A talk given at Guangxi Noraml University (Guilin, Oct. 23, 2015) and Center for Combinatorics (Nankai Univ., Dec. 3, 2015)

On $g_n(x)=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{k}x^k$ and related topics

Zhi-Wei Sun

Nanjing University
Nanjing 210093, P. R. China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Dec. 3, 2015

Abstract

The polynomials $g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k$ ($n = 0, 1, 2, \ldots$) introduced by the speaker are closely related to Franel polynomials and Apéry polynomials. In this talk we report recent results on some surprising properties of $g_n(x)$ and related topics. We will also mention some related conjectures posed by the speaker.

Part I. Relations among $g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k$ and Franel polynomials and Apéry polynomials

Apéry Numbers

In 1978 Apéry proved that $\zeta(3) = \sum_{n=1}^{\infty} 1/n^3$ is irrational! During his proof he used the sequence $\{B_n/A_n\}_{n=1}^{\infty}$ of rational numbers to approximate $\zeta(3)$, where

$$A_0 = 1,\ A_1 = 5,\ B_0 = 0,\ B_1 = 6,$$

and both $\{A_n\}_{n\geqslant 0}$ and $\{B_n\}_{n\geqslant 0}$ satisfy the recurrence

$$(n+1)^3 u_{n+1} = (2n+1)(17n^2+17n+5)u_n - n^3 u_{n-1}\ (n = 1, 2, \ldots).$$

In fact,

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 = \sum_{k=0}^{n} \binom{n+k}{2k}^2 \binom{2k}{k}^2$$

and these numbers are called *Apéry numbers*.

Franel numbers

In 1894 J. Franel introduced the Franel numbers

$$f_n = \sum_{k=0}^{n} \binom{n}{k}^3 \quad (n = 0, 1, 2, \ldots)$$

and noted the recurrence relation

$$(n+1)^2 f_{n+1} = (7n(n+1)+2)f_n + 8n^2 f_{n-1} \ (n = 1, 2, 3, \ldots).$$

In 2008 D. Callan gave a combinatorial interpretation of the Franel numbers.

V. Strehl's Identity:

$$A_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} f_k.$$

Barrucand's Identity:

$$\sum_{k=0}^{n} \binom{n}{k} f_k = g_n \quad \text{where } g_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

Connection with modular forms

Don Zagier (2009) investigated what integer sequence $\{u_n\}$ satisfies $u_{-1} = 0$, $u_0 = 1$, and the Apéry-like recurrence relation

$$(k+1)^2 u_{k+1} = (Ak^2 + Ak + B)u_k + Ck^2 u_{k-1} \ (k = 1, 2, 3, \ldots).$$

When $(A, B, C) = (7, 2, 8)$, u_n is just the Franel number f_n, and Zagier noted that

$$\sum_{n=0}^{\infty} f_n \left(\frac{\eta(\tau)^3 \eta(6\tau)^9}{\eta(2\tau)^3 \eta(3\tau)^9} \right)^n = \frac{\eta(2\tau)\eta(3\tau)^6}{\eta(\tau)^2 \eta(6\tau)^3}$$

for any complex number τ with $\mathrm{Im}(\tau) > 0$, where

$$\eta(\tau) := e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}).$$

Apéry polynomials

For $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, I introduced the Apéry polynomial

$$A_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k.$$

Note that $A_n = A_n(1)$ for all $n \in \mathbb{N}$.

In 2010 I posed conjectures on $\sum_{k=0}^{p-1} A_k(x) \bmod p^2$, where p is an odd prime and $x \in \{1, -4, 9, -18^2, 99^2, 99^4, -882^2\}$. Below is an example.

Conjecture (Z. W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Remark. I [JNT, 2012] proved the mod p version of the conjectural congruence. The conjecture still remains open!

Some easy facts about arithmetic means

Let $n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Then

$$\frac{1}{n}\sum_{k=0}^{n-1} 1 = 1 \in \mathbb{Z},$$

$$\frac{2}{n}\sum_{k=0}^{n-1} k = n - 1 \in \mathbb{Z},$$

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1) = 1 \in \mathbb{Z}.$$

For $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}$, their arithmetic mean is given by

$$\frac{a_0 + a_1 + \cdots + a_{n-1}}{n} = \frac{1}{n}\sum_{k=0}^{n-1} a_k.$$

Arithmetic means involving Apéry polynomials

Theorem. (i) (Z. W. Sun [J. Number Theory 132(2012)])

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x)\in\mathbb{Z}[x]\quad\text{for all } n=1,2,3,\ldots$$

For any prime $p>3$, we have

$$\sum_{k=0}^{p-1}(2k+1)A_k\equiv p+\frac{7}{6}p^4B_{p-3}\pmod{p^5}$$

where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers.

(ii) (Conjectured by Z. W. Sun and proved by V.J.W. Guo and J. Zeng [JNT, 2012])

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k(x)\in\mathbb{Z}[x]\quad\text{for all } n=1,2,3,\ldots.$$

The polynomials $f_n(x)=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{n}x^k$

Motivated by Strehl's identity $\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{n}=f_n$, we define the polynomials

$$f_n(x):=\sum_{k=0}^n\binom{n}{k}^2\binom{2k}{n}x^k=\sum_{k=0}^n\binom{n}{k}\binom{k}{n-k}\binom{2k}{k}x^k\ (n=0,1,\ldots).$$

Theorem (Sun [Adv. Appl. Math. 51(2013)]) Let p be an odd prime and let r be any p-adic integer. Then

$$\sum_{l=0}^{p-1}(-1)^l\binom{l+r}{l}f_l(x)\equiv\sum_{k=0}^{p-1}\binom{2k}{k}x^k\binom{k+r}{k}^2\pmod{p^2}.$$

To prove this we need an auxiliary identity

$$\sum_{l=k}^{2k}(-1)^l\binom{l}{k}\binom{k}{l-k}\binom{x+l}{l}=\binom{x+k}{x}^2.$$

The polynomials $g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k$

Recall Barrucand's identity

$$\sum_{k=0}^{n} \binom{n}{k} f_k = g_n, \text{ where } g_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k}.$$

I introduced the polynomials

$$g_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k \ (n = 0, 1, \ldots).$$

In 2011, I found some conjectural series for $1/\pi$ involving $f_n(x)$ or $g_n(x)$. For example,

$$\sum_{n=0}^{\infty} \frac{1054n + 233}{3840^n} \binom{2n}{n} f_n(-64) = \frac{520}{\pi},$$

$$\sum_{n=0}^{\infty} \frac{16n + 5}{324^n} \binom{2n}{n} g_n(-20) = \frac{189}{25\pi}.$$

The 520-series with $520 prize was confirmed by M. Rogers and A. Straub [Int. J. Number Theory 9(2013)].

Relations among $A_n(x), f_n(x)$ and $g_n(x)$

Theorem (Z.-W. Sun [Ramanujan J., in press]) Let n be any nonnegative integer. Then

$$\sum_{k=0}^{n}\binom{n}{k}f_k(x)=g_n(x),\quad f_n(x)=\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}g_k(x),$$

and

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}f_k(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}g_k(x).$$

Also, for any $n\in\mathbb{Z}^+$ we have

$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}(-1)^k g_k(x)$$

and

$$\frac{(-1)^{n-1}}{n}\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k(x)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}f_k(x).$$

A general result

Theorem (Z.-W. Sun [Ramanujan J., in press]) Let

$$X_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x_k \quad \text{and} \quad y_n = \sum_{k=0}^{n} \binom{n}{k} x_k \quad \text{for all } n \in \mathbb{N}.$$

Then

$$X_n = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}(-1)^{n-k} y_k \quad \text{for every } n \in \mathbb{N}.$$

Also, for any $n \in \mathbb{Z}^+$ we have

$$\frac{(-1)^{n-1}}{n} \sum_{k=0}^{n-1}(2k+1)X_k = \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{n+k}{k}(-1)^k y_k$$

and

$$\frac{(-1)^{n-1}}{n} \sum_{k=0}^{n-1}(2k+1)(-1)^k X_k = \sum_{k=0}^{n-1} \binom{n-1}{k}\binom{n+k}{k} x_k.$$

Irreducibilities of $A_n(x), f_n(x)$ and $g_n(x)$

Conjecture (Z.-W. Sun, 2013). All the polynomials

$$A_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 x^k,$$
$$x^n f_n\left(\frac{1}{x}\right) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{n} x^{n-k},$$
$$g_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} x^k$$

are irreducible over the field of rational numbers.

Part II. Supercongruences involving $g_n(x)$ and related things

Wolstenholme's Theorem

Wolstenholme's Theorem (1862): For any prime $p > 3$,

$$\binom{2p-1}{p-1} \equiv 1 \pmod{p^3}, \sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2}, \sum_{k=0}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p}.$$

Proof. $\sum_{k=1}^{p-1} 1/k^2 \equiv 0 \pmod{p}$ since $\sum_{j=1}^{p-1} 1/(2j)^2 \equiv \sum_{k=1}^{p-1} 1/k^2$ (mod p). Also,

$$2\sum_{k=1}^{p-1} \frac{1}{k} = \sum_{k=1}^{p-1} \left(\frac{1}{k} + \frac{1}{p-k}\right) = \sum_{k=1}^{p-1} \frac{p}{k(p-k)} \equiv -p\sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod{p^2}$$

and hence

$$\begin{aligned}\binom{2p-1}{p-1} = \prod_{k=1}^{p-1}\left(1+\frac{p}{k}\right) &\equiv 1 + \sum_{k=1}^{p-1}\frac{p}{k} + \frac{p^2}{2}\sum_{1\le j<k\le p-1}\frac{2}{jk} \\ &\equiv 1 + \frac{p^2}{2}\left(\left(\sum_{k=1}^{p-1}\frac{1}{k}\right)^2 - \sum_{k=1}^{p-1}\frac{1}{k^2}\right) \\ &\equiv 1 \pmod{p^3}.\end{aligned}$$

Wolstenholme-type congruences

Skula-Granville Congruence (2004): For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{2^k}{k^2} \equiv -\left(\frac{2^{p-1}-1}{p}\right)^2 \pmod p.$$

A Congruence for Lucas Numbers (conjectured by R. Tauraso in 2010 and proved by H. Pan and Z.-W. Sun [Sci. China Math. 57(2014)]). For any prime $p > 5$, we have

$$\sum_{k=1}^{p-1} \frac{L_k}{k^2} \equiv 0 \pmod p,$$

where the Lucas numbers $L_0, L_1, L_2, \ldots$ are given by

$$L_0 = 2,\ L_1 = 1, \text{ and } L_{n+1} = L_n + L_{n-1}\ (n = 1, 2, 3, \ldots).$$

Supercongruences involving Franel numbers

Theorem (Z. W. Sun [Adv. Appl. Math., 2013]). Let $p > 3$ be a prime. For any p-adic integer r we have

$$\sum_{k=0}^{p-1}(-1)^k\binom{k+r}{k}f_k \equiv \sum_{k=0}^{p-1}\binom{2k}{k}\binom{k+r}{k}^2 \pmod{p^2}.$$

In particular,

$$\sum_{k=0}^{p-1}(-1)^k f_k \equiv \left(\frac{p}{3}\right) \pmod{p^2},\ \sum_{k=0}^{p-1}(-1)^k k f_k \equiv -\frac{2}{3}\left(\frac{p}{3}\right) \pmod{p^2}.$$

We also have

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k}f_k \equiv 0 \pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}f_k \equiv 0 \pmod{p}.$$

Wolstenholme-type congruences involving $g_n(x)$

Theorem (Z.-W. Sun [Ramanujan J., in press]). For any prime $p > 5$, we have

$$\sum_{k=1}^{p-1} \frac{g_k(-1)}{k} \equiv 0 \pmod{p^2}$$

and

$$\sum_{k=1}^{p-1} \frac{g_k(-1)}{k^2} \equiv 0 \pmod{p}.$$

Applying the Zeilberger algorithm via `Mathematica 9` we find the complicated recurrence for $s_n = g_n(-1)$ $(n = 0, 1, 2, \ldots)$:

$$(n+3)^2(4n+5)s_{n+3} + (20n^3 + 125n^2 + 254n + 165)s_{n+2} + (76n^3 + 399n^2 + 678n + 375)s_{n+1} - 25(n+1)^2(4n+9)s_n = 0.$$

Proof of $\sum_{k=1}^{p-1} g_k(-1)/k^2 \equiv 0 \pmod p$

Observe that

$$\sum_{l=1}^{p-1}\frac{1}{l^2}\sum_{k=1}^{l}\binom{l}{k}^2\binom{2k}{k}x^k=\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{l=k}^{p-1}\binom{l-1}{k-1}^2$$

$$=\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{k+j-1}{j}^2=\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{-k}{j}^2.$$

So we have

$$\sum_{l=1}^{p-1}\frac{g_l(x)-1}{l^2}\equiv\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\sum_{j=0}^{p-1-k}\binom{p-k}{j}^2 \pmod p.$$

Recall that $\sum_{l=1}^{p-1} 1/l^2 \equiv 0 \pmod p$. Also, for any $k=1,\ldots,p-1$ we have

$$\sum_{j=0}^{p-1-k}\binom{p-k}{j}^2=\sum_{j=0}^{p-k}\binom{p-k}{j}\binom{p-k}{p-k-j}-1=\binom{2(p-k)}{p-k}-1$$

by the Chu-Vandermonde identity.

Proof of $\sum_{k=1}^{p-1} g_k(-1)/k^2 \equiv 0 \pmod p$

Thus

$$\sum_{k=1}^{p-1}\frac{g_k(x)}{k^2}\equiv\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k\left(\binom{2(p-k)}{p-k}-1\right)\equiv-\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^2}x^k \pmod p$$

(Note that $\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv 0 \pmod p$ for $k=1,\ldots,p-1$.)

R. Tauraso [JNT 130(2010)] obtained $\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}$ mod p^4 via putting $n=p$ in the following identity

$$\sum_{k=1}^{n}\binom{2k}{k}\frac{k^2}{4n^4+k^4}\prod_{j=1}^{k-1}\frac{n^4-j^4}{4n^4+j^4}=\frac{2}{5n^2}$$

conjectured by J. M. Borwein and D. M. Bradley (1997) and proved by G. Almkvist and A. Granville (1999). Z.-W. Sun [JNT 134(2014)] proved $\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv\frac{56}{15}pB_{p-3} \pmod{p^2}$ via several new identities including

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(2(n+k)+1)16^k}=\frac{\binom{2n}{n}^2}{16^n}\sum_{k=0}^{2n}\frac{1}{2k+1}.$$

Proof of $\sum_{k=1}^{p-1} g_k(-1)/k \equiv 0 \pmod{p^2}$

Note that $\sum_{l=1}^{p-1}(g_l(x)-1)/l$ coincides with

$$\sum_{l=1}^{p-1}\frac{1}{l}\sum_{k=1}^{l}\binom{l}{k}^2\binom{2k}{k}x^k=\sum_{k=1}^{p-1}\binom{2k}{k}x^k\sum_{l=k}^{p-1}\frac{1}{k}\binom{l-1}{k-1}\binom{l}{k}$$

$$=\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}x^k\sum_{j=0}^{p-1-k}\binom{k+j-1}{j}\binom{k+j}{j}.$$

For $1\leqslant k\leqslant p-1$ and $p-k<j\leqslant p-1$, clearly

$$\binom{k+j-1}{j}\binom{k+j}{j}=\frac{(k+j-1)!(k+j)!}{(k-1)!k!(j!)^2}\equiv 0 \pmod{p^2}.$$

If $j=p-k$ with $1\leqslant k\leqslant p-1$, then

$$\binom{k+j-1}{j}\binom{k+j}{j}=\binom{p-1}{j}\binom{p}{j}=\frac{p}{j}\binom{p-1}{j-1}\binom{p-1}{j}$$
$$\equiv-\frac{p}{j}\equiv\frac{p}{k} \pmod{p^2}.$$

Proof of $\sum_{k=1}^{p-1} g_k(-1)/k \equiv 0 \pmod{p^2}$

Recall that $\sum_{l=1}^{p-1} 1/l \equiv 0 \pmod{p^2}$. So we have

$$\begin{aligned}\sum_{k=1}^{p-1} \frac{g_k(x)}{k} &\equiv \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} x^k \left(\sum_{j=0}^{p-1} \binom{k+j-1}{j} \binom{k+j}{j} - \frac{p}{k} \right) \\ &= \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} x^k \sum_{j=0}^{p-1} \binom{-k}{j} \binom{-k-1}{j} - p \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} x^k \\ &\equiv \sum_{k=1}^{p-1} \frac{x^k}{k^2} p - p \sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k^2} x^k = p \sum_{k=1}^{p-1} \frac{1-\binom{2k}{k}}{k^2} x^k \pmod{p^2}\end{aligned}$$

with the help of the following lemma.

Lemma. For any prime p, we have

$$k\binom{2k}{k} \sum_{r=0}^{p-1} \binom{-k}{r} \binom{-k-1}{r} \equiv p \pmod{p^2} \text{ for all } k = 1, \ldots, p-1.$$

Under this lemma, we finally obtain the desired result since $\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \equiv 0 \pmod{p}$ and $\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2} \binom{2k}{k} \equiv 0 \pmod{p}$.

Proof of the lemma

Define

$$u_k = \sum_{r=0}^{p-1} \binom{-k}{r}\binom{-k-1}{r} \quad \text{for all } k \in \mathbb{N}.$$

Applying the Zeilberger algorithm via `Mathematica 9`, we find the recurrence

$$\begin{aligned}
&k(k+1)^2(2(2k+1)u_{k+1} - ku_k) \\
=&(p+k)(p+k-1)(2kp+p+3k^2+3k+1)\binom{-1-k}{p-1}\binom{-k}{p-1} \\
=&p^2\binom{p+k}{p}\binom{p+k-1}{p}(2kp+p+3k^2+3k+1).
\end{aligned}$$

Thus, for each $k = 1, \ldots, p-2$, we have

$$2(2k+1)u_{k+1} \equiv ku_k \pmod{p^2}$$

Proof of the lemma (continued)

and hence

$$\begin{aligned}(k+1)\binom{2(k+1)}{k+1}u_{k+1} =&2(k+1)\binom{2k+1}{k+1}u_{k+1}\\ =&2(2k+1)\binom{2k}{k}u_{k+1} \equiv k\binom{2k}{k}u_k \pmod{p^2}.\end{aligned}$$

So it remains to prove $\binom{2}{1}u_1 \equiv p \pmod{p^2}$. With the help of the Chu-Vandermonde identity, we actually have

$$\begin{aligned}u_1 =&\sum_{r=0}^{p-1}(-1)^r\binom{-2}{r} = (-1)^{p-1}\sum_{r=0}^{p-1}\binom{-1}{p-1-r}\binom{-2}{r}\\ =&(-1)^{p-1}\binom{-3}{p-1} = \binom{p+1}{p-1} = \frac{p^2+p}{2}.\end{aligned}$$

This concludes the proof.

Some other congruences involving $g_n(x)$

Theorem (Sun, Ramanujan J., in press). Let $p > 3$ be a prime.

(i) We have

$$\sum_{k=0}^{p-1} g_k(x)(1-p^2H_k^{(2)}) \equiv \sum_{k=0}^{p-1} \frac{p}{2k+1}\left(1-2p^2H_k^{(2)}\right)x^k \pmod{p^4},$$

where $H_k^{(2)} = \sum_{0<j\leqslant k} 1/j^2$. Consequently,

$$\sum_{k=1}^{p-1} g_k \equiv p^2\sum_{k=1}^{p-1} g_kH_k^{(2)} + \frac{7}{6}p^3B_{p-3} \pmod{p^4},$$

$$\sum_{k=0}^{p-1} g_k(-1) \equiv \left(\frac{-1}{p}\right) + p^2\left(\sum_{k=0}^{p-1} g_k(-1)H_k^{(2)} - E_{p-3}\right) \pmod{p^3},$$

$$\sum_{k=0}^{p-1} g_k(-3) \equiv \left(\frac{p}{3}\right) \pmod{p^2}.$$

Some other congruences involving $g_n(x)$

(ii) We also have

$$\sum_{k=1}^{p-1}\frac{g_k(x)}{k}\equiv 0 \pmod p,$$

$$\sum_{k=1}^{p-1}\frac{g_{k-1}}{k}\equiv -\left(\frac{p}{3}\right)2q_p(3) \pmod p,$$

$$\sum_{k=1}^{p-1}kg_k\equiv -\frac{3}{4} \pmod{p^2},$$

and moreover

$$\frac{1}{3n^2}\sum_{k=0}^{n-1}(4k+3)g_k = h_{n-1}$$

for all $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$, where

$$h_n := \int_0^1 g_n(x)dx = \sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\frac{1}{k+1}.$$

Further congruences involving g_n

Theorem (G.-S. Mao and Z.-W. Sun, IJNT, in press). Let $p > 3$ be a prime, and let $H_k = \sum_{0<j\leqslant k} 1/j$ for $k = 0, 1, 2, \ldots$. Then

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_k \equiv \frac{1}{3}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} H_{2k} \equiv \frac{7}{12}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p},$$

where $B_n(x)$ denotes the Bernoulli polynomial of degree n. Also,

$$\frac{1}{p^2}\sum_{k=1}^{p-1} g_k \equiv \sum_{k=1}^{p-1} g_k H_k^{(2)} \equiv \frac{5}{8}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p}$$

and

$$\frac{1}{p^2}\sum_{k=1}^{p-1} h_k \equiv \sum_{k=1}^{p-1} h_k H_k^{(2)} \equiv \frac{3}{4}\left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) \pmod{p}.$$

A conjecture for $g_n(x)$

Conjecture (Z. W. Sun [Ramanujan J., in press]). For every $n = 1, 2, 3, \ldots$, we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x) \in \mathbb{Z}[x]$$

and the number

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(8k^2+12k+5)g_k(-1)$$

is always an odd integer. Also, for any prime p we have

$$\sum_{k=0}^{p-1}(8k^2+12k+5)g_k(-1) \equiv 3p^2 \pmod{p^3}.$$

Progress on the conjecture

In 2015, V. J. W. Guo, G.-S. Mao and H. Pan made important progress on the last conjecture in their preprint "*Proof of a conjecture involving Sun polynomials*" (arXiv:1511.04005). [They called those $g_n(x)$ $(n = 0, 1, 2, \ldots)$ Sun polynomials.]

Theorem (Guo, Mao and Pan, 2015). For any positive integer n, we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)g_k(x) \in \mathbb{Z}[x]$$

and

$$\sum_{k=0}^{n-1}(8k^2+12k+5)g_k(-1) \equiv 0 \pmod{n}.$$

Sketch of their proof of the last congruence

How Guo, Mao and Pan proved that $n \mid \sum_{k=0}^{n-1}(8k^2+12k+5)$?
They define

$$S_m = \sum_{k=0}^{m-1}(-1)^k \binom{2k}{k}\binom{m-3}{k}\binom{k}{m-3-k} = f_{m-3}(-1)$$

for all $m = 0, 1, 2, \ldots$, and show that

$$\begin{aligned}
&\sum_{m=0}^{n-1}(8m^2+12m+5)g_m(-1)\\
\equiv&\sum_{m=1}^{n}\binom{n}{m}(S_{m+2}+12S_{m+1}+16S_m)\\
=&n\sum_{m=1}^{n-1}\binom{n-1}{m-1}\frac{S_{m+2}+12S_{m+1}+16S_m}{m} \pmod{2n^2}.
\end{aligned}$$

Sketch of their proof of the last congruence

Zeilberger's algorithm yields the following recurrence for S_m:

$$\begin{aligned}&(5m^3-8m^2)S_{m+3}+(45m^3-117m^2+90m-24)S_{m+2}\\&+(200m^3-720m^2+824m-288)S_{m+1}\\&+(160m^3-736m^2+1024m-384)S_m\\&=0.\end{aligned}$$

This equality modulo $24m$ gives the congruence

$$-24S_{m+2}-288S_{m+1}-384S_m\equiv 0 \pmod{24m},$$

i.e.,

$$S_{m+2}+12S_{m+1}+16S_m\equiv 0 \pmod{m}.$$

So, they get the congruence

$$\sum_{m=0}^{n-1}(8m^2+12m+5)g_m(-1)\equiv 0 \pmod{n}.$$

Connection between $p = x^2 + 3y^2$ and Franel numbers

Z.W. Sun [J. Number Theory 133(2013)]: Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}.$$

If $p \equiv 2 \pmod 3$, then

$$\sum_{k=0}^{p-1} \frac{f_k}{2^k} \equiv -2 \sum_{k=0}^{p-1} \frac{f_k}{(-4)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

Conjecture (Z. W. Sun): For any prime $p = x^2 + 3y^2$ with $x \equiv 1 \pmod 3$, we have

$$x \equiv \frac{1}{4} \sum_{k=0}^{p-1} (3k+4) \frac{f_k}{2^k} \equiv \frac{1}{2} \sum_{k=0}^{p-1} (3k+2) \frac{f_k}{(-4)^k} \pmod{p^2}.$$

Connection between $p = x^2 + 3y^2$ and the numbers g_n

Conjecture (Z. W. Sun [J. Number Theory 2013]): Let $p > 3$ be a prime. When $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, we have

$$\sum_{k=0}^{p-1} \frac{g_k}{3^k} \equiv \sum_{k=0}^{p-1} \frac{g_k}{(-3)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}$$

and

$$x \equiv \sum_{k=0}^{p-1} (k+1) \frac{g_k}{3^k} \equiv \sum_{k=0}^{p-1} (k+1) \frac{g_k}{(-3)^k} \pmod{p^2}.$$

If $p \equiv 2 \pmod 3$, then

$$2 \sum_{k=0}^{p-1} \frac{g_k}{3^k} \equiv - \sum_{k=0}^{p-1} \frac{g_k}{(-3)^k} \equiv \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

On $\sum_{k=0}^{p-1} g_k/(\pm 3)^k$ modulo p

Let m be 3 or -3. Then $m-1 \in \{2,-4\}$. Observe that

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{g_n}{m^n} &= \sum_{n=0}^{p-1}\frac{1}{m^n}\sum_{k=0}^{n}\binom{n}{k}f_k = \sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{n=k}^{p-1}\binom{n}{k}\frac{1}{m^{n-k}}\\ &= \sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{k+j}{j}\frac{1}{m^j} = \sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{-k-1}{j}\frac{1}{(-m)^j}\\ &\equiv \sum_{k=0}^{p-1}\frac{f_k}{m^k}\sum_{j=0}^{p-1-k}\binom{p-1-k}{j}\left(-\frac{1}{m}\right)^j = \sum_{k=0}^{p-1}\frac{f_k}{m^k}\left(1-\frac{1}{m}\right)^{p-1-k}\\ &\equiv \sum_{k=0}^{p-1}\frac{f_k}{m^k}\left(\frac{m}{m-1}\right)^k = \sum_{k=0}^{p-1}\frac{f_k}{(m-1)^k} \pmod{p}.\end{aligned}$$

So we obtain $\sum_{k=0}^{p-1} g_k/(\pm 3)^k$ modulo p since

$$\sum_{k=0}^{p-1}\frac{f_k}{2^k} \equiv \sum_{k=0}^{p-1}\frac{f_k}{(-4)^k} \equiv \begin{cases} 2x \pmod{p} & \text{if } p = x^2+3y^2\ (3 \mid x-1),\\ 0 \pmod{p} & \text{if } p \equiv 2 \pmod{3}.\end{cases}$$

Congruences involving $h_n = \int_0^1 g_n(x)dx$

Let

$$h_n := \int_0^1 g_n(x)dx = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \frac{1}{k+1} \quad (n = 0, 1, 2, \ldots).$$

Theorem (Sun, JNT 132(2012)). For any odd prime p, we have

$$h_1 + h_2 + \ldots + h_{p-1} \equiv 0 \pmod{p^2}.$$

Conjecture (Sun, August 2013). For $n = 0, 1, 2, \ldots$ let

$$H_n = |h_{i+j}|_{0 \leqslant i,j \leqslant n}.$$

Let p be any odd prime. If $p \equiv 1 \pmod 3$ and $p = x^2 + 3y^2$ with $x, y \in \mathbb{Z}$ and $x \equiv 1 \pmod 3$, then

$$H_{p-1} \equiv (-1)^{(p-1)/2} \left(2x - \frac{p}{2x}\right) \pmod{p^2}.$$

If $p \equiv 2 \pmod 3$ then

$$H_{p-1} \equiv (-1)^{(p+1)/2} \frac{3p}{\binom{(p+1)/2}{(p+1)/6}} \pmod{p^2}.$$

Some conjectures on f_n and g_n

Conjecture (Z.-W. Sun, 2012). (i) For any integer $n > 1$, we have

$$\sum_{k=0}^{n-1}(9k^2+5k)(-1)^k f_k \equiv 0 \pmod{(n-1)n^2}$$

For any positive integer n, we have

$$\sum_{k=0}^{n-1}(4k+1)g_k 9^{n-1-k} \equiv 0 \pmod{n^2}.$$

(ii) For any prime $p > 3$, we have

$$\sum_{k=0}^{p-1}\frac{g_k}{9^k} \equiv \left(\frac{p}{3}\right) \pmod{p^2}, \quad \sum_{k=1}^{p-1}\frac{g_{k-1}}{k} \equiv -\left(\frac{p}{3}\right)\frac{9^{p-1}-1}{p} \pmod{p^2}.$$

Remark. The speaker's conjecture that $n^2 \mid \sum_{k=0}^{n-1}(3k+2)(-1)^k f_k$ was later confirmed by V.J.W. Guo [Integral Transforms. Spec. Funct. 24(2013)].

Another conjecture

Conjecture (Z.-W. Sun [Ramanujan J., in press]). For $n \in \mathbb{N}$ define

$$F_n := \sum_{k=0}^{n} \binom{n}{k}^3 (-8)^k \quad \text{and} \quad G_n := \sum_{k=0}^{n} \binom{n}{k}^2 (6k+1) C_k,$$

where C_k refers to the Catalan number $\binom{2k}{k}/(k+1)$. For any $n \in \mathbb{Z}^+$, the number

$$\frac{1}{n} \sum_{k=0}^{n-1} (6k+5)(-1)^k F_k$$

is always an odd integer. Also, for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1} (-1)^k F_k \equiv \left(\frac{p}{3}\right) \pmod{p^2} \text{ and } \sum_{k=1}^{p-1} G_k \equiv -\frac{4}{3} p^3 B_{p-3} \pmod{p^4}.$$

Conjectures involving $\sum_{k=0}^n \binom{n}{k}^4 x^k$

In 2011 the author introduced the polynomials $S_n(x) = \sum_{k=0}^n \binom{n}{k}^4 x^k$ $(n = 0, 1, 2, \ldots)$ and posed 13 related conjectures. Here is one of them.

Conjecture (Z. W. Sun) For any prime $p > 2$ we have

$$\sum_{n=0}^{p-1} S_n(12)$$
$$\equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod{12}\ \&\ p = x^2 + y^2\ (3 \nmid x), \\ (\frac{xy}{3})4xy \pmod{p^2} & \text{if } p \equiv 5 \pmod{12}\ \&\ p = x^2 + y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4, \end{cases}$$

$$\sum_{k=0}^{p-1}(4k+3)S_k(12) \equiv p\left(1 + 2\left(\frac{3}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(4k+3)S_k(12) \in \mathbb{Z} \qquad \text{for all } n = 1, 2, 3, \ldots.$$

Conjectures involving $\sum_{k=0}^{n}\binom{n}{k}^4 x^k$

Here is another conjecture.

Conjecture (Z. W. Sun). Let p be an odd prime. Then

$$\sum_{k=0}^{p-1} S_k(-20)$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,9 \pmod{20}\ \&\ p=x^2+y^2\ (5\nmid x),\\ 4xy \pmod{p^2} & \text{if } p\equiv 13,17\,(\mathrm{mod} 20),\ p=x^2+y^2\ (5\mid x-y),\\ 0 \pmod{p^2} & \text{if } p\equiv 3 \pmod 4. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(6k+5)S_k(-20)\equiv p\left(\frac{-1}{p}\right)\left(2+3\left(\frac{-5}{p}\right)\right) \pmod{p^2}.$$

Moreover,

$$\frac{1}{n}\sum_{k=0}^{n-1}(6k+5)S_k(-20)\in\mathbb{Z} \qquad \text{for all } n=1,2,3,\ldots.$$

Thank you!