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Supercongruences Motivated by e

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Abstract

In this talk we introduce the speaker's recent work [J. Number Theory 147(2015), 326-341] on supercongruences motivated by the well-known fact $\lim_{n\to\infty}(1+1/n)^n = e$. Let $p > 3$ be a prime. We prove that

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv \frac{2}{3}p^4 B_{p-3} \pmod{p^5},$$

and

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv \frac{p^5}{18}B_{p-3} \pmod{p^6} \ \text{ if } p > 5,$$

where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. We also show the new congruence

$$\sum_{k=1}^{p-1}\frac{(k+1)^k}{k^{k+1}} \equiv -1 \pmod{p}.$$

In addition, we mention the speaker's recent discoveries about $\zeta(3)$ and $L(2,(\cdot/3))$ motivated by congruences.

Supercongruences

A p-adic congruence (with p prime) is called a *supercongruence* if it happens to hold modulo higher powers of p. Here is a classical example due to J. Wolstenholme:

$$\sum_{k=1}^{p-1}\frac{1}{k}\equiv 0 \pmod{p^2} \text{ and } \binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$$

for every prime $p>3$. In 1900 Glaisher [G1, G2] showed further that

$$\binom{2p-1}{p-1}\equiv 1-\frac{2}{3}p^3B_{p-3} \pmod{p^4}$$

for any prime $p>3$, where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. In this talk we introduce some new supercongruences modulo prime powers motivated by the well-known formula

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e.$$

Main Results

Theorem 1. Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1} \binom{-1/(p+1)}{k}^{p+1} \equiv 0 \pmod{p^5}.$$

Moreover, if $p > 5$ then

$$\sum_{k=0}^{p-1} \binom{-1/(p+1)}{k}^{p+1} \equiv \frac{p^5}{18} B_{p-3} \pmod{p^6}.$$

Theorem 2. Let $p > 3$ be a prime and let m be an integer not divisible by p. Then we have

$$\sum_{k=0}^{p-1} (-1)^{km} \binom{p/m-1}{k}^{m} \equiv \frac{(m-1)(7m-5)}{36m^2} p^4 B_{p-3} \pmod{p^5}.$$

In particular,

$$\sum_{k=0}^{p-1} \binom{1/(p-1)}{k}^{p-1} \equiv \frac{2}{3} p^4 B_{p-3} \pmod{p^5}.$$

Main results (continued)

Theorem 3. Let $p > 3$ be a prime and let m be an integer not divisible by p.
(i) If $p > 5$, then

$$\sum_{k=1}^{p-1} \frac{(-1)^{km}}{k^2} \binom{p/m - 1}{k}^m \equiv \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{k} \pmod{p^3}.$$

Also, for any $n = 1, \ldots, (p-3)/2$ we have

$$\sum_{k=1}^{p-1} \frac{(-1)^{km}}{k^{2n}} \binom{p/m - 1}{k}^m \equiv -\frac{p}{2n+1} B_{p-1-2n} \pmod{p^2}.$$

(ii) For every $n = 1, \ldots, (p-3)/2$, we have

$$\sum_{k=1}^{p-1} \frac{(-1)^{km}}{k^{2n-1}} \binom{p/m - 1}{k}^m$$
$$\equiv \left(1 + \frac{1-m}{2m}(2n+1)\right) \frac{p^2 n}{2n+1} B_{p-1-2n} \pmod{p^3}.$$

Main results (continued)

Remark. If n is a positive integer and $p > 2n+1$ is a prime, then the last congruences with $m = p \pm 1$ yields the congruences

$$\sum_{k=1}^{p-1} \frac{1}{k^{2n-1}} \binom{1/(p-1)}{k}^{p-1} \equiv -\frac{2p^2n^2}{2n+1} B_{p-1-2n} \pmod{p^3}$$

and

$$\sum_{k=1}^{p-1} \frac{1}{k^{2n-1}} \binom{-1/(p+1)}{k}^{p+1} \equiv \frac{p^2n}{2n+1} B_{p-1-2n} \pmod{p^3}.$$

Theorem 4. Let p be an odd prime and let $a \in \mathbb{Z}$ with $p \nmid a$. Then

$$\sum_{k=1}^{p-1} \frac{1}{k} \left(1 + \frac{a}{k}\right)^k \equiv -1 \pmod{p}.$$

If $p > 3$, then

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \left(1 + \frac{a}{k}\right)^k \equiv 1 + \frac{1}{2a} \pmod{p}.$$

Lemma 1

For $m = 1, 2, 3, \ldots$ and $n = 0, 1, 2, \ldots$, we define

$$H_n^{(m)} := \sum_{0<k\leqslant n} \frac{1}{k^m}$$

and call it a harmonic number of order m. Those $H_n = H_n^{(1)}$ $(n = 0, 1, 2, \ldots)$ are usually called *harmonic numbers*.

Lemma 1. Let $p > 3$ be a prime. Then

$$H_{p-1} \equiv -\frac{p^2}{3}B_{p-3} \pmod{p^3} \text{ and } H_{p-1}^{(2)} \equiv \frac{2}{3}pB_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{p-1} H_k^{(2)} \equiv p^2 B_{p-3} \pmod{p^3}, \quad \sum_{k=1}^{p-1} H_k^{(3)} \equiv -\frac{2}{3}pB_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{p-1} H_k^{(4)} \equiv H_{p-1}^{(3)} \equiv 0 \pmod{p}.$$

Remark. The first two congruences are due to Glaisher.

Lemma 2

Lemma 2. Let $p > 3$ be a prime. Set

$$\Sigma_1 = \sum_{k=1}^{p-1} \sum_{1 \leqslant i < j \leqslant k} \left(\frac{1}{ij^2} + \frac{1}{i^2 j} \right), \quad \Sigma_2 = \sum_{k=1}^{p-1} \sum_{1 \leqslant i < j \leqslant k} \frac{1}{i^2 j^2},$$

$$\Sigma_3 = \sum_{k=1}^{p-1} \sum_{1 \leqslant i < j \leqslant k} \left(\frac{1}{ij^3} + \frac{1}{i^3 j} \right) \quad \text{and} \quad \Sigma_4 = \sum_{k=1}^{p-1} \sum_{1 \leqslant i < j \leqslant k} \frac{H_k^{(2)}}{ij}.$$

Then we have

$$\Sigma_1 \equiv pB_{p-3} \pmod{p^2}, \qquad \Sigma_2 \equiv B_{p-3} \pmod{p}$$

and

$$\Sigma_3 \equiv \Sigma_4 \equiv -B_{p-3} \pmod{p}.$$

Proof of $\Sigma_3 \equiv -B_{p-3} \pmod{p}$.

Recall the congruence $\sum_{k=1}^{p-1} H_k/k^2 \equiv B_{p-3} \pmod{p}$ (due to Sun and Tauraso [Adv. Appl. Math., 2010]). Note that

$$\Sigma_3 = \sum_{1\leqslant i<j\leqslant p-1} \left(\frac{1}{ij^3}+\frac{1}{i^3j}\right)\sum_{k=j}^{p-1} 1 = \sum_{1\leqslant i<j\leqslant p-1} \left(\frac{1}{ij^3}+\frac{1}{i^3j}\right)(p-j)$$

$$\equiv -\sum_{1\leqslant i<j\leqslant p-1} \left(\frac{1}{ij^2}+\frac{1}{i^3}\right) = -\sum_{j=1}^{p-1}\frac{H_j-1/j}{j^2} - \sum_{i=1}^{p-1}\frac{p-1-i}{i^3}$$

$$\equiv -\sum_{k=1}^{p-1}\frac{H_k}{k^2} + 2H_{p-1}^{(3)} + H_{p-1}^{(2)} \equiv -B_{p-3} \pmod{p}.$$

Proof of $\Sigma_4 \equiv -B_{p-3} \pmod{p}$

$$\Sigma_4 = \sum_{1\leqslant i<j\leqslant p-1} \frac{1}{ij}\left(\sum_{k=1}^{p-1} H_k^{(2)} - \sum_{s=1}^{j-1} H_s^{(2)}\right)$$

$$\equiv - \sum_{1\leqslant i<j\leqslant p-1} \frac{1}{ij} \sum_{1\leqslant t\leqslant s<j} \frac{1}{t^2} = - \sum_{1\leqslant i<j\leqslant p-1} \frac{1}{ij} \sum_{t=1}^{j} \frac{j-t}{t^2}$$

$$= - \sum_{j=1}^{p-1} H_{j-1} H_j^{(2)} + \sum_{j=1}^{p-1} \frac{H_j}{j}\left(H_j - \frac{1}{j}\right) \pmod{p^2}.$$

For every $k = 1, \ldots, p-1$, we have

$$H_{p-k} = H_{p-1} - \sum_{0<j<k} \frac{1}{p-j} \equiv H_{k-1} \pmod{p},$$

$$H_{p-k}^{(2)} = H_{p-1}^{(2)} - \sum_{0<j<k} \frac{1}{(p-j)^2} \equiv -H_{k-1}^{(2)} \pmod{p}.$$

Proof of $\Sigma_4 \equiv -B_{p-3} \pmod{p}$ (continued)

$$\sum_{k=1}^{p-1} H_{k-1} H_k^{(2)} \equiv \sum_{k=1}^{p-1} H_{p-k} H_k^{(2)} = \sum_{k=1}^{p-1} H_k H_{p-k}^{(2)}$$

$$\equiv -\sum_{k=1}^{p-1} H_k H_{k-1}^{(2)} = -\sum_{k=1}^{p-1} H_k \left(H_k^{(2)} - \frac{1}{k^2} \right) \pmod{p}$$

and hence

$$\Sigma_4 \equiv \sum_{k=1}^{p-1} H_k H_k^{(2)} - 2\sum_{k=1}^{p-1} \frac{H_k}{k^2} + \sum_{k=1}^{p-1} \frac{H_k^2}{k} \pmod{p}.$$

Note that

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} = \sum_{k=1}^{p-1} \frac{H_{p-k}^2}{p-k}$$

$$\equiv -\sum_{k=1}^{p-1} \frac{1}{k} \left(H_k - \frac{1}{k} \right)^2 = -\sum_{k=1}^{p-1} \frac{H_k^2}{k} + 2\sum_{k=1}^{p-1} \frac{H_k}{k^2} - H_{p-1}^{(3)} \pmod{p}.$$

Proof of $\Sigma_4 \equiv -B_{p-3} \pmod p$.

So we have

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k} \equiv \sum_{k=1}^{p-1} \frac{H_k}{k^2} \equiv B_{p-3} \pmod p.$$

Since

$$\sum_{k=1}^{p-1} H_k H_k^{(2)} \equiv 0 \pmod p$$

(cf. [Sun, Proc. Amer. Math. Soc. 2012]), we obtain

$$\Sigma_4 \equiv -B_{p-3} \pmod p.$$

This concludes the proof.

Proof of Theorem 1

For any p-adic integer x, we can write $(1+px)^m$ as the p-adic series $\sum_{n=0}^{\infty}\binom{m}{n}p^n x^n$. Thus, for each $k = 1, \ldots, p-1$, we have

$$\binom{-1/(p+1)}{k}^{p+1} = \binom{p/(p+1)-1}{k}^{p+1} = \prod_{j=1}^{k}\left(1-\frac{p}{(p+1)j}\right)^{p+1}$$

$$\equiv \prod_{j=1}^{k}\left(1-\frac{p}{j}+\sum_{r=2}^{4}(-1)^r\binom{p+1}{r}\frac{p^r}{(p+1)^r j^r}\right)$$

$$= \prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{p^3}{2(p+1)j^2}-\frac{p^4(p-1)}{6(p+1)^2 j^3}+\frac{p^5(p-1)(p-2)}{24(p+1)^3 j^4}\right)$$

$$\equiv \prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{p^3(p^2-p+1)}{2j^2}-\frac{p^4}{6j^3}(p-1)(1-2p)+\frac{2p^5}{24j^4}\right)$$

$$\equiv \prod_{j=1}^{k}\left(1-\frac{p}{j}+\frac{p^5-p^4+p^3}{2j^2}+\frac{p^4-3p^5}{6j^3}+\frac{p^5}{12j^4}\right) \pmod{p^6}.$$

Proof of Theorem 1 (continued)

Hence

$$\begin{aligned}&\binom{-1/(p+1)}{k}^{p+1}-\prod_{j=1}^{k}\left(1-\frac{p}{j}\right)\\ \equiv&\frac{p^5-p^4+p^3}{2}H_k^{(2)}+\frac{p^4-3p^5}{6}H_k^{(3)}+\frac{p^5}{12}H_k^{(4)}\\ &+\frac{p(p^4-p^3)}{2}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^2}+\frac{1}{i^2j}\right)-\frac{p^5}{6}\sum_{1\leqslant i<j\leqslant k}\left(\frac{1}{ij^3}+\frac{1}{i^3j}\right)\\ &+\frac{p^5}{2}\sum_{1\leqslant i_1<i_2\leqslant k}\frac{1}{i_1i_2}\left(\sum_{j=1}^{k}\frac{1}{j^2}-\frac{1}{i_1^2}-\frac{1}{i_2^2}\right)\pmod{p^6}.\end{aligned}$$

Proof of Theorem 1 (continued)

Thus, in view of Lemmas 1 and 2, we obtain

$$\begin{aligned}
&\sum_{k=1}^{p-1}\binom{-1/(p+1)}{k}^{p+1}-\sum_{k=1}^{p-1}(-1)^k\binom{p-1}{k}\\
\equiv&\frac{p^5-p^4+p^3}{2}p^2B_{p-3}+\frac{p^4-3p^5}{6}\left(-\frac{2}{3}pB_{p-3}\right)+\frac{p^5}{12}\times 0\\
&+\frac{p^5-p^4}{2}pB_{p-3}-\frac{p^5}{6}(-B_{p-3})+\frac{p^5}{2}(-B_{p-3}-(-B_{p-3}))\\
\equiv&\frac{p^5}{18}B_{p-3} \pmod{p^6}
\end{aligned}$$

and hence the desired result follows since

$$\sum_{k=0}^{p-1}(-1)^k\binom{p-1}{k}=(1-1)^{p-1}=0.$$

Lemma 3

Lemma 3 Let $n > m \geqslant 0$ be integers. Then

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k k^m = 0.$$

This is a known identity in combinatorics. Write

$$k^m = \sum_{j=0}^{m} S(m,j)(k)_j = \sum_{j=0}^{m} c_j \binom{k}{j},$$

where $S(m,j)$ ($j = 0, \dots, m$) are Stirling numbers of the second kind, and $(x)_j = \prod_{0 \leqslant i < j}(x - i)$. Then

$$\begin{aligned}\sum_{k=0}^{n} \binom{n}{k} (-1) k^m =& \sum_{k=0}^{n} \binom{n}{k} (-1)^k \sum_{j=0}^{m} c_j \binom{k}{j} \\ =& \sum_{j=0}^{m} c_j \binom{n}{j} (-1)^j \sum_{k=j}^{n} \binom{n-j}{k-j} (-1)^{k-j} \\ =& 0.\end{aligned}$$

Lemma 4

Lemma 4. Let p be an odd prime. Then, for any positive integers d and r with $d + r < p$, we have $\sum_{k=r}^{p-1} \binom{k}{r}/k^{r+d} \equiv 0 \pmod{p}$.

Proof. Observe that

$$\begin{aligned}
\sum_{k=r}^{p-1} \binom{k}{r} k^{-r-d} &= \sum_{s=0}^{p-1-r} \binom{r+s}{s} (r+s)^{-r-d} \\
&= \sum_{s=0}^{p-1-r} (-1)^s \binom{-r-1}{s} (r+s)^{-r-d} \\
&\equiv \sum_{s=0}^{p-1-r} \binom{p-1-r}{s} (-1)^s (s-1-(p-1-r))^{p-1-r-d} \\
&= \sum_{k=0}^{p-1-r} \binom{p-1-r}{k} (-1)^{p-1-r-k} (-1-k)^{p-1-r-d} \\
&= 0 \pmod{p}
\end{aligned}$$

with the help of Lemma 3.

Proof of Theorem 4

Let $d \in \{1, 2\}$. With the help of Lemma 4, we have

$$\begin{aligned}
\sum_{k=1}^{p-1} \frac{1}{k^d}\left(1+\frac{a}{k}\right)^k &= \sum_{k=1}^{p-1} \frac{(k+a)^k}{k^{k+d}} \\
&= \sum_{k=1}^{p-1} \frac{1}{k^{k+d}}\left(k^k + \sum_{r=1}^{k}\binom{k}{r}k^{k-r}a^r\right) \\
&= \sum_{k=1}^{p-1} \frac{1}{k^d} + \sum_{r=1}^{p-1} a^r \sum_{k=r}^{p-1} \frac{\binom{k}{r}}{k^{r+d}} \\
&\equiv \sum_{k=1}^{p-1} \frac{1}{k^d} + \sum_{r=p-d}^{p-1} a^r \sum_{k=r}^{p-1} \frac{\binom{k}{r}}{k^{r+d}} \pmod{p}.
\end{aligned}$$

Thus

$$\sum_{k=1}^{p-1} \frac{1}{k}\left(1+\frac{a}{k}\right)^k \equiv \sum_{k=1}^{(p-1)/2}\left(\frac{1}{k}+\frac{1}{p-k}\right) + \frac{a^{p-1}}{(p-1)^p} \equiv -1 \pmod{p}$$

Proof of Theorem 4 (continued)

When $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p},$$

and hence

$$\begin{aligned}
&\sum_{k=1}^{p-1} \frac{1}{k^2} \left(1 + \frac{a}{k}\right)^k \\
\equiv &\sum_{k=1}^{p-1} \frac{1}{k^2} + \frac{a^{p-1}}{(p-1)^{p+1}} + a^{p-2}\left(\frac{\binom{p-2}{p-2}}{(p-2)^p} + \frac{\binom{p-1}{p-2}}{(p-1)^p}\right) \\
\equiv &0 + 1 + \frac{1}{a}\left(\frac{1}{-2} + 1\right) = 1 + \frac{1}{2a} \pmod{p}
\end{aligned}$$

as desired. This concludes the proof.

Fundamental congruences for harmonic numbers

Z. W. Sun [Proc. AMS 140(2012)] Let $p > 5$ be a prime. Then

$$\sum_{k=1}^{p-1} H_k^2 \equiv 2p-2 \pmod{p^2},$$

$$\sum_{k=1}^{p-1} H_k^3 \equiv 6 \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}.$$

Remark. It is known that

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17\pi^4}{360}.$$

Further congruences involving harmonic numbers

Sun and Zhao [Colloq. Math. 2013]. Let $p > 5$ be a prime. Then

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \frac{7}{24} pB_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k2^k} \equiv -\frac{3}{8} B_{p-3} \pmod{p} \quad \left(\text{in contrast, } \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k2^k} = \frac{5}{8}\zeta(3)\right).$$

If $p > 6n + 1$ then

$$\sum_{k=1}^{p-1} \frac{(H_k^{(2n)})^2}{k^{2n}} \equiv \frac{\binom{6n+1}{2n-1} + n}{6n+1} pB_{p-1-6n} \quad (\text{mod } p^2).$$

One more congruence:

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv \frac{4}{5} pB_{p-5} \pmod{p^2}$$

[Conjectured by Sun and proved by R. Meštrović (IJNT, 2012)].

Apéry's series and its p-adic analogues

Apéry used the following series to prove the irrationality of $\zeta(3)$:

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k^3\binom{2k}{k}}=-\frac{2}{5}\zeta(3).$$

This was obtained by letting $n\to+\infty$ in the following identity:

$$5\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}}=\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{n}{k}\binom{n+k}{k}}-2H_n^{(3)}.$$

Taking $n=p-1$ Tauraso [JNT 130(2010)] got the following p-adic analogue for any prime $p>5$.

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^3\binom{2k}{k}}\equiv-\frac{2}{5}\cdot\frac{H_{p-1}}{p^2}\pmod{p^3}\equiv\frac{2}{15}B_{p-3}\pmod{p^2}.$$

Z.W. Sun [JNT 134(2014)]. For any prime $p>5$, we have

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^3\binom{2k}{k}}\equiv-2B_{p-3}\pmod{p}.$$

Two new formulas for $\zeta(3)$

Motivated by related congruences, on October 22, 2014 I found the following new formula for $\zeta(3)$:

$$\zeta(3) = 3\sum_{k=1}^{\infty} \frac{H_k}{k^2\binom{2k}{k}} - \sum_{k=1}^{\infty} \frac{1}{k^3\binom{2k}{k}}.$$

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Conjecture (Sun, Oct. 22, 2014). We have

$$\sum_{k=1}^{\infty} \frac{H_{2k}+2H_k}{k^2\binom{2k}{k}} = \frac{5}{3}\zeta(3).$$

Also, for any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{H_{2k}+2H_k}{k^2\binom{2k}{k}} \equiv -\frac{8}{3}\cdot\frac{H_{p-1}}{p^2} - \frac{17}{30}p^2 B_{p-5} \pmod{p^3}.$$

Related congruences

The formula in the last conjecture has the following equivalent version:

$$\sum_{k=1}^{\infty} \frac{H_{2k} + 2/(3k)}{k^2\binom{2k}{k}} = \zeta(3).$$

Conjecture (Z.-W. Sun, Oct. 27, 2014). For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{3H_{2k} + 2/k}{k^2\binom{2k}{k}} \equiv -4\frac{H_{p-1}}{p^2} - \frac{9}{10}p^2 B_{p-5} \pmod{p^3}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{3H_{2k} + 2/k}{k^2\binom{2k}{k}} \equiv B_{p-3} \pmod{p}.$$

Remark. It is known that $H_{p-1}/p^2 \equiv -B_{p-3}/3 \pmod{p}$ for any prime $p > 3$.

A new series for $L(2,(\cdot/3))$ and related congruences

Conjecture (Z.-W. Sun, 2009). For any prime $p > 3$, we have

$$\sum_{k=1}^{p-1} \frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k} \equiv 0 \pmod{p^2}.$$

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Conjecture (i) (Sun, July 9, 2014). For any prime $p>3$, we have

$$\sum_{k=1}^{p-1}\frac{\binom{4k}{2k+1}\binom{2k}{k}}{48^k}\equiv \frac{5}{12}p^2B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

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(ii) (August 12, 2014) We have

$$\sum_{k=1}^{\infty}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}=\frac{15}{2}\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2}.$$

Also, for any prime $p>3$ we have

$$p^2\sum_{k=1}^{p-1}\frac{48^k}{k(2k-1)\binom{4k}{2k}\binom{2k}{k}}\equiv 4\left(\frac{p}{3}\right)+4p \pmod{p^2}.$$

Thank you!