that $A = \{a_s(n_s)\}_{s=1}^k$ is a cover if it covers integers in
$[1, 2^{k-l}(l+1)]$ where $0 \leq l < \min\{k, n_1, \ldots, n_k\}$. In contrast with this conjecture, we noted that ‘b) $\Rightarrow$ a)’ yields the following

**Theorem 7.** (21) is an $m$-cover of $\mathbb{Z}$ if it covers $2^{k-M}(M+1)$ consecutive integers
at least $m$ times, where

$$M = \max_{1 \leq i \leq k} |\{1 \leq s \leq k : \beta_s = \beta_i\}|.$$  

That a) $\iff$ c) is useful, we derived from it many new properties of the moduli in
an $m$-cover.

By part (ii) of Theorem 4 in the paper, if $A = \{a_s(n_s)\}_{s=1}^k$ forms an exact $m$-
cover then for any $\emptyset \neq J \subset \{1, \ldots, k\}$ there exists an $I \subset \{1, \ldots, k\}$ with $I \neq J$
such that $\sum_{s \in I} 1/n_s = \sum_{s \in J} 1/n_s$. Actually this also follows from our work in
the paper in Israel J. Math. (1992). By the first part of Theorem 3, if (21) is
an $m$-cover of $\mathbb{Z}$ and $J$ is a subset of $\{1, \ldots, k\}$ with $\sum_{s \in I} 1/\beta_s = \sum_{s \in J} 1/\beta_s$ for
no other $I \subset \{1, \ldots, k\}$, then there are at least $m$ nonzero integers in the form
$\sum_{s \in I} 1/\beta_s - \sum_{s \in J} 1/\beta_s$ with $I \subset \{1, \ldots, k\}$.


Let $A = \{a_s(n_s)\}_{s=1}^k$ and $n_1 \leq \cdots \leq n_{k-l} < n_{k-l+1} = \cdots = n_k$ where $0 < l \leq k$.

By applying results in Part I [Acta Arith. 72(1995)] to the system $A = \{a_s + \frac{n_s}{m_s} \mathbb{Z}\}_{s=1}^k$ where $m_1, \ldots, m_k \in \mathbb{Z}^+$, we obtained lots of results on the moduli in an
$m$-cover $A$.

By Theorem 1(i), $A = \{a_s(n_s)\}_{s=1}^k$ forms an $m$-cover if it covers $W$ consecutive integers where

$$W = \min_{m_1, \ldots, m_k \in \mathbb{Z}} \left| \left\{ \left\{ \sum_{s \in I} \frac{m_s}{n_s} : I \subseteq \{1, \ldots, k\} \right\} \right. \right|.$$  

In Example 2 we noted that if $n_1 = n_2 = 4$ and $n_3 = n_4 = n_5 = 6$ then
$|S(\{a_s(n_s)\}_{s=1}^5)| = 10 > W = 9.$
Here we listed several properties of \( m \)-cover \( A \) proved in the paper:

1) For any \( m_1, \ldots, m_k \in \mathbb{Z}^+ \) there are at least \( m \) positive integers in the form \( \sum_{s \in I} m_s/n_s \) where \( I \subseteq \{1, \ldots, k\} \). [This is an extension of Zhang’s result in 1989.]

2) If \( l \neq k \), then either \( l \geq n_k/n_{k-1} \) or \( \sum_{s=1}^{k-1} 1/n_s \geq m \). [This improves the Davenport-Mirsky-Newman-Rado result which says that if \( l = 1 \) then \( \sum_{s=1}^{k} 1/n_s > 1 \) (i.e. \( A \) is not a disjoint cover).]

3) If \( a_t(n_t) \) is essential (i.e. \( A_t = \{a_s(n_s)\}_{s \neq t} \) fails to be an \( m \)-cover), then for any \( a \in \mathbb{Z} \) there exist \( I, J \subseteq \{1, \ldots, k\} \) such that

\[
\frac{a}{n_t} \equiv \sum_{s \in I} \frac{1}{n_s} - \sum_{s \in J} \frac{1}{n_t} \pmod{1}.
\] (26)

14. Zhi-Wei Sun, Exact \( m \)-covers and the linear form \( \sum_{s=1}^{k} x_s/n_s \), Acta Arith., 81 (1997), no. 2, 175–198. MR 98h:11019; Zbl. M. 871.11011

In this paper we characterized exact \( m \)-covers in several ways. One of the characterizations is

**Theorem 4.2.** \( A = \{a_s(n_s)\}_{s=1}^{k} \) forms an exact \( m \)-cover if and only if we have

\[
\sum_{J \subseteq \{1, \ldots, k\} \setminus I} (-1)^{|J|} e^{2\pi i \sum_{s \in J} a_s/n_s} \{\sum_{s \in I} 1/n_s\} = \theta
\]

\[
= \sum_{x_s \in R(n_s) \ (s \in I)} e^{2\pi i \sum_{s \in I} a_s x_s/n_s} \{\sum_{s \in I} x_s/n_s\} = \theta
\] (27)

for all \( \theta \in [0, 1) \) and \( I \subseteq \{1, \ldots, k\} \) with \( |I| = m \).

From the characterizations we deduced some properties of exact \( m \)-covers. Here we summarize the central results.

**Theorem.** Let \( A = \{a_s(n_s)\}_{s=1}^{k} \) be an exact \( m \)-cover. Then

1. For \( a = 0, 1, 2, \ldots \) and \( t = 1, \ldots, k \) we have

\[
\left| \left\{ I \subseteq \{1, \ldots, k\} : t \notin I \land \sum_{s \in I} \frac{1}{n_s} = \frac{a}{n_t} \right\} \right| \geq \binom{m-1}{[a/n_t]}
\] (28)

where the lower bounds are best possible.

2. If \( \emptyset \neq I \subseteq \{1, \ldots, k\} \) and \( (n_s, n_t) | a_s - a_t \) for all \( s, t \in I \), then we have

\[
\left\{ \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} : J \subseteq I \right\} \supseteq \left\{ \frac{r}{[n_s]_{s \in I}} : r \in R([n_s]_{s \in I}) \right\}
\] (29)
where \( \bar{I} = \{1, \ldots, k\} \setminus I \), moreover for any \( r \in R([n_s]_{s \in I}) \) we have

\[
\left| \left\{ J \subseteq \bar{I} : \left\{ \sum_{s \in J} \frac{1}{n_s} \right\} = \frac{r}{[n_s]_{s \in I}} \right\} \right| \geq \prod_{s \in I} n_s \quad \left( n_s \right)_{s \in I}.
\] (30)

(III) The number of solutions of the equation \( \sum_{s=1}^{k} x_s/n_s = c \) with \( x_s \in R(n_s) \) for \( s = 1, \ldots, k \), is the sum of finitely many (not necessarily distinct) prime factors of \( n_1, \ldots, n_k \) if \( c \neq 0, 1, 2, \ldots \), and at least \( \left( k-m \right) \) if \( c \) equals a nonnegative integer \( n \).


In the paper we essentially determined all covers \( A = \{a_s(n_s)\}_{s=1}^{k} \) with \( k < 10 \), in the Appendix we listed out all those irreducible minimal covers \( \{a_s(n_s)\}_{s=1}^{k} \) (up to possible changes of the residues) with \( k < 10 \) for which \( n_1, \ldots, n_k \) are distinct, and \( \{2, 3, 6\} \not\subseteq \{n_1, \ldots, n_k\} \) if \( k = 9 \). Our algorithm is based on the recent results of Sun and actually valid for any \( k \in \mathbb{Z}^+ \). As an application of the data yielded by the algorithm, for any minimal cover \( A = \{a_s(n_s)\}_{s=1}^{k} \) with \( k < 10 \), \( n_1 < \cdots < n_k \) and \( \{3, 6\} \not\subseteq \{n_1, \ldots, n_k\} \), we found (uniformly) an explicit residue class \( a(2p_0p_1 \cdots p_k) \) containing no integers of the form \( 2^h + p \) (\( p \) is a prime) where \( 2 \nmid a \), \( p_0 \in \{31, 43\} \) and \( p_1, \ldots, p_k \) are distinct primes different from 2 and \( p_0 \).

The second theorem in the paper is as follows:

**Theorem 2.** Let \( A = \{a_s(n_s)\}_{s=1}^{k} \) be a minimal cover with \( n_1 < \cdots < n_k \). Suppose that for each \( s = 1, \ldots, k \) prime \( p_s \) divides \( 2^{n_s} - 1 \) but not \( 2^n - 1 \) with \( 0 < n < n_s \). Let \( X = \bigcap_{s=1}^{k} 2^{a_s}(p_s) = a(p_1 \cdots p_k) \), and \( c \) be any integer divisible by a unique prime among \( p_1, \ldots, p_k \). Then there exists an \( n \in \mathbb{Z}^+ \) such that \( 2^n + cp \in X \) for infinitely many primes \( p \).


**Theorem.** Let \( A = \{a_s(n_s)\}_{s=1}^{k} \) form an \( m \)-cover and \( J \subseteq \{1, \ldots, k\} \). Put \( \bar{J} = \{1, \ldots, k\} \setminus J \).

(i) For any \( m_1, \ldots, m_k \in \mathbb{Z} \) we have

\[
\left| \left\{ I \subseteq \{1, \ldots, k\} : I \neq \bar{J} \& \left\{ \sum_{s \in I} \frac{m_s}{n_s} \right\} = \left\{ \sum_{s \in J} \frac{m_s}{n_s} \right\} \right\} \right| \geq m.
\] (31)

(ii) Suppose that there is an \( x \in \bigcap_{s \in J} a_s(n_s) \) (\( J \neq \emptyset \)) with \( w_A(x) = m \). For each \( s \in \bar{J} \) let \( m_s \) be a positive integer prime to \( n_s \). Then there exists an \( \alpha \in [0, 1) \) such
that
\[
\left\{ \sum_{s \in I} \frac{m_s}{n_s} : I \subseteq J, \left[ \sum_{s \in I} \frac{m_s}{n_s} \right] \geq m - |J| \right\} \supseteq \left\{ \frac{a}{|n_s|_{s \in J}} : 0 \leq a < |n_s|_{s \in J}, \{a\} = \alpha \right\}.
\]

In view of this theorem, an \( m \)-cover \( A = \{a_s(n_s)\}_{s=1}^k \) possesses the following properties:

(a) For each \( J \subseteq \{1, \cdots, k\} \), there exist at least \( m \) subsets \( I \) of \( \{1, \cdots, k\} \) with \( I \neq J \) such that \( \sum_{s \in I} 1/n_s - \sum_{s \in J} 1/n_s \in \mathbb{Z} \).

(b) If \( a_t(n_t) \) is essential, then the set
\[
S_t(A) = \left\{ \left\{ \sum_{s \in I} \frac{1}{n_s} : I \subseteq \{1, \cdots, k\} \setminus \{t\}, \left[ \sum_{s \in I} \frac{1}{n_s} \right] \geq m - 1 \right\} \right.
\]
contains an arithmetic progression of length \( n_t \) with common difference \( 1/n_t \).

Part (a) is another generalization of Zhang’s result. In the case \( J = \emptyset \), this and 1) in item 13 are implied by our following conjecture: If \( A \) forms an \( m \)-cover and \( m_1, \cdots, m_k \in \mathbb{Z}^+ \), then there exist a chain \( \emptyset \neq I_1 \subset \cdots \subset I_m \subseteq \{1, \cdots, k\} \) such that \( \sum_{s \in I_t} m_s/n_s \in \mathbb{Z} \) for all \( t = 1, \cdots, m \).

Part b) yields 3) in item 13 and the inequality \( |S_t(A)| \geq n_t \). Now we conjecture that for any minimal \( m \)-cover \( A = \{a_s(n_s)\}_{s=1}^k \), \( |S(A)| \leq n_1 + \cdots + n_k \), and \( S(A) \supseteq \{r/d : r \in R(d)\} \) if \( d \in \mathbb{Z}^+ \) and \( 1/d \in S(A) \).


In 1975 F. Cohen and J.L. Selfridge found a 94-digit positive integer which cannot be written as the sum or difference of two prime powers. Following their basic construction and introducing a new method to avoid a bunch of extra congruences, we proved that if
\[ P = \{2, 3, 5, 7, 11, 13, 17, 19, 31, 37, 41, 61, 73, 97, 109, 151, 241, 257, 331\} \]
and
\[ x \equiv 47867742232066880047611079 \left( \mod \prod_{p \in P} p \right) \]
then \( x \) is not of the form \( \pm p^a \pm q^b \) where \( p, q \) are primes and \( a, b \) are nonnegative integers.
Let $G$ be a group and $A = \{a_iG_i\}_{i=1}^k$ be a finite system of left cosets in $G$ where each $G_i$ is a subgroup of $G$.

The main results in the paper are as follows:

(I) Let $A$ be an exact $m$-cover of $G$ with all the $G_i$ subnormal in $G$. Then $k \geq m + d(G, \bigcap_{i=1}^k G_i)$, and the lower bound is best possible. Moreover, for any subgroup $K$ of $G$ not contained in all the $G_i$ we have

$$|\{1 \leq i \leq k : K \not\subseteq G_i\}| \geq 1 + d\left(K, K \cap \bigcap_{i=1}^k G_i\right).$$

(This generalizes the author’s work in 1990.)

(II) Let $A$ be an exact $m$-cover of $G$. Whenever $G/(G_i)_G$ is solvable, we have $k \geq m + f([G : G_i])$ and hence $[G : G_i] \leq 2^{k-m}$.

Concerning result (II) we have a further conjecture.

**Conjecture.** Let $A$ be an exact $m$-cover of a group $G$ with all the $G/(G_i)_G$ solvable. Then $k \geq m + f(N)$ where $N$ is the least common multiple of the indices $[G : G_1], \ldots, [G : G_k]$.

For a cover $A$ of $G$, if it doesn’t form an exact $m$-cover for any $m = 1, 2, 3, \ldots$, then we don’t have a similar inequality in general. When $G$ is cyclic, or $|G|$ is square-free and all the $G_i$ are subnormal in $G$, if $m(A') < m(A)$ for any proper subsystem $A'$ of $A$ then we can show that $k \geq m(A) + f([G : \bigcap_{i=1}^k G_i])$.

For a subgroup $H$ of $G$ we let $G/H$ denote the set of all left cosets of $H$ in $G$. To prove Results (I) and (II), we need the following

**Theorem 2.1.** Assume that $A$ forms an exact $m$-cover of $G$ For a subgroup $H$ of $G$ we have

$$\{C \in G/H : C \supseteq a_iG_i \text{ for some } i = 1, \ldots, k\} = \emptyset \text{ or } G/H.$$

in the following cases:

(a) $H$ is the group $G$ or a normal subgroup of prime index in $G$;
(b) $G_1, \ldots, G_k$ are normal in $G$ and $H$ is maximal in $G$;
(c) $G_1, \ldots, G_k$ are subnormal and $H$ is maximal normal in $G$.

The paper also includes an interesting application in group theory.

**Corollary 4.6.** Let $H$ be a subnormal subgroup of a group $G$ with $[G : H] < \infty$. Then $H$ is normal in $G$ if and only if

$$|N_G(H)/H| + d(H, H_G) \geq [G : H].$$
When $H$ is a subnormal subgroup of a group $G$ with $[G : H] < \infty$, we conjectured that $|N_G(H)/H| \geq d(G, H)$.


Let $m$ be an integer and $M$ an additive abelian group. Let $f$ be a map from a subset of $\mathbb{C} \times \mathbb{C}$ into $M$. If for any ordered pair $(x, y)$ in the domain $\text{Dom}(f)$ of $f$ and each positive integer $n$ prime to $m$, we have

$$\left\{ \binom{x + mr}{n}, ny \right\} : r = 0, 1, \ldots, n - 1 \subseteq \text{Dom}(f)$$

and

$$\sum_{r=0}^{n-1} f\left(\binom{x + mr}{n}, ny\right) = f(x, y),$$

then we call $f$ an $m$-uniform map (into $M$).

The following result presented in the paper is an extension of the Main Theorem on Equivalence mentioned in item 2.

**Theorem 2.1.** Let $m$ be an integer and $M$ a left $R$-module where $R$ is a ring with identity. Let $f$ be a map into $M$ with $\text{Dom}(f) \subseteq \mathbb{C} \times \mathbb{C}$ such that (*) holds for any $(x, y) \in \text{Dom}(f)$ and $n \in \mathbb{Z}^+$ with $(m, n) = 1$. Then the following two statements are equivalent:

(a) $f$ is an $m$-uniform map into $M$.

(b) Whenever

$$\sum_{1 \leq s \leq k} \sum_{x \in a_s(n_s)} \lambda_s = \sum_{1 \leq t \leq l} \sum_{x \in b_t(m_t)} \mu_t \quad \text{for all } x \in \mathbb{Z}$$

(with $\lambda_s, \mu_t \in R$, $a_s, n_s, b_t, m_t \in \mathbb{Z}$, $0 \leq a_s < n_s$, $0 \leq b_t < m_t$ and $(n_s m_t, m) = 1$), we have

$$\sum_{s=1}^{k} \lambda_s f\left(\binom{x + ma_s}{n_s}, n_s y\right) = \sum_{t=1}^{l} \mu_t f\left(\binom{x + mb_t}{m_t}, m_t y\right) \quad \text{for } (x, y) \in \text{Dom}(f).$$

The proof of this result is very simple. As for examples of uniform maps, we gave
Proposition 2.1. (i) Let \( m \in \mathbb{Z} \). Then the function \([ ]_m : \mathbb{R} \times \mathbb{R} \to \mathbb{Q} \) given by

\[
[ ]_m(x, y) = [x] + \frac{1 - m}{2}
\]
is an \( m \)-uniform map into the rational field \( \mathbb{Q} \).

(ii) For each \( m = 0, 1, 2, \cdots \) the functions \( b_m : \mathbb{C} \times \mathbb{C}^* \to \mathbb{C} \) and \( e_m : \mathbb{C} \times \mathbb{Z} \to \mathbb{C} \) given by

\[
b_m(x, y) = y^{m-1}B_m(x)
\]
and

\[
e_m(x, y) = \begin{cases} 
    e^{\pi i x y^{m}}E_m(x) & \text{if } y \text{ is odd}, \\
    -\frac{2}{m+1}e^{\pi i x y^{m}}B_{m+1}(x) & \text{if } y \text{ is even},
\end{cases}
\]
are 1-uniform maps into the complex field \( \mathbb{C} \), where \( B_m(x) \) and \( E_m(x) \) are the \( m \)th Bernoulli polynomial and the \( m \)th Euler polynomial respectively.

Proposition 2.2. Let \( p \) be an odd prime. For \( x \geq 0 \) and \( m \in \mathbb{Z} \setminus p\mathbb{Z} \) let

\[
q(x, m) = \frac{q_p(m)}{m} + \sum_{0 < j < [x]} \frac{1}{jm} \text{ where } q_p(m) = \frac{m^{p-1} - 1}{p}.
\]

Then the function \( \bar{q}(x, m) = q(x, m) \mod p \) is a \( p \)-uniform map into the finite field \( \mathbb{Z}/p\mathbb{Z} \).

From the above, we deduced the following

Theorem 1.2. Let \( p \) be an odd prime. Let \( A = \{a_s(n_s)\}_{s=1}^k \) \((0 \leq a_s < n_s)\) and \( B = \{b_t(m_t)\}_{t=1}^l \) \((0 \leq b_t < m_t)\) be covering equivalent systems with the moduli \( n_s \) and \( m_t \) not divisible by \( p \) but dividing integer \( N \). Then for any \( x \in [0, p) \) we have

\[
P \prod_{s=1}^k \left( p^{n_s} - 1 \right) / \prod_{t=1}^l \left( p^{m_t} - 1 \right) 
\equiv (-1)^{(k-l)\frac{p-1}{2}} \left( 1 + pN \left( \sum_{s=1}^k \frac{q_p(n_s)}{n_s} - \sum_{t=1}^l \frac{q_p(m_t)}{m_t} \right) \right) \pmod{p^2}.
\]

Actually we may not require the integer \( N \) in Theorem 1.2 to be a common multiple of those moduli \( n_s \) and \( m_t \). For example \( N = 1 \) is allowed if we don’t mind using \( x \not\in \mathbb{Z} \) in the notation \( \binom{x}{n} \).

Corollary 1.2. Let \( A = \{a_s(n_s)\}_{s=1}^k \) \((0 \leq a_s < n_s)\) be an exact \( m \)-cover of \( \mathbb{Z} \). Let \( N \) be the least common multiple of \( n_1, \cdots, n_k \) and \( p \) an odd prime not dividing \( N \). Then

\[
P \prod_{s=1}^k \left( p^{n_s} - 1 \right) \equiv (-1)^{(k-m)\frac{p-1}{2}} \left( 1 + pN \sum_{s=1}^k \frac{q_p(n_s)}{n_s} \right) \pmod{p^2}.
\]
Applying Corollary 1.2 to the trivial disjoint cover \( A = \{ r(n) \}_{r=0}^{n-1} \) we then get a congruence due to A. Granville.


A residue class \( a + n \mathbb{Z} \) with weight \( \lambda \) is denoted by \( \langle \lambda, a, n \rangle \). For a finite system \( A = \{ \langle \lambda_s, a_s, n_s \rangle \}_{s=1}^{k} \) of such triples, the periodic map \( w_A(x) = \sum_{n_s \mid x-a_s} \lambda_s \) is called the covering map of \( A \). Some interesting identities for those \( A \) with a fixed covering map have been known, in this paper we mainly determine out all those functions \( f : \Omega \rightarrow \mathbb{C} \) such that 

\[
\sum_{r=0}^{n_s-1} f(r + n_s \mathbb{Z}) e^{2\pi i a_r x / n_s} \neq 0 \quad \text{for all } a \in \mathbb{Z} \text{ and } s = 1, \ldots, k,
\]

if and only if for any \( \psi_1 \in \mathbb{P}(\mathbb{C}) \) periodic mod \( n_1, \ldots, \psi_k \in \mathbb{P}(\mathbb{C}) \) periodic mod \( n_k \) we have

\[
\psi_1 + \cdots + \psi_k = 0 \iff f \circ \psi_1 + \cdots + f \circ \psi_k = 0.
\]
This was announced by me in 1989 (see item 3).

Now we state two more results in this paper:

I. Let \( \lambda_1, \ldots, \lambda_k \in \mathbb{C}^* = \mathbb{C} \setminus \{0\} \), and \( \xi_1, \ldots, \xi_k \) be distinct roots of unity. Then the smallest (positive) period of the arithmetical function \( \psi(x) = \sum_{s=1}^{k} \lambda_s \xi_s^x \), coincides with \( [n_1, \ldots, n_k] \) where \( n_s \) is the least \( n \in \mathbb{Z}^+ \) with \( \xi_s^n = 1 \) (i.e., \( \xi_s \) is a primitive \( n_s \)th root of unity).

II. Let \( A = \{a_s(n_s)\}_{s=1}^{k} \) be a system of residue classes with \( n_1, \ldots, n_k \) square-free and \( n_1 \leq \cdots \leq n_{k-l} < n_{k-l+1} = \cdots = n_k \) (0 < \( l < k \)). If \(|\{1 \leq s \leq k : (x + a_s, n_s) = 1\}| = m \) for all \( x \in \mathbb{Z} \), then \( l \geq \min_{1 \leq s \leq k-l} n_k/(n_s, n_k) \), furthermore

\[
\frac{l}{n_k} = \sum_{s=1}^{k-l} \frac{x_s}{(n_s, n_k)} \quad \text{for some } x_1, \ldots, x_{k-l} \in \mathbb{N} = \{0, 1, 2, \cdots\}.
\]


In this paper we characterize covering equivalence in various ways, our characterizations involve the Γ-function, the Hurwitz ζ-function, trigonometric functions, the greatest integer function and Egyptian fractions.

Here we collect some results of the paper.

**Theorem.** Let \( n_s, m_t \in \mathbb{Z}^+ \), \( a_s \in R(n_s) \) and \( b_t \in R(m_t) \) for \( s = 1, \ldots, k \) and \( t = 1, \ldots, l \). Then the following statements are equivalent:

\[
A = \{a_s(n_s)\}_{s=1}^{k} \sim B = \{b_t(m_t)\}_{t=1}^{l};
\]

\[
\sum_{J \subseteq \{1, \ldots, k\}} (-1)^{|J|} e^{2\pi i \sum_{s \in J} \frac{a_s}{n_s}} = \sum_{J \subseteq \{1, \ldots, l\}} (-1)^{|J|} e^{2\pi i \sum_{t \in J} \frac{b_t}{m_t}} \quad \text{for all } c \geq 0;
\]

\[
2^k \prod_{s \in S_z} \frac{a_s - z}{n_s} \cdot \prod_{s \in S_z} \left( -1 \left\{ \frac{a_s}{n_s} \right\} \right) = 2^l \prod_{t \in T_z} \frac{b_t - z}{m_t} \cdot \prod_{t \in T_z} \left( -1 \left\{ \frac{b_t}{m_t} \right\} \right) \quad \text{for } z \in \mathbb{C}
\]

where \( S_z = \{1 \leq s \leq k : z \in a_s(n_s)\} \) and \( T_z = \{1 \leq t \leq l : z \in b_t(m_t)\} \);

\[
\frac{\prod_{s \in U_z} \Gamma \left( \frac{a_s - z}{n_s} \right) n_s^{a_s - z - \frac{1}{2}}}{\prod_{t \in V_z} \Gamma \left( \frac{b_t - z}{m_t} \right) m_t^{b_t - z - \frac{1}{2}}} = (2\pi)^{k-l} \frac{\prod_{s \in U_z} \left( -1 \left\{ \frac{a_s}{n_s} \right\} \right) n_s^{a_s} \left( \frac{a_s}{n_s} \right)^{-\frac{1}{2}}}{\prod_{t \in V_z} \left( -1 \left\{ \frac{b_t}{m_t} \right\} \right) m_t^{b_t} \left( \frac{b_t}{m_t} \right)^{-\frac{1}{2}}} \quad \text{for } z \in \mathbb{C}
\]
where \( U_z = \{ 1 \leq s \leq k : z \in a_s + n s \mathbb{N} \} \) and \( V_z = \{ 1 \leq t \leq l : z \in b_t + m t \mathbb{N} \} \);

\[
\prod_{s=1}^{k} F\left( \frac{u}{n_s}, \frac{v}{n_s}, \frac{w+a_s}{n_s}, 1 \right) = \prod_{t=1}^{l} F\left( \frac{u}{m_t}, \frac{v}{m_t}, \frac{w+b_t}{m_t}, 1 \right)
\]

for \( u, v, w \in \mathbb{C} \) with \( \text{Re}(w) > \text{Re}(u+v) \) and \( w, w-u, w-v \not\in -\mathbb{N} \),

where \( F(\alpha, \beta, \gamma, z) \) \((\text{Re}(\gamma) > \text{Re}(\alpha+\beta) \text{ and } \gamma \not\in -\mathbb{N})\) denotes the hypergeometric series given by

\[
F(\alpha, \beta, \gamma, z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1) \cdots (\alpha+n-1) \beta(\beta+1) \cdots (\beta+n-1)}{n! (\gamma+1) \cdots (\gamma+n-1)} z^n
\]

which converges absolutely for \(|z| \leq 1\);

\[
\sum_{i=1}^{k} n_i^{-s} \zeta\left( s, \frac{z+a_i}{n_i} \right) = \sum_{j=1}^{l} m_j^{-s} \zeta\left( s, \frac{z+b_j}{m_j} \right) \text{ for all } z \in \mathbb{C} \setminus (-\infty, 0]
\]

where \( \zeta(s, v) \) is the well-known Hurwitz zeta function and \( s \) is a complex number different from \( 1, 0, -1, -2, \cdots \).


In 1950 P. Erdős proved that if \( x \equiv 2036812 \) (mod 5592405) and \( x \equiv 3 \) (mod 62) then \( x \) is not of the form \( 2^n + p \) where \( n \) is a nonnegative integer and \( p \) is a prime.

Our main result in this note is as follows:

**Theorem.** Let \( A = \{ a_s(n_s) \}_{s=1}^{k} \) be a minimal cover with \( 0 \leq a_s < n_s \) for \( s = 1, \cdots, k \). Suppose that distinct primes \( p_1, \cdots, p_k \) are primitive divisors of \( 2^{n_1} - 1, \cdots, 2^{n_k} - 1 \) respectively. Put \( \bigcap_{s=1}^{k} 2^{a_s} \) \((\text{mod } p_s) = a(\text{mod } d) \) where \( a \in \mathbb{Z} \) and \( d = p_1 \cdots p_k \), and write

\[
\left( a_t(\text{mod } n_t) \setminus \bigcup_{s=1}^{k} a_s(\text{mod } n_s) \right) \cap \{ 0, 1, \cdots, N - 1 \} = \{ b_1^{(t)}, \cdots, b_{l_t}^{(t)} \}
\]

for \( t = 1, \cdots, k \) where \( N \) is the least common multiple \([n_1, \cdots, n_k] \) of the moduli \( n_1, \cdots, n_k \). Set

\[
S(A) = \bigcup_{t=1}^{k} \bigcup_{j=1}^{l_t} \frac{a - 2^{b_j^{(t)}}}{p_t} \left( \text{mod } \frac{d}{p_t} \right)
\]
where all the \((a - 2^j^{(i)})/p_i\) are integers. Then an integer \(c\) divisible by none of \(p_1, \cdots, p_k\) belongs to \(S(A)\) if and only if \(a \pmod{d}\) contains integers of the form \(2^n + cp\) where \(n \geq 0\) is an integer and \(p\) is a prime.

It follows from the theorem that for any integer \(c \in [-3150, 20054]\) divisible by none of \(3, 5, 7, 13, 17, 241\), the residue class \(20036812 \pmod{5592405}\) contains no integers of the form \(2^n + cp\) where \(n \geq 0\) is an integer and \(p\) is a prime.


Covers of \(\mathbb{Z}\) by finitely many residue classes have been investigated for many years. The Newman-Znám result asserts that if \(\{a_s \pmod{n_s}\}_{s=1}^k\) forms a disjoint cover of \(\mathbb{Z}\) with \(1 < n_1 \leq \cdots \leq n_{k-1} < n_k\) then \(n_{k-p+1} = \cdots = n_{k-1} = n_k\) where \(p\) is the least prime dividing \(n_k\). In this paper we study covers of \(\mathbb{Z}^n\) in the form \(A = \{\bar{a}_s(\bar{m}_s)\}_{s=1}^k\) where \(\bar{a}_s(\bar{m}_s) = \{\bar{x} = (x_1, \cdots, x_n) : x_t \equiv a_{st} \pmod{m_{st}}\}\) for \(t = 1, \cdots, n\). Some classical results on covers of \(\mathbb{Z}\) are generalized. In particular, we show that if each \(\bar{x} \in \mathbb{Z}^n\) is covered by \(A\) exactly \(m\) times and the moduli \(\bar{m}_s\) are not all equal, then any maximal modulus \(\bar{m}_r\) with respect to divisibility \((\bar{m}_r | \bar{m}_s) \iff \bar{m}_s = \bar{m}_r\) occurs at least \(p\) times among \(\bar{m}_1, \cdots, \bar{m}_k\) where \(p\) is the smallest prime divisor of \(m_{r_1} \cdots m_{r_n}\).

24. Zhi-Wei Sun, *On the function \(w(x) = |\{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}|\)*, Combinatorica **23** (2003), no. 4, 681–691. MR 2004m:11013; Zbl. M. 1047.11014.

The main results of the paper are included in the following theorem.

**Theorem.** Let \(A = \{a_s(n_s)\}_{s=1}^k\) and \(n_0 \in \mathbb{Z}^+\) be a period of the covering function \(w_A(x)\). Then we have the following results:

(a) For each \(r \in R(n_k/(n_0, n_k))\) there exists a \(J \subseteq \{1, \cdots, k - 1\}\) such that \(\sum_{s \in J} 1/n_s = r/n_k\).

(b) If \(n_1 \leq \cdots \leq n_{k-l} < n_{k-l+1} = \cdots = n_k\ (0 < l < k)\), then for any positive integer \(r < n_k/n_{k-l}\) with \(r \not\equiv 0 \pmod{n_k/(n_0, n_k)}\), the binomial coefficient \(\binom{l}{r}\) can be written as the sum of some (not necessarily distinct) prime divisors of \(n_k\).

(c) \(M(A) = \max_{x \in \mathbb{Z}} w_A(x)\) can be written in the form \(\sum_{s=1}^k m_s/n_s\) where \(m_1, \cdots, m_k \in \mathbb{Z}^+\).


In the early 1930s P. Erdös initiated the study of covers of \(\mathbb{Z}\) by finitely many residue classes, he posed the famous question whether for any arbitrarily large
\(c > 0\) there exists a cover \(A = \{a_s(n_s)\}_{s=1}^k\) of \(Z\) with all the moduli \(n_1, \ldots, n_k\) distinct and greater than \(c\). In the 1950s B. H. Neumann investigated covers of general groups by cosets, in 1974 M. Herzog and J. Schönheim conjectured that if \(A = \{a_i G_i\}_{i=1}^k\) forms a partition of a group \(G\) into \(k > 1\) left cosets then at least two of the (finite) indices \([G : G_1], \ldots, [G : G_k]\) are equal.

The main result of this paper can be summarized as follows.

**Theorem.** Let \(A = \{a_i G_i\}_{i=1}^k\) \(([G : G_1] \leq \cdots \leq [G : G_k]\) be a uniform cover of a group \(G\) by left cosets (i.e. it covers every element of \(G\) with the same multiplicity). Assume that \(G_1, \ldots, G_k\) are all subnormal in \(G\) and not all equal to \(G\). Then the indices \([G : G_1], \ldots, [G : G_k]\) cannot be distinct. Moreover, if each of the indices occurs at most \(M > 1\) times then we have the following (i)–(iii) where \(\gamma\) is the Euler constant and the \(O\)-constants are absolute.

(i) \(M\) is greater than or equal to the smallest prime divisor of the indices \([G : G_1], \ldots, [G : G_k]\).

(ii) All prime divisors of \([G : G_1], \ldots, [G : G_k]\) are smaller than \(e^{\gamma} M \log M + O(M \log \log M)\).

(iii) The number of distinct prime divisors of \([G : G_1], \ldots, [G : G_k]\) does not exceed \(e^{\gamma} M + O(M/\log M)\).

We emphasize that, for uniform covers of groups by cosets of subnormal subgroups, the above theorem confirms the generalized Herzog–Schönheim conjecture and answers the analogous question of Erdös negatively!


We introduce two main results in the paper.

**Theorem 1.1.** Let \(F\) be a field of characteristic \(p\) where \(p\) is zero or a prime. Let \(n_1, \ldots, n_k\) be positive integers not divisible by \(p\), and let \(\psi_1, \ldots, \psi_k\) be maps from \(Z\) to \(F\) with periods \(n_1, \ldots, n_k\) respectively. Then \(\psi_1 + \cdots + \psi_k = 0\) if \(\psi_1(x) + \cdots + \psi_k(x) = 0\) for \(\sum_{d \in D} \varphi(d)\) consecutive integers \(x\), where \(\varphi\) is Euler’s totient function, \(D = \bigcup_{s=1}^k D(n_s)\), and \(D(n)\) denotes the set of positive divisors of \(n \in Z^+\). (Actually \(\sum_{d \in D} \varphi(d)\) also equals \(\lvert \bigcup_{s=1}^k \{r/n_s : r = 0, \ldots, n_s - 1\}\rvert\)).

By this local-global theorem, if \(A = \{a_s(n_s)\}_{s=1}^k\) (where \(a_s \in Z, n_s \in Z^+\) and \(a_s(n_s) = a_s + n_sZ\)) covers \(\{r/n_s : r = 0, 1, \ldots, n_s - 1; s = 1, \ldots, k\}\) consecutive integers exactly \(m\) times then it covers all the integers exactly \(m\) times.

The following result characterizes the least positive period of the covering function of a finite system of residue classes with weights.
**Theorem 1.3.** Let \( \lambda_s \in \mathbb{C} \), \( a_s \in \mathbb{Z} \) and \( n_s \in \mathbb{Z}^+ \) for \( s = 1, \ldots, k \). Then the smallest positive period \( n_0 \) of the (weighted) covering function

\[
 w(x) = \sum_{1 \leq s \leq k} \lambda_s \quad \text{for} \quad x \in a_s(n_s)
\]

is the least \( n \in \mathbb{Z}^+ \) such that \( \alpha n \in \mathbb{Z} \) for all those \( \alpha \in [0, 1) \) with

\[
 \sum_{1 \leq s \leq k} \frac{\lambda_s e^{2\pi i \alpha a_s}}{n_s} \neq 0.
\]


Let \( \{a_s \pmod{n_s}\}_{s=1}^k \) be a finite system of residue classes with the moduli \( n_1, \ldots, n_k \) distinct. By means of algebraic integers we show that the range of the covering function \( w(x) = |\{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}| \) is not contained in any residue class with modulus greater one. In particular, the values of \( w(x) \) cannot have the same parity.


Suppose that \( A = \{a_s \pmod{n_s}\}_{s=1}^k \) covers all the integers at least \( m \) times with \( a_k \pmod{n_k} \) irredudant. We show that if \( n_k \) is a period of the covering function \( w_A(x) = |\{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}| \) then for any \( r = 0, 1, \ldots, n_k - 1 \) there are at least \( m \) integers in the form \( \sum_{s \in I} 1/n_s - r/n_k \) with \( I \subseteq \{1, \ldots, k-1\} \).