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EQUALITIES AND INEQUALITIES RELATED TO COVERS OF $\mathbb{Z}$ OR GROUPS

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Abstract. This is an introduction to equalities and inequalities related to coverings of the integers by residue classes and covers of groups by cosets or subgroups. The field is connected with number theory, combinatorics, algebra and analysis.

1. The Davenport-Mirsky-Newman-Rado result and its generalizations

Let $\mathbb{N} = \{0, 1, 2, \cdots\}$ and $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$. For $a \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$ we let

$$a(n) = a + n\mathbb{Z} = \{x \in \mathbb{Z} : \ x \equiv a \pmod{n}\} = \{\cdots, a-n, a, a+n, \cdots\}.$$

It will be called a residue class with modulus n, or an arithmetic sequence with difference n. Note that $0(1) = \mathbb{Z}$ is considered as a residue class with modulus 1.

A finite system

$$A = \{a_s(n_s)\}_{s=1}^{k} \tag{1.1}$$

of such sets is said to be a *cover* or *covering system* (CS) (of $\mathbb{Z}$) if each integer lies in at least one of the classes in (1.1).

The concept of cover was first introduced by P. Erdös in 1934 when he answered a question of Romanoff.

Erdös [1950, Summa Brasil. Math.; MR 13,437]: *There is an infinite arithmetic progression of odd integers no term of which is of the form $2^k + p$, where k is a positive integer and p is an odd prime.*

Here are some other important applications of covers of $\mathbb{Z}$.

F. Cohen and J.L. Selfrige [1975, Math. Comput.; MR 51#12758], Z. W. Sun [2000, Proc. Amer. Math. Soc.]: *The residue class*

$$47867742232066880047611079(66483084961588510124010691590)$$

contains no integers of the form $\pm p^a \pm q^b$ *where* p, q *are primes and* $a, b \in \mathbb{N}$.

R. Crocker [1971, Pacific J. Math.]: *There are infinitely many positive odd integers not of the form* $2^u + 2^v + p$ $(u, v \geq 1$, p *odd prime).*

J.L. Selfrige [R.K. Guy, 1981, Unsolved Problems in Number Theory]: *One of* $3, 5, 7, 13, 17, 19, 73$ *always divides* $78557 \cdot 2^n + 1$.

L.J. Stockmeyer and A.R. Meyer [1973, Proc. 5th. Ann. ACM Symp. on Theory of Computing, Assoc. for Computing Machinery]: *The question whether a given* $A = \{a_s(n_s)\}_{s=1}^k$ *is a cover is co-NP- complete.*

The most famous open problem in this area is the following question of P. Erdös.

Erdös' Question [$1000 for a solution]: *Whether for any* $c > 0$ *there always exists a cover of* $\mathbb{Z}$

$$A = \{a_s(n_s)\}_{s=1}^k \quad (1 < n_1 < n_2 < \cdots < n_k) \tag{1.2}$$

with $n_1 > c$*?*

Erdös [1952, Mat. Lapok; MR 17,14]: *The positive solution of the above problem implies that for every* $r \geq 1$ *there exists an infinite arithmetic progression of positive odd integers no term of which is of the form* $2^k + \theta_r$ *where* θ_r *has at most* r *distinct prime factors.*

Examples of covers in the form (1.2):

Erdös: $\{0(2),\ 0(3),\ 1(4),\ 5(6),\ 7(12)\}$;

$$\{0(3), 0(4), 0(5), 1(6), 6(8), 3(10), 5(12), 11(15),$$
$$7(20), 10(24), 2(30), 34(40), 59(60), 98(120)\}.$$

R. Morikawa [1981, Bull. Fac. Lib. Arts; MR 84j:10064]: *There is a cover of* $\mathbb{Z}$ *with all the moduli distinct and the smallest moduli being 24.*

Let N be the least common multiple $[n_1, \cdots, n_k]$ of the moduli $n_1, \cdots, n_k$. Observe that

$$\begin{aligned} &\left|\left\{0 \leqslant x < N:\ x \in \bigcup_{s=1}^k a_s(n_s)\right\}\right| \\ \leqslant &\sum_{s=1}^k |\{0 \leqslant x < N:\ x \in a_s(n_s)\}| = \sum_{s=1}^k \frac{N}{n_s}. \end{aligned}$$

So, when (1.1) forms a cover we have the inequality

$$\sum_{s=1}^k \frac{1}{n_s} \geq 1, \tag{1.3}$$

and equality holds if and only if (1.1) forms a *disjoint cover* (that is, (1.1) covers each integer exactly once).

Soon after his invention of CS, Erdös made the following conjecture.

Erdös' Conjecture: *If (1.2) is a cover of* $\mathbb{Z}$ *then* $\sum_{s=1}^{k}\frac{1}{n_s}>1$, *i.e. it covers some integer more than once.*

Davenport-Mirsky-Newman-Rado [1950's]. *Let*

$$A=\{a_s(n_s)\}_{s=1}^{k}\quad(1<n_1\leqslant\cdots\leqslant n_{k-1}\leqslant n_k)\tag{1.4}$$

be a disjoint cover of $\mathbb{Z}$. *Then* $n_{k-1}=n_k$.

Proof. Without loss of generality we let $0\leqslant a_s<n_s$ $(1\leqslant s\leqslant k)$. For $|z|<1$ we have

$$\sum_{s=1}^{k}\frac{z^{a_s}}{1-z^{n_s}}=\sum_{s=1}^{k}\sum_{q=0}^{\infty}z^{a_s+qn_s}=\sum_{n=0}^{\infty}z^n=\frac{1}{1-z}.$$

If $n_{k-1}<n_k$, then

$$\infty=\lim_{\substack{z\to e^{2\pi i/n_k}\\|z|<1}}\frac{z^{a_k}}{1-z^{n_k}}=\lim_{\substack{z\to e^{2\pi i/n_k}\\|z|<1}}\left(\frac{1}{1-z}-\sum_{s=1}^{k-1}\frac{z^{a_s}}{1-z^{n_s}}\right)<\infty,$$

which yields a contradiction! □

The Davenport-Mirsky-Newman-Rado result can be strengthened as follows:

Theorem 1.1. *Let* $A=\{a_s(n_s)\}_{s=1}^{k}$ *be a disjoint cover of* $\mathbb{Z}$ *with*

$$1<n_1\leqslant\cdots\leqslant n_{k-l}<n_{k-l+1}=\cdots=n_k\ (0<l\leqslant k).\tag{1.5}$$

Then

(i) (Š. Znám (1969) & M. Newman (1971)) $l\geq p(n_k)$ *where* $p(n_k)$ *is the least prime factor of* n_k.

(iii) (Z.W. Sun, Chin. Quart. J. Math. 1991) $l\geq\min_{1\leqslant s\leqslant k-l}\frac{n_k}{(n_s,n_k)}$.

(iv) (Y.G. Chen and Š. Porubský, 1995) *There are* $x_1,\cdots,x_{k-l}\in\mathbb{N}$ *such that*

$$l=\sum_{s=1}^{k-l}\frac{n_s}{(n_s,n_k)}x_s.\tag{1.6}$$

The following deep result also implies the Davenport-Mirsky-Newman-Rado result.

Theorem 1.2 (Z. W. Sun [Trans. Amer. Math. Soc. 348(1996)]). *Let* $A=\{a_s(n_s)\}_{s=1}^{k}$ *be a cover of* $\mathbb{Z}$ *with (1.5). Then we have*

$$l\geq\frac{n_k}{n_{k-l}}>1\ \textit{or}\ \sum_{s=1}^{k-l}\frac{1}{n_s}\geq 1.\tag{1.7}$$

In the 1950's B. H. Neumann investigated covers of general groups by cosets. A nice generalization of Erdös' conjecture is

The Herzög-Schönheim Conjecture (Canad. Math. Bull., 1974). *Let G be a group and $G_1, \cdots, G_k$ $(k > 1)$ its subgroups of distinct indices. Then, for any $a_1, \cdots, a_k \in G$ system*

$$\mathcal{A} = \{a_s G_s\}_{s=1}^k \tag{1.8}$$

cannot be a disjoint cover (*i.e. a partition*) *of G.*

Recently I [J. Algebra, in press] proved that the conjecture holds if all the k subgroups $G_1, \cdots, G_k$ are subnormal in G. The proof involves analytic number theory, combinatorics and group theory.

2. Covering functions and equalities for covers of $\mathbb{Z}$

For system $A = \{a_s(n_s)\}_{s=1}^k$ we define the *covering function* $w_A : \mathbb{Z} \to \mathbb{N}$ as follows:

$$w_A(x) = |\{1 \leqslant s \leqslant k : x \in a_s(n_s)\|. \tag{2.1}$$

Clearly w_A is periodic modulo $N = [n_1, \cdots, n_k]$. Observe that

$$\frac{1}{N}\sum_{x=0}^{N-1} w_A(x) = \sum_{s=1}^k \frac{1}{N} \sum_{\substack{0 \leqslant x < N \\ x \in a_s(n_s)}} 1 = \sum_{s=1}^k \frac{1}{N} \cdot \frac{N}{n_s} = \sum_{s=1}^k \frac{1}{n_s}.$$

If $w_A(x) \geq m$ for all $x \in \mathbb{Z}$, then we call A an m-cover; if $w_A(x) = m$ for all $x \in \mathbb{Z}$, then we call A an *exact m-cover.* For an m-cover $A = \{a_s(n_s)\}_{s=1}^k$ we have $\sum_{s=1}^k 1/n_s \geq m$, and equality holds if and only if A forms an exact m-cover.

Let $A = \{a_s(n_s)\}_{s=1}^k$ and $B = \{b_t(m_t)\}_{t=1}^l$ be two systems of residue classes. If they have the same covering function, then we say that they are *covering equivalent* and denote this by $A \sim B$.

Many known results concerning finite systems of residue classes with number weights can be expressed in the following form:

$$\{a_s(n_s)\}_{s=1}^k \sim \{b_t(m_t)\}_{t=1}^l \Rightarrow \sum_{s=1}^k f(a_s + n_s\mathbb{Z}) = \sum_{t=1}^l f(b_t + m_t\mathbb{Z}). \tag{2.2}$$

Here are some examples of such results:

(a) (Erdős [E2]) $\{a_s(n_s)\}_{s=1}^k \sim \{0(1)\} \Longrightarrow \sum_{s=1}^k \frac{1}{n_s} = 1$.

(b) (B. Novák and Znám [NZ]) $\{a_s(n_s)\}_{s=1}^k \sim \{0(1)\} \Longrightarrow \sum_{s=1}^k \frac{z^{a_s}}{1-z^{n_s}} = \frac{1}{1-z}$ where z is any complex number with $|z| \neq 1$.

(c) (Porubský [P]) $\{a_s(n_s)\}_{s=1}^k \sim \{0(1),\cdots,0(1)\} \implies \sum_{s=1}^k n_s^{n-1}B_n(\frac{a_s}{n_s}) = mB_n$ where m is the number of repetitions of $0(1)$, and $B_n(x)$ is the Bernoulli polynomial of degree n and $B_n = B_n(0)$.

It is interesting to determine all those functions satisfying (2.2).

Definition 2.1. Let m be an integer and M an additive abelian group. Let F be a map from a subset of $\mathbb{C}\times\mathbb{C}$ into M. If for any ordered pair $\langle x,y\rangle$ in the domain $\mathrm{Dom}(F)$ of F and each positive integer n, we have

$$\left\{\left\langle \frac{x+r}{n}, ny\right\rangle : r=0,1,\cdots,n-1\right\} \subseteq \mathrm{Dom}(F) \tag{2.3}$$

and

$$\sum_{r=0}^{n-1} F\left(\frac{x+r}{n}, ny\right) = F(x,y), \tag{2.4}$$

then we call F a *uniform map* (into M).

In 1989 Z. W. Sun [Nanjing Univ. J. Math. Biquarterly] showed the following Fundamental Theorem on Covering Equivalence:

Theorem 2.1(Z. W. Sun, 1989)**.** *Let M be a left R-module where R is a ring with identity. Let F be a map into M with* $\mathrm{Dom}(F)\subseteq\mathbb{C}\times\mathbb{C}$ *such that* (2.3) *holds for any* $\langle x,y\rangle\in\mathrm{Dom}(F)$ *and $n\in\mathbb{Z}^+$. Then the following two statements are equivalent:*

(a) *F is a uniform map into M.*

(b) *Whenever*

$$\sum_{\substack{1\leqslant s\leqslant k\\ x\in a_s(n_s)}} \lambda_s = \sum_{\substack{1\leqslant t\leqslant l\\ x\in b_t(m_t)}} \mu_t \qquad \text{for all } x\in\mathbb{Z} \tag{2.5}$$

(*with* $\lambda_s,\mu_t\in R$, $a_s,n_s,b_t,m_t\in\mathbb{Z}$, $0\leqslant a_s<n_s$, $0\leqslant b_t<m_t$), *we have*

$$\sum_{s=1}^k \lambda_s F\left(\frac{x+a_s}{n_s}, n_s y\right) = \sum_{t=1}^l \mu_t F\left(\frac{x+b_t}{m_t}, m_t y\right) \quad \text{for } \langle x,y\rangle\in\mathrm{Dom}(F). \tag{2.6}$$

If $F(x,y)$ is a uniform function, then so is $F^-(x,y) = F(\{x\},y)$. Uniform functions are rich in examples.

An identity of Hermite is as follows:

$$\sum_{r=0}^{n-1}\left[x+\frac{r}{n}\right] = [nx] \qquad \text{for } x\in\mathbb{R} \text{ and } n\in\mathbb{Z}^+.$$

This shows that $F(x,y)=[x]$ is an equivalent function.

Let $B_m(x)$ denote the Bernoulli polynomial of degree m. A theorem of Raabe states that

$$\sum_{r=0}^{n} B_m\left(z+\frac{r}{n}\right)=n^{1-m}B_m(nz)\quad \text{for } n\in\mathbb{Z}^+ \text{ and } z\in\mathbb{C},$$

i.e. $b_m(x,y)$ is an equivalent function where $b_m(x,y)=B_m(x)y^{m-1}$. Note that $b_0(x,y)=1/y,\ b_1(x,y)=x-1/2$ and $b_1(x,y)-b_1^-(x,y)=[x]$.

For $s\in\mathbb{C}$ with $Re(s)>1$, the Hurwitz zeta function is given by

$$\zeta(s,z)=\sum_{n=0}^{\infty}\frac{1}{(n+z)^s}.$$

It is easy to verify that $\zeta_s(x,y)=y^{-s}\zeta(s,x)$ is a uniform function.

The multiplication formula of Gauss says that

$$\prod_{r=0}^{n-1}\Gamma\left(z+\frac{r}{n}\right)=(2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-nz}\Gamma(nz)\quad \text{for } n\in\mathbb{Z}^+ \text{ and } z\neq 0,-1,-2,\cdots.$$

Equivalently, $\log\gamma(x,y)$ is a uniform function where

$$\gamma(x,y)=\Gamma(x)y^{x-\frac{1}{2}}/\sqrt{2\pi}\qquad \text{for} x\neq 0,-1,-2,\cdots \text{ and } y>0.$$

$\log(2\sin\pi x)$ and $\frac{1}{y}\cot\pi x$ are also uniform functions.

In view of Theorem 2.1 and various different kinds of uniform functions, it seems that we can not give a unified form for functions satisfying (2.2). However, we have

Theorem 2.2 (Z. W. Sun, Adv. Math. China, 1989; J. Algebra, 2001)**.** *For a function $f:\bigcup_{n\in\mathbb{Z}^+}\mathbb{Z}/n\mathbb{Z}\to\mathbb{C}$, (2.2) holds if and only if f has the following form:*

$$f(a+n\mathbb{Z})=\frac{1}{n}\sum_{m=0}^{n-1}\psi\left(\frac{m}{n}\right)e^{2\pi i\frac{m}{n}a}. \tag{2.7}$$

3. Lower bound for the number of cosets in a cover

A fundamental result on covers of groups is as follows:

Theorem 3.1 (B. H. Neumann, Publ. Math. Debrecen, 1954; M. J. Tomkinson, Comm. Algebra, 1987)**.** *If $\{a_iG_i\}_{i=1}^k$ forms a cover of a group G but none of its proper subsystems does, then*

$$\left[G:\bigcap_{i=1}^{k}G_i\right]\le k!<\infty \tag{3.1}$$

where the bound $k!$ is best possible.

In a cover of group G, if $\bigcap_{i=1}^k G_i$ equals a given subnormal subgroup H of G with finite index, what is the lower bound of k? In this direction we need two functions.

In 1966 J. Mycielski [JS] introduced the following function $f:\mathbb{Z}^+\to\mathbb{N}$.

$$f(p)=p-1 \text{ for any prime } p \ \text{ and } f(mn)=f(m)+f(n) \text{ for all } m,n\in\mathbb{Z}^+.$$

Evidently

$$f(p_1^{\alpha_1}\cdots p_r^{\alpha_r})=\sum_{i=1}^r \alpha_i(p_i-1) \tag{3.2}$$

where $p_1,\cdots,p_r$ are distinct primes and $\alpha_1,\cdots,\alpha_r$ are nonnegative integers.

Let G be a group. A subgroup H of G is said to be *subnormal* if there is a chain $H_0=H\subseteq H_1\subseteq\cdots\subseteq H_n=G$ of subgroups of G such that H_i is normal in H_{i+1} for all $0\leqslant i<n$.

Let H be a subnormal subgroup of a group G with finite index, and

$$H_0=H\subset H_1\subset\cdots\subset H_n=G \tag{3.3}$$

be a composition series from H to G (i.e. H_i is maximal normal in H_{i+1} for each $0\leqslant i<n$). If the length n is zero (i.e. $H=G$), then we set $d(G,H)=0$, otherwise we put

$$d(G,H)=\sum_{i=0}^{n-1}([H_{i+1}:H_i]-1). \tag{3.4}$$

By the Jordan–Hölder theorem, $d(G,H)$ does not depend on the choice of the composition series from H to G. Clearly $d(G,H)=0$ if and only if $H=G$. If K is a subnormal subgroup of H with $[H:K]<\infty$, then

$$d(G,H)+d(H,K)=d(G,K).$$

When H is normal in G, the 'distance' $d(G,H)$ was first introduced by I. Korec [K74]. The current general notion is due to Z. W. Sun [S90].

Theorem 3.2 (Z. W. Sun, Fund. Math. 1990; European J. Combin. 2001)**.** *Let G be a group and H a subnormal subgroup of G with finite index. Then*

$$[G:H]-1\geq d(G,H)\geq f([G:H])\geq \log_2[G:H]. \tag{3.5}$$

Moreover, $d(G,H)=f([G:H])$ if and only if G/H_G is solvable where $H_G=\bigcap_{g\in G} gHg^{-1}$ is the largest normal subgroups of G contained in H.

Mycielski's Conjecture [1966, Fund. Math.]: *Let G be an abelian group and $G_1, \cdots, G_k$ be subgroups of G (with finite indices). If $A = \{a_iG_i\}_{s=1}^k$ forms an exact cover of G then*

$$k \geq 1 + f([G : G_s]) \quad \text{for every} \quad s = 1, \cdots, k.$$

Š. Znám [1966, Colloq. Math.]: *Mycielski's conjecture holds for the additive group $\mathbb{Z}$ of integers.*

Znam's Conjecture [1968, Coll. Math. Soc. János Bolyai]: *If $A = \{a_s(n_s)\}_{s=1}^k$ is a disjoint cover then*

$$k \geq 1 + f([n_1, \cdots, n_k]) \quad \text{and hence } [n_1, \cdots, n_k] \leqslant 2^{k-1}.$$

I. Korec [1974, Fund. Math.]: *Let $\{a_iG_i\}_{i=1}^k$ be a partition of a group into left cosets of normal subgroups then $[G : \bigcap_{i=1}^k G_i] < \infty$ and $k \geq 1 + f([G : \bigcap_{i=1}^k G_i])$.*

Theorem 3.3 [Sun, Fund. Math., 134(1990); European J. Combin. 22(2001)]**.** *Let G be a group and $\{a_iG_i\}_{i=1}^k$ be an exact m-cover of G with all the G_i subnormal in G. Then*

$$k \geq m + d\Big(G, \bigcap_{i=1}^k G_i\Big) \tag{3.6}$$

where the lower bound can be attained. Moreover, for any subgroup K of G not contained in all the G_i we have

$$|\{1 \leqslant i \leqslant k : K \not\subseteq G_i\}| \geq 1 + d\Big(K, K \cap \bigcap_{i=1}^k G_i\Big). \tag{3.7}$$

Berger-Felzenbaum-Fraenkel [1988, Coll. Math.]: *If $\{a_iG_i\}_{i=1}^k$ is a disjoint cover of a finite solvable group G, then $k \geq 1 + f([G : G_i])$ for $i = 1, \cdots, k$.*

Theorem 3.4 [Z. W. Sun, European J. Combin. 2001]**.** *Let $\{a_iG_i\}_{i=1}^k$ be an exact m-cover of a group G. Whenever $G/(G_i)_G$ is solvable, we have $k \geq m + f([G : G_i])$ and hence $[G : G_i] \leqslant 2^{k-m}$.*

We have a further conjecture.

Sun's Conjecture 3.1. *Let $\{a_iG_i\}_{i=1}^k$ be an exact m-cover of a group G with all the $G/(G_i)_G$ solvable. Then $k \geq m + f(N)$ where N is the least common multiple of the indices $[G : G_1], \cdots, [G : G_k]$.*

Znám [1969, Colloq. Math.]: *$k \geq 1 + f(n_t)$ if $A = \{a_s(n_s)\}_{s=1}^k$ forms a cover in which $a_t(n_t)$ is disjoint with all the remaining classes.*

Znám [1975, Acta Arith.]: *If* $A = \{a_s(n_s)\}_{s=1}^k$ *is a cover with* $a_t(n_t)$ *irredundant then* $k \geq 1 + f(n_t)$.

Znám [1975, Acta Arith.] *If* $A = \{a_s(n_s)\}_{s=1}^k$ *forms a minimal cover of* $\mathbb{Z}$*, then the above inequalities hold.*

R.J. Simpson [1985, Acta Arith.]: *Let* $A = \{a_s(n_s)\}_{s=1}^k$ *be a minimal cover. Then for any divisor* $d < N = [n_1, \cdots, n_k]$ *of* N *we have*

$$|\{1 \leqslant i \leqslant k : \ n_i \nmid d\}| \geq 1 + f(N/d). \tag{3.8}$$

(Letting $d = 1$ *we then obtain* $k \geq 1 + f(N)$*.)*

In 1990 Z. W. Sun extended Simpson's result to minimal m-cover of cyclic groups. Simpson's result cannot be extended to abelian groups.

Berger-Felzenbaum-Fraenkel [1988, Coll. Math.]: *Let* $\{a_iG_i\}_{i=1}^k$ *be a minimal cover of a group* G *of squarefree order. If all the* G_i *are normal in* G *then* $k \geq 1 + f([G : \cap_{i=1}^k G_i])$.

Again, we can extend this slightly.

Corollary 3.1 [Z. W. Sun, Fund. Math. 1990; European J. Combin. 2001]**.** *Let* H *be a subnormal subgroup of a group* G *with* $[G : H] < \infty$*. Then* $[G : H] \geq 1 + d(G, H_G)$ *and consequently* $|G/H_G| \leqslant 2^{[G:H]-1}$*. Moreover,* H *is normal in* G *if and only if*

$$|N_G(H)/H| + d(H, H_G) \geq [G : H] \tag{3.9}$$

where $N_G(H)$ *denotes the normalizer of* H *in* G*.*

Let $\{Ha_1, \cdots, Ha_k\}$ be the right decomposition of group G. Then the system $\{a_i(a_i^{-1}Ha_i)\}_{i=1}^k$ forms a disjoint cover of G with $\bigcap_{i=1}^k a_i^{-1}Ha_i = H_G$.

Sun's Conjecture 3.2. *Let* G *be a group and* $G_1, \cdots, G_k$ *its subnormal subgroups such that system* $\{G_i\}_{i=1}^k$ *forms a minimal* m*-cover of* G *where* $m \in \mathbb{Z}^+$*. Assume that* $[G : \cap_{i=1}^k G_i] = \prod_{t=1}^r p_t^{\alpha_t}$ *where* $p_1, \cdots, p_r$ *are distinct primes and* $\alpha_1, \cdots, \alpha_r$ *are positive integers. Then*

$$k \geq m + \sum_{t=1}^r p_t(\alpha_t - 1). \tag{3.10}$$