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p-adic Congruences Motivated by Series

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Part A. Philosophy about p-adic Congruences

Analytic number theory

In analytic number theory, if an arithmetic function f is not regular then number theorists turn to study the main term of the arithmetic mean $\frac{1}{x}\sum_{n\leqslant x} f(n)$ or the partial sum $\sum_{n\leqslant x} f(n)$. If $f(n)$ takes 1 or 0 according as n is a prime or not, then f is irregular but we have

$$\pi(x) = \sum_{n\leqslant x} f(n) \sim \frac{x}{\log x} \quad \text{(Prime Number Theorem)}.$$

The divisor function $d(n)$ has the value 2 if and only if n is a prime. Dirichlet showed that

$$\sum_{n\leqslant x} d(n) = x\log x + (2\gamma - 1)x + O(\sqrt{x}).$$

It is also known that

$$\sum_{n\leqslant x} d(n)^2 = \frac{x\log^3 x}{\pi^2} + O(x\log^2 x).$$

General philosophy about p-adic congruences

Given a *natural* sequence $a_0, a_1, a_2, \ldots$ of p-adic integers, we may consider $\sum_{k=0}^{p-1} a_k$ or $\sum_{k=0}^{(p-1)/2} a_k$ modulo power of p because such a sum usually behaves better than a general term. When a_{p-1} and $a_{(p-1)/2}$ modulo powers of p obey certain patterns, the partial sum $\sum_{k=0}^{p-1} a_k$ or $\sum_{k=0}^{(p-1)/2} a_k$ should also have patterns modulo powers of p.

Example. Let $p > 3$ be a prime. Then

$$\binom{2p}{p} = 2\binom{2p-1}{p-1} \equiv 2 \pmod{p^3} \ \text{(J. Wolstenholme, 1863)},$$

$$\binom{p-1}{(p-1)/2} \equiv (-1)^{(p-1)/2} 4^{p-1} \pmod{p^3} \ \text{(F. Morley, 1895)},$$

$$\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left(\frac{p}{3}\right) \pmod{p^2} \ \text{(Z. W. Sun and R. Tauraso, 2011)}.$$

Another example involving Apéry numbers

In his proof of the irrationality of $\zeta(3)$, Apéry introduced

$$A_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \ (n = 0, 1, 2, \ldots).$$

Beukers' Conjecture (1985) [proved by S. Ahlgren and K. Ono in 2000]. For any prime $p > 3$ we have the super congruence

$$A_{(p-1)/2} \equiv a(p) \pmod{p^2},$$

where $a(n)$ $(n = 1, 2, 3, \ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau) = q \prod_{n=1}^{\infty} (1 - q^{2n})^4 (1 - q^{4n})^4 = \sum_{n=1}^{\infty} a(n) q^n.$$

Conjecture (Z. W. Sun, 2010). For any odd prime p, we have

$$\sum_{k=0}^{p-1} A_k \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p = x^2 + 2y^2, \\ 0 \pmod{p^2} & \text{if } p \equiv 5, 7 \pmod{8}. \end{cases}$$

Part B. Arithmetic Theory of Harmonic Numbers

Harmonic numbers

Harmonic numbers are given by

$$H_0 = 0 \text{ and } H_n := \sum_{k=1}^{n} \frac{1}{k} \quad (n = 1, 2, 3, \ldots).$$

Euler's constant:

$$\lim_{n \to +\infty} (H_n - \log n) = \gamma = 0.577\ldots.$$

Harmonic numbers of order m:

$$H_n^{(m)} := \sum_{0 < k \leqslant n} \frac{1}{k^m}.$$

The Riemann zeta function:

$$\lim_{n \to +\infty} H_n^{(m)} = \zeta(m) \quad (m = 2, 3, 4, \ldots).$$

Bernoulli numbers

Bernoulli numbers $B_0, B_1, B_2, \ldots$ are given by

$$B_0 = 1, \quad \sum_{k=0}^{n} \binom{n+1}{k} B_k = 0 \quad (n = 1, 2, 3, \ldots).$$

It is known that $B_1 = -1/2$ and $B_{2n+1} = 0$ for $n = 1, 2, 3, \ldots$.
van Staudt-Clausen Theorem. For $m = 2, 4, 6, \ldots$ we have

$$B_m + \sum_{p-1 \mid m} \frac{1}{p} \in \mathbb{Z}$$

and hence

$$pB_m \equiv \begin{cases} -1 \pmod{p} & \text{if } p-1 \mid m, \\ 0 \pmod{p} & \text{if } p-1 \nmid m. \end{cases}$$

Euler: For $m = 1, 2, 3, \ldots$ we have

$$2\zeta(2m) = (-1)^{m-1} \frac{B_{2m}}{(2m)!} (2\pi)^{2m}.$$

$$\zeta(2) = \frac{\pi^2}{6} \text{ and } \zeta(4) = \frac{\pi^4}{90}.$$

Series involving harmonic numbers

$$\sum_{k=1}^{\infty} \frac{H_k}{k^2} = 2\zeta(3) \text{ (Euler)},$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k^3} = \frac{\pi^4}{72} \text{ (Goldbach, 1742)},$$

$$\sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17}{360}\pi^4 \text{ (D. Borwein and J.M. Borwein, 1995)},$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12} \text{ (S.W. Coffman, 1987)},$$

$$\sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k2^k} = \frac{5}{8}\zeta(3) \text{ (B. Cloitre, 2004)}.$$

Classical congruences for harmonic numbers

Let $p > 3$ be a prime.

J. Wolstenholme (1863):

$$H_{p-1} \equiv 0 \pmod{p^2} \text{ and } H_{p-1}^{(2)} \equiv 0 \pmod{p}.$$

J.W.L. Glaisher (1900):

$$H_{p-1}^{(m)} \equiv \begin{cases} \frac{pm}{m+1} B_{p-1-m} \pmod{p^2} & \text{if } m \in \{2, 4, \ldots, p-3\}, \\ -\frac{p^2 m(m+1)}{2(m+2)} B_{p-2-m} \pmod{p^3} & \text{if } m \in \{1, 3, \ldots, p-4\}. \end{cases}$$

In particular,

$$\begin{aligned} H_{p-1} &\equiv -\frac{p^2}{3} B_{p-3} \pmod{p^3}, \\ H_{p-1}^{(2)} &\equiv \frac{2}{3} p B_{p-3} \pmod{p^2}, \\ H_{p-1}^{(3)} &\equiv -\frac{6}{5} p^2 B_{p-5} \pmod{p^3} \text{ for } p > 5. \end{aligned}$$

On $H^{(m)}_{(p-1)/2}$ mod p^2

E. Lemma (1938) For any prime $p > 3$ we have

$$H_{(p-1)/2} \equiv -2q_p(2) + p\,q_p(2)^2 \pmod{p^2},$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1} - 1)/p$.

Z. H. Sun (2000) Let $p > 5$ be a prime. Then

$$H_{(p-1)/2} \equiv -2q_p(2) + p\,q_p(2)^2 - p^2\left(\frac{2}{3}q_p(2)^2 + \frac{7}{12}B_{p-3}\right) \pmod{p^2},$$

and

$$H^{(m)}_{(p-1)/2} \equiv \begin{cases} \frac{m(2^{m+1}-1)}{2(m+1)} pB_{p-1-m} \pmod{p^2} & \text{if } m \in \{2, 4, \ldots, p-5\}, \\ \frac{2-2^m}{m} B_{p-m} \pmod{p} & \text{if } m \in \{3, 5, \ldots, p-4\}. \end{cases}$$

Fundamental congruences for harmonic numbers

Z. W. Sun (2009) [Proc. AMS 140(2012)] Let $p > 5$ be a prime. Then

$$\sum_{k=1}^{p-1} H_k^2 \equiv 2p-2 \pmod{p^2},$$

$$\sum_{k=1}^{p-1} H_k^3 \equiv 6 \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv 0 \pmod{p},$$

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv 0 \pmod{p}.$$

Recall that

$$\sum_{k=1}^{\infty} \frac{H_k}{k2^k} = \frac{\pi^2}{12} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{H_k^2}{k^2} = \frac{17\pi^4}{360}.$$

Further congruences involving Bernoulli numbers

Sun and Zhao (arXiv:0911.4433). Let $p > 5$ be a prime. Then

$$\sum_{k=1}^{p-1} \frac{H_k^{(2)}}{k2^k} \equiv -\frac{3}{8} B_{p-3} \pmod p \quad \left(\text{in contrast, } \sum_{k=1}^{\infty} \frac{H_k^{(2)}}{k2^k} = \frac{5}{8}\zeta(3)\right)$$

$$\sum_{k=1}^{p-1} \frac{H_k}{k2^k} \equiv \frac{7}{24} pB_{p-3} \pmod{p^2} \quad \text{(conjectured by Sun).}$$

If $p > 6n + 1$ then

$$\sum_{k=1}^{p-1} \frac{(H_k^{(2n)})^2}{k^{2n}} \equiv \frac{n + \binom{6n+1}{2n-1}}{6n+1} pB_{p-1-6n} \pmod{p^2}.$$

One more congruence:

$$\sum_{k=1}^{p-1} \frac{H_k^2}{k^2} \equiv \frac{4}{5} pB_{p-5} \pmod{p^2}$$

[Conjectured by Sun and proved by R. Meštrović [IJNT 8(2012)]].

More congruences involving Bernoulli numbers

Z. W. Sun & Tauraso [Adv. in Appl. Math. 45(2010)].

$$\sum_{k=1}^{p-1}\frac{H_k}{k^2}\equiv B_{p-3}\pmod p\quad\text{for any prime } p>3.$$

R. Meštrović [IJNT 8(2012)]. For any prime $p>5$ we have

$$\sum_{k=1}^{p-1}\frac{H_k^3}{k}\equiv\frac{3}{2}pB_{p-5}\pmod{p^2}.$$

Z. W. Sun (arXiv:1011.3487). Let m and n be positive integers with $m\leqslant 2n+1$, and let $p>2n+1$ be a prime. Then

$$\sum_{k=1}^{p-1}\frac{H_k^{(m)}}{k^{2n+1-m}}\equiv\frac{(-1)^{m-1}}{2n+1}\binom{2n+1}{m}B_{p-1-2n}\pmod p.$$

When $m<2n$ we have

$$\sum_{k=1}^{p-1}\frac{H_k^{(m)}}{k^{2n-m}}\equiv\frac{pB_{p-1-2n}}{2n+1}\left(n+(-1)^m\frac{n-m}{m+1}\binom{2n+1}{m}\right)\pmod{p^2}.$$

An application

Motivation.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$$

Z. W. Sun (arXiv:1011.3487). Let $p > 3$ be a prime. Then

$$\sum_{k=0}^{p-1}\binom{-1/(p+1)}{k}^{p+1} \equiv 0 \pmod{p^5}$$

and

$$\sum_{k=0}^{p-1}\binom{1/(p-1)}{k}^{p-1} \equiv 0 \pmod{p^4}.$$

If $p > 2n+1$ is a prime then

$$\sum_{k=1}^{p-1}\frac{1}{k^{2n-1}}\binom{1/(p-1)}{k}^{p-1} \equiv -\frac{2p^2n^2}{2n+1}B_{p-1-2n} \pmod{p^3},$$

$$\sum_{k=1}^{p-1}\frac{1}{k^{2n-1}}\binom{-1/(p+1)}{k}^{p+1} \equiv \frac{p^2n}{2n+1}B_{p-1-2n} \pmod{p^3}.$$

Part C. p-adic Congruences from π-series or the ζ-function

On $\sum_{k=1}^{p-1}\binom{2k}{k}/k$ mod powers of p

Z. W. Sun & Tauraso [Adv. in Appl. Math 45(2010)]. For any prime $p > 5$ we have

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k} \equiv \frac{8}{9}p^2 B_{p-3} \pmod{p^3}$$

and

$$\sum_{k=1}^{p-1}(-1)^k\frac{\binom{2k}{k}}{k} \equiv -\frac{5}{p}F_{p-(\frac{p}{5})} \pmod{p}.$$

The first congruence can be deduced by taking $n = p-1$ in Staver's identity

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k} = \frac{n+1}{3}\binom{2n+1}{n}\sum_{k=1}^{n}\frac{1}{k^2\binom{n}{k}^2}.$$

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k} \equiv \frac{p}{3}\binom{2p-1}{p-1}\sum_{k=1}^{p-1}\frac{1}{k^2} \equiv 0 \pmod{p^2}.$$

The proof of the second congruence is more sophisticated.

Connections to Euler numbers

$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\equiv\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0 \pmod p.$$

How to determine $\sum_{k=1}^{(p-1)/2}\binom{2k}{k}/k \bmod p^2$?

Recall that Euler numbers $E_0, E_1, \ldots$ are given by

$$E_0=1,\ \sum_{2\mid k}\binom{n}{k}E_{n-k}=0\ (n=1,2,3,\ldots).$$

It is known that $E_1=E_3=E_5=\cdots=0$ and

$$\sec x=\sum_{n=0}^{\infty}(-1)^nE_{2n}\frac{x^{2n}}{(2n)!}\ \left(|x|<\frac{\pi}{2}\right).$$

Z. W. Sun [Sci. China Math. 54(2011)]. For any prime $p>3$ we have

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv(-1)^{(p+1)/2}\frac{8}{3}pE_{p-3} \pmod{p^2}.$$

More congruences involving Euler numbers

Z. W. Sun [Sci. China Math., 54(2011)]: Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2\binom{2k}{k}} \equiv (-1)^{(p-1)/2}\frac{4}{3}E_{p-3} \pmod{p},$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2}+p^2E_{p-3} \pmod{p^3},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv (-1)^{(p-1)/2}-p^2E_{p-3} \pmod{p^3}.$$

Related known results:

$$\sum_{k=1}^{\infty}\frac{1}{k^2\binom{2k}{k}} = \frac{\zeta(2)}{3} = \frac{\pi^2}{18},$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

Apéry's series and its p-adic analogue

In Apéry's proof of the irrationality of $\zeta(3)$, the following series plays a key role.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^3 \binom{2k}{k}} = -\frac{2}{5}\zeta(3).$$

This was obtained by letting $n \to +\infty$ in the following identity:

$$5\sum_{k=1}^{n} \frac{(-1)^k}{k^3 \binom{2k}{k}} = \sum_{k=1}^{n} \frac{(-1)^k}{k^3 \binom{n}{k}\binom{n+k}{k}} - 2H_n^{(3)}.$$

Taking $n = p-1$ Tauraso [JNT 130(2010)] got the following p-adic analogue for any prime $p > 5$.

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^3 \binom{2k}{k}} \equiv -\frac{2}{5} \cdot \frac{H_{p-1}}{p^2} \pmod{p^3}$$
$$\equiv \frac{2}{15} B_{p-3} \pmod{p^2}.$$

On $\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2}\binom{2k}{k} \bmod p^2$

A curious identity (conjectured by J.M. Borwein and D.M. Bradley in 1997 and proved by G. Almkvist and A. Granville in 1999).

$$\sum_{k=1}^{n} \binom{2k}{k} \frac{k^2}{4n^4+k^4} \prod_{j=1}^{k-1} \frac{n^4-j^4}{4n^4+j^4} = \frac{2}{5n^2}.$$

Taking $n=p$ Tauraso [JNT 130(2010)] obtained the following congruence:

$$\sum_{k=1}^{p-1} \frac{(-1)^k}{k^2}\binom{2k}{k} \equiv \frac{4}{5} \cdot \frac{H_{p-1}}{p} \pmod{p^3}$$

$$\equiv -\frac{4}{15} p B_{p-3} \pmod{p^2}$$

for any prime $p > 5$.

Another contribution of Tauraso

Let $p = 2n+1$ be an odd prime. As van Hamme observed,

$$\binom{n}{k}\binom{n+k}{k}(-1)^k = \binom{(p-1)/2}{k}\binom{(-p-1)/2}{k}$$

$$\equiv \binom{-1/2}{k}^2 = \left(\frac{\binom{2k}{k}}{(-4)^k}\right)^2 = \frac{\binom{2k}{k}^2}{16^k} \pmod{p^2}.$$

Taking $n = (p-1)/2$ in the combinatorial identity

$$\sum_{k=1}^{n} \frac{(-1)^k}{k}\binom{n}{k}\binom{n+k}{k} = -2H_n,$$

we see that

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k} \equiv -2H_{(p-1)/2} \pmod{p^2}.$$

R. Tauraso's Conjecture (2009): For any prime $p > 3$ we have

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{k16^k} \equiv -2H_{(p-1)/2} \pmod{p^3}.$$

My discoveries in Feb. 2010

Conjecture (Sun, 2010). Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3\binom{2k}{k}} \equiv -2B_{p-3} \pmod{p},$$

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^2}\binom{2k}{k} \equiv \frac{56}{15}pB_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k} \equiv -2H_{(p-1)/2} - \frac{7}{2}p^2B_{p-3} \pmod{p^3},$$

$$\sum_{k=(p+1)/2}^{p-1} \frac{\binom{2k}{k}^2}{k16^k} \equiv \frac{7}{2}p^2B_{p-3} \pmod{p^3}.$$

Remark. In Feb. 2010 I could prove that all the four congruences are equivalent.

A useful transformation

Let p be an odd prime. Then for any $k = 0, 1, \ldots, p-1$ we have

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2}.$$

In particular, if $p/2 < k < p$ then

$$\frac{\binom{2k}{k}}{p} \equiv \frac{2}{k\binom{2(p-k)}{p-k}} \pmod{p}.$$

This technique was often used by Tauraso and Z.W. Sun.

Example. For any odd prime $p = 2n+1$ we have

$$\frac{1}{p}\sum_{k=n+1}^{p-1} \frac{(-1)^k}{k^2}\binom{2k}{k} \equiv \sum_{k=n+1}^{p-1} \frac{2(-1)^k}{k^3\binom{2(p-k)}{p-k}} \equiv 2\sum_{k=1}^{n} \frac{(-1)^k}{k^3\binom{2k}{k}} \pmod{p}.$$

On $\sum_{p/2<k<p}\binom{2k}{k}^2/(k16^k) \bmod p^3$

Apéry's identity

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{n}{k}\binom{n+k}{k}} = 5\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}} + 2H_n^{(3)}.$$

Taking $n=(p-1)/2$ we get

$$\sum_{k=1}^{n}\frac{16^k}{k^3\binom{2k}{k}} \equiv 5\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}} + 2H_n^{(3)} \pmod{p^2}.$$

It is known that $H_{(p-1)/2}^{(3)} \equiv -2B_{p-3} \pmod{p}$. Also,

$$\frac{1}{p^2}\sum_{p/2<k<p}\frac{\binom{2k}{k}^2}{k16^k} \equiv -\frac{1}{4}\sum_{k=1}^{n}\frac{16^k}{k^3\binom{2k}{k}^2} \pmod{p}.$$

So

$$\frac{-4}{p^2}\sum_{p/2<k<p}\frac{\binom{2k}{k}^2}{k16^k} \equiv 5\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}} - 4B_{p-3} \pmod{p}.$$

On $\sum_{0<k<p/2}\binom{2k}{k}^2/(k16^k) \bmod p^3$

A new identity (Sun, Feb. 2010) [found by the software `Sigma`]:

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k}(H_{n+k}-H_{n-k})=\frac{5}{2}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{2k}{k}+2H_n^{(2)}.$$

Taking $n=(p-1)/2$ and noting that

$$\binom{n}{k}\binom{n+k}{k}(-1)^k\left(1-\frac{p}{4}(H_{n+k}-H_{n-k})\right)\equiv\frac{\binom{2k}{k}^2}{16^k}\pmod{p^4}$$

and that

$$\sum_{k=1}^{n}\binom{n}{k}\binom{n+k}{k}\frac{(-1)^k}{k}=-2H_n,$$

we get

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}^2}{k16^k}+2H_n\equiv-\frac{p}{4}\left(\frac{5}{2}\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{2k}{k}+2H_n^{(2)}\right)\pmod{p^4}.$$

It is known that $H^{(2)}_{(p-1)/2}\equiv 7pB_{p-3}/3 \pmod{p^2}$.

Tauraso's proof of his conjecture

Let $(x)_k = \prod_{0\leqslant j<k}(x+j)$ for $k = 0, 1, 2, \ldots$. It is known that

$$\sum_{k=0}^{n-1} \frac{(x)_k(1-x)_k}{(n-k)(1)_k^2} = \frac{(x)_n(1-x)_n}{(1)_n^2} \sum_{k=0}^{n-1} \left(\frac{1}{x+k} + \frac{1}{1-x+k} \right).$$

Taking $x = 1/2$ we get

$$\sum_{k=0}^{n-1} \frac{\binom{2k}{k}^2}{(n-k)16^k} = \frac{\binom{2n}{n}^2}{16^n} \sum_{k=0}^{n-1} \frac{4}{2k+1}.$$

As

$$\frac{1}{p-k} = \frac{p^2+pk+k^2}{p^3-k^3} \equiv -\frac{p^2}{k^3} - \frac{p}{k^2} - \frac{1}{k} \pmod{p^3},$$

to obtain $\sum_{k=1}^{p-1} \binom{2k}{k}^2/(k16^k)$ mod p^3 it suffices to determine

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}^2}{k^r 16^k} \equiv \sum_{k=1}^{n} \binom{n}{k}\binom{n+k}{k} \frac{(-1)^k}{k^r} \bmod p^{4-r}$$

for $r = 2, 3$ which are relatively easy. (Tauraso, May 2011).

My lost proof of Tauraso's conjecture

Let $n=(p-1)/2$. Recall that I ever proved

$$\sum_{p/2<k<p}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv 2p\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}}\pmod{p^2},$$

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}^2}{k16^k}\equiv -\frac{5}{4}p^2\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}}+p^2B_{p-3}\pmod{p},$$

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}^2}{k16^k}+2H_n\equiv -\frac{5}{8}p\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{2k}{k}-\frac{7}{6}p^2B_{p-3}\pmod{p^3}.$$

It follows that

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k16^k}+2H_n\equiv&-\frac{5}{8}p\sum_{k=1}^{p-1}\frac{(-1)^k}{k^2}\binom{2k}{k}-\frac{p^2}{6}B_{p-3}\\ \equiv&0\pmod{p^3}.\end{aligned}$$

On $\sum_k \binom{2k}{k}/((2k+1)16^k)$

It is known that

$$\sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)16^k} = \frac{\pi}{3}, \quad \sum_{k=0}^{\infty} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} = \frac{\pi^2}{10}.$$

I [JNT 131(2011)] proved that for any prime $p > 3$ we have

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv 0 \pmod{p^2},$$

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}}{(2k+1)16^k} \equiv \frac{p}{3} E_{p-3} \pmod{p^2}.$$

And I conjectured that for any prime $p > 5$ we have

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv -\frac{p}{15} B_{p-3} \pmod{p^2},$$

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv -\frac{p}{4} B_{p-3} \pmod{p^2}.$$

Find new series for π^3

There are very few interesting series for π^3. The only well-known series for π^3 is the following one:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^3}=\frac{\pi^3}{32}.$$

Conjecture (Z. W. Sun [JNT 131(2011)]). For any prime $p>5$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}\equiv(-1)^{(p-1)/2}\left(\frac{H_{p-1}}{4p^2}+\frac{p^2}{36}B_{p-5}\right)\pmod{p^3}.$$

Motivated by this conjecture, I guessed that

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{(2k+1)^3 16^k}=\frac{7}{216}\pi^3.$$

After I announced this conjecture, Olivier Gerard pointed out there is a computer proof via `Mathematica` (version 7).

Find new series for π^3

Let p be an odd prime. I proved that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv(-1)^{(p-1)/2}-p^2E_{p-3}\pmod{p^3},$$

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\frac{\binom{2k}{k}}{(-2)^k}\equiv(-1)^{(p-1)/2}2^{p-1}\pmod{p^3}.$$

For $k=0,\ldots,p-1$, it is easy to see that

$$\binom{p-1}{k}(-1)^k\equiv 1+pH_k+\frac{p^2}{2}(H_k^2-H_k^{(2)})\pmod{p^3}.$$

So, it is natural to investigate $\sum_{k=0}^{p-1}\binom{2k}{k}H_k^{(2)}/2^k \bmod p$.

Theorem (Sun, 2010) Let $p>3$ be a prime. Then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}H_k^{(2)}\equiv -E_{p-3}\pmod{p}.$$

Note that $\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{2^k}H_k^{(2)}$ is divergent!

Find new series for π^3

Let p be an odd prime. Recall that for any $k = 1, \ldots, p-1$ we have

$$k\binom{2k}{k}\binom{2(p-k)}{p-k} \equiv (-1)^{\lfloor 2k/p \rfloor - 1} 2p \pmod{p^2}.$$

Note also that

$$H_{p-k}^{(2)} = H_{p-1}^{(2)} - \frac{1}{(p-k+1)^2} - \cdots - \frac{1}{(p-1)^2} \equiv -H_{k-1}^{(2)} \pmod{p}.$$

Thus via the transformation $k \to p-k$ we should investigate $\sum_k 2^k H_{k-1}^{(2)}/(k\binom{2k}{k})$ which cannot be evaluated via `Mathematica`.

Theorem (Sun, Sept. 2010). We have

$$\sum_{k=1}^{\infty} \frac{2^k H_{k-1}^{(2)}}{k\binom{2k}{k}} = \frac{\pi^3}{48}.$$

A sketch of the proof

Using the fact that

$$B(a,b) := \int_0^1 x^{a-1}(1-x)^{b-1}dx = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} \quad \text{for any } a,b>0,$$

and the dilogarithm function $\mathrm{Li}_2(x)$ given by

$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad (|x|<1),$$

I deduced that

$$\sum_{k=1}^{\infty} \frac{2^{k-1}H_{k-1}^{(2)}}{k\binom{2k}{k}} = \int_{-1}^{1} \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt$$

$$= \frac{\pi^3}{96} \quad \text{(by Mathematica 7).}$$

Remark. The indefinite integral

$$\int \frac{\arctan t}{1+t} \log \frac{1+t^2}{2} dt$$

is **very very** complicated. It occurs more than two pages!

On $\sum_k \binom{2k}{k}/((2k+1)^2(-16)^k)$

Let $p > 5$ be a prime. In Oct. 2011 R. Tauraso proved my conjecture

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv \frac{H_{p-1}}{5p} \pmod{p^3}.$$

On Nov. 4 I asked whether he could prove my conjectural congruence

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv -\frac{p}{4}B_{p-3} \pmod{p^2}.$$

He said that he had not tried that.

On Nov. 19 I considered the above unsolved problem. First, I noted that

$$\frac{1}{p}\sum_{p/2<k<p} \frac{\binom{2k}{k}}{(2k+1)^2(-16)^k} \equiv -\sum_{k=0}^{(p-3)/2} \frac{(-16)^k}{(2k+1)^3\binom{2k}{k}} \pmod{p}.$$

On $\sum_{k=0}^{(p-3)/2}(-16)^k/((2k+1)^3\binom{2k}{k})$ mod p

For $n=(p-1)/2$ clearly $\binom{n}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod{p}$. Thus

$$\begin{aligned}\sum_{k=0}^{n-1}\frac{(-16)^k}{(2k+1)^3\binom{2k}{k}} &\equiv\sum_{k=0}^{n-1}\frac{4^k}{(2k+1)^3\binom{n}{k}}\\ &\equiv\sum_{k=1}^{n}\frac{4^{n-k}}{(2(n-k)+1)^3\binom{n}{k}}\\ &\equiv-\frac{1}{8}\sum_{k=1}^{n}\frac{(-1)^k}{k^3\binom{2k}{k}}\pmod{p}.\end{aligned}$$

A further conjecture (Z. W. Sun, Nov. 19, 2011) For any prime $p>5$ we have

$$\sum_{k=0}^{(p-3)/2}\frac{(-16)^k}{(2k+1)^3\binom{2k}{k}}\equiv-\frac{3}{4}\cdot\frac{H_{p-1}}{p^2}-\frac{47}{400}p^2B_{p-5}\pmod{p^3}.$$

On $\sum_{k=0}^{(p-3)/2}\binom{2k}{k}^2/((2k+1)^2 16^k) \bmod p^2$

Let $p>3$ be a prime. Tauraso used the identity

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\binom{n}{k}\binom{n+k}{k}=-2H_n^2$$

to obtain $\sum_{k=1}^{p-1}\binom{2k}{k}^2/(k^2 16^k) \bmod p^2$. I found that

$$\sum_{k=0}^{n}\frac{(-1)^k}{(2k+1)^2}\binom{n}{k}\binom{n+k}{k}=\frac{1}{(2n+1)^2}+\frac{2}{2n+1}\sum_{k=1}^{n}\frac{1}{2k-1}.$$

It led me to deduce

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)^2 16^k}\equiv -2q_p(2)^2+\frac{2}{3}p\,q_p(2)^3-\frac{p}{6}B_{p-3}\pmod{p^2}.$$

Note also that

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)^3 16^k}\equiv -\frac{1}{8}\sum_{k=1}^{p-1}\frac{\binom{2k}{k}^2}{k^3 16^k}\equiv -\frac{4}{3}q_p(2)^3-\frac{B_{p-3}}{6}\pmod{p}.$$

On $\sum_{k=0}^{(p-3)/2}\binom{2k}{k}^2/((2k+1)16^k) \bmod p^3$

Taking $n=(p-1)/2$ in the identity

$$\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(n-k)16^k}=\frac{\binom{2n}{n}^2}{16^n}\sum_{k=0}^{n-1}\frac{4}{2k+1}$$

and noting that

$$\begin{aligned}\frac{1}{p-(2k+1)} &= \frac{p^2+p(2k+1)+(2k+1)^2}{p^3-(2k+1)^3}\\ &\equiv -\frac{p^2}{(2k+1)^3}-\frac{p}{(2k+1)^2}-\frac{1}{2k+1} \pmod{p^3},\end{aligned}$$

on Nov. 20 I finally got the congruence

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}^2}{(2k+1)16^k}\equiv -2q_p(2)-p\,q_p(2)^2+\frac{5}{12}p^2B_{p-3} \pmod{p^3}.$$

On $\sum_k \binom{2k}{k}^2/((2k+1)16^k)$

Series

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}^2}{(2k+1)16^k}=\frac{4G}{\pi}$$

where G is the Catalan constant given by

$$G=L\left(2,\left(\frac{-1}{\cdot}\right)\right)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}.$$

I also observed that

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}^2}{(2k+1)16^k}\equiv\frac{p^2}{8}\sum_{k=1}^{(p-1)/2}\frac{16^k}{k^3\binom{2k}{k}^2}$$

$$\equiv-\frac{1}{2}\sum_{p/2<k<p}\frac{\binom{2k}{k}^2}{k16^k}\pmod{p^3}.$$

On Nov. 21 I wrote an article on such things and posted it to arXiv. The paper contains many equivalent versions of $\sum_{k=1}^{(p-1)/2}(-1)^k/(k^3\binom{2k}{k})\equiv-2B_{p-3}\pmod{p}$.

On Nov. 22, 2011 (before 12:00)

In the morning of Nov. 22, my preprint appeared on arXiv publicly and I talked about the results at our seminar.

I worried that other experts on congruences might be able to show my conjectural congruences after seeing various equivalent versions in my paper. So **it is urgent that I should find a proof as soon as possible!**

On the way for lunch I suddenly realized a possible way to prove the desired congruences. Recall the identity

$$\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(n-k)16^k}=\frac{\binom{2n}{n}^2}{16^n}\sum_{k=0}^{n-1}\frac{4}{2k+1}.$$

If we have a similar identity for

$$S_n:=\sum_{k=0}^{n-1}\frac{\binom{2k}{k}^2}{(2n-2k+1)16^k},$$

then by taking $n=(p-1)/2$ we will be able to determine $\sum_{k=1}^{n}\binom{2k}{k}^2/(k16^k) \bmod p^3$.

On Nov. 22, 2011 (12:01–23:30)

When I got home after lunch, I immediately tried to find an identity for S_n via `Sigma`. Unfortunately `Sigma` in my computer could not run correctly for over one year. Then I sleep for about two hours and tried again after I waked up. `Sigma` still failed to work correctly.

On 17:00 I took our campus bus to Xianlin since I should give a lecture there in the evening. At Xianlin, I changed my way and used the Zeilberger algorithm to find the recursion for S_n first.

$$4(n+1)^2 S_n - (2n+3)^2 S_{n+1} = -\frac{4n+3}{4^{2n-1}}\binom{2n-1}{n}^2 \ (n = 1, 2, \ldots).$$

I could not solve this recursion via `Sigma` in my computer.

I got home at 22:00. Around 22:30 I decided to ask Prof. Qinghu Hou for help. Using another software he found a complicated solution to the recurrence relation.

$$S_n = \frac{64 \times 16^n}{\binom{2n}{n}^2 (n+1)^2} \sum_{k=1}^{n} \frac{(4k-1)k^2}{(2k-1)^2}\binom{2k-1}{k}^4.$$

On Nov. 22, 2011 (23:30–23:58)

The expression for S_n is too complicated and hence not useful! I felt frustrated! Then, I called Qinghu and asked him to send me a correct software for solving recurrence relations in case I might need it. To my surprise he sent me `Sigma`. But after a try I was glad to find that it works well.

I wished to find a proof and included it in the new version of my arXiv paper before 24:00 so that the paper could appear publicly on Nov. 23. It was close to 24:00 but I still had no idea.

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Before I went to bed I decided to make the final try. Instead of $2n+1-2k$ in S_n I used $2n+1+k$ and found that the recursion became simpler though the solution is still complicated. Then I tried the sum $\sum_{k=0}^{n-1}\binom{2k}{k}^2/((2n+2k+1)16^k)$, and found via `Sigma` the new identity

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(2n+2k+1)16^k}=\frac{\binom{2n}{n}^2}{16^n}\sum_{k=0}^{2n}\frac{1}{2k+1}.$$

This is why I wanted! It is easy to investigate the right-hand side modulo p^3 with $n=(p-1)/2$.

On Nov. 23, 2011 (00:00–5:00)

Thus I could determine $\sum_{k=1}^n \binom{2k}{k}^2/(k16^k)$ mod p^3 and hence prove the long-desired congruence

$$\sum_{k=1}^{(p-1)/2} \frac{(-1)^k}{k^3\binom{2k}{k}} \equiv -2B_{p-3} \pmod p,$$

which had puzzled me nearly two years.

I felt very very happy!!!

Then I decided to spend the whole night to revise my article to include this result. I finished the revision and posted it to arXiv just before 5:00. Four hours later (from 9:00 am, Nov. 23, 2011), my proof in the preprint is available publicly!

Unlike the case with Tauraso's conjecture, this time I was lucky and had no loss!

A further conjecture

Conjecture (Z. W. Sun). Let $p > 5$ be a prime. Then

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}^2}{k16^k} \equiv -\frac{21}{2} H_{p-1} \pmod{p^4}.$$

Also,

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{k64^k} \equiv -3H_{(p-1)/2} + \frac{7}{4} p^2 B_{p-3} \pmod{p^3},$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}\binom{4k}{2k}}{k64^k} \equiv -3H_{(p-1)/2} - 2\left(\frac{-1}{p}\right) pE_{p-3} \pmod{p^2},$$

and

$$p\sum_{k=1}^{(p-1)/2} \frac{64^k}{k^3\binom{2k}{k}\binom{4k}{2k}} \equiv 32\left(\frac{-1}{p}\right) E_{p-3} \pmod{p}.$$

Another conjecture on congruences

Conjecture (Z. W. Sun). Let $p > 3$ be a prime. Then

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k}(H_{2k} - H_k) \equiv -\frac{7}{3}pB_{p-3} \pmod{p^2},$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}\binom{4k}{2k}}{k64^k}(H_{2k} - H_k) \equiv -4\left(\frac{-1}{p}\right)E_{p-3} \pmod{p},$$

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}\binom{3k}{k}}{k27^k}(H_{2k} - H_k) \equiv -\frac{1}{2}\left(\frac{p}{3}\right)E_{P-3}\left(\frac{1}{3}\right) \pmod{p}.$$

I have proved that

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}^2}{k16^k}H_k \equiv 4\left(\frac{-1}{p}\right)E_{p-3} \pmod{p}.$$

A conjectural identity

So the first congruence in the conjecture implies that

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{16^k} H_{2k} \equiv 4\left(\frac{-1}{p}\right) E_{p-3} \pmod{p},$$

which has the equivalent form

$$\sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}^2}{(2k+1)16^k} H_{2k} \equiv -2\left(\frac{-1}{p}\right) E_{p-3} \pmod{p}.$$

Conjecture (Z. W. Sun, Jan. 2012) We have the identity

$$\sum_{k=1}^{\infty} \frac{\binom{2k}{k}^2}{k16^k}(H_{2k}-H_k) = \frac{2}{3}\sum_{k=0}^{\infty} \frac{\binom{2k}{k}^2}{(2k+1)16^k} H_{2k}.$$

Both series on the two sides of the identity converge very slowly. On my request, Don Zagier verified the identity up to 200 digits.

Conjectural series motivated by congruences

Based on my investigations of congruences, I have formulated 170 conjectural series for powers of π and other important constants. For example, motivated by my conjectural congruence

$$\sum_{k=0}^{p-1}\frac{28k^2+18k+3}{(-64)^k}\binom{2k}{k}^4\binom{3k}{k}\equiv 3p^2-\frac{7}{2}p^5B_{p-3}\pmod{p^6}$$

for any odd prime p, I conjectured that

$$\sum_{k=1}^{\infty}\frac{(28k^2-18k+3)(-64)^k}{k^5\binom{2k}{k}^4\binom{3k}{k}}=-14\zeta(3).$$

A full list of my conjectural series is available from the preprint Zhi-Wei Sun, *List of conjectural series for powers of π and other constants*, arXiv:1102.5649
`http://arxiv.org/abs/1102.5649`

Thank you!